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Lie powers of relation modules for groups

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ABSTRACT

Motivated by applications to abstract group theory, we study Lie powers of relation modules. The relation module associated to a free presentation $G = F/N$ of a group G is the abelianization $N_{ab} = N/[N, N]$ of N, with G-action given by conjugation in F. The degree n Lie power is the homogeneous component of degree n in the free Lie ring on N_{ab} (equivalently, it is the relevant quotient of the lower central series of N). We show that after reduction modulo a prime p this becomes a projective G-module, provided $n > 1$ and n is not divisible by p.

1. Introduction

Let G be a group given by a free presentation $G = F/N$ where F is a non-cyclic free group and N is a normal subgroup of F. The free abelian group $N_{ab} = N/N'$, regarded as a G-module with action induced by conjugation in F, is known as the relation module for G stemming from the given free presentation.

For an arbitrary commutative ring K with 1 and a free K-module V, let $\mathcal{L}(V)$ denote the free Lie algebra on V over K, and let $\mathcal{L}_n(V)$ denote its degree n homogeneous component. If V carries the structure of a G-module, the action of G extends to the whole of $\mathcal{L}(V)$ with G acting diagonally on Lie products. Thus each $\mathcal{L}_n(V)$ becomes a KG-module called the n-th Lie power of V. The n-th metabelian Lie power $\mathcal{M}_n(V)$ of V is the degree n homogeneous component of the free metabelian Lie algebra $\mathcal{M}(V) = \mathcal{L}(V)/\mathcal{L}''(V)$.

Let p be a prime, let $\mathbb{Z}_{(p)}$ denote the ring of integers localized at p, and let $M = N_{ab} \otimes \mathbb{Z}_{(p)}$. The aim of this paper is to establish the following result.

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Theorem. *Let p be a prime, n an integer, and G a group given by a free presentation $G = F/N$ with F non-cyclic free and N normal in F. Furthermore, let $M = N_{ab} \otimes \mathbb{Z}_{(p)}$. If $n \geqslant 2$ and p does not divide n, then both the Lie power $\mathcal{L}_n(M)$ and the metabelian Lie power $\mathcal{M}_n(M)$ are projective $\mathbb{Z}_{(p)}G$-modules.*

Reduction modulo p gives the following.

Corollary. *In the notation of the Theorem, let $\overline{M} = N_{ab} \otimes (\mathbb{Z}/p\mathbb{Z})$. If $n \geqslant 2$ and p does not divide n, then both the Lie power $\mathcal{L}_n(\overline{M})$ and the metabelian Lie power $\mathcal{M}_n(\overline{M})$ are projective $(\mathbb{Z}/p\mathbb{Z})G$-modules.*

For $n = 2$ and $n = 3$, where $\mathcal{M}_n(\overline{M}) = \mathcal{L}_n(\overline{M})$, this was recently proved in [5], and used to obtain a rather surprising result on certain central extensions of groups. In the most prominent special case, the result in question states that the free centre-by-nilpotent-by abelian groups $F/[\gamma_c(F'), F]$ are torsion-free for $c = 6$. This is in startling contrast to the cases where c is a prime or $c = 4$, when for sufficiently large ranks these relatively free groups do contain elements of finite order: see [1,7–9]. The present paper was motivated by these applications to abstract group theory, and indeed, our Corollary can be used to extend the results of [5] to other values of c. This will be carried out in a subsequent paper.

In the case where G is a finite group of order prime to n, our Theorem was proved a long time ago [3, Proposition 9.2] using Tate cohomology. Our present approach is based on exploring filtrations of Lie powers and symmetric powers, and relies on results of [8] and [6]. We mention that the Theorem cannot be extended to arbitrary n: results in [2] and [3] show that it is not true for $n = p$, and one can see from [9] that it is not true when $p = 2$ and $n = 4$.

2. A lemma on projective modules

We begin preparations with a general lemma which may have some independent interest. For any K-free KG-module V, we let V^n denote the n-th symmetric power of V, with the convention that $V^0 = K$.

Lemma 2.1. *Let K be a commutative ring with 1, G a group, V a K-free projective KG-module, and n a positive integer. Assume that, for each prime divisor q of n, either q is a unit in K or G has no element of order q. Then the symmetric power V^n, the Lie power $\mathcal{L}_n(V)$ and the metabelian Lie power $\mathcal{M}_n(V)$ are also projective.*

We shall use this with $K = \mathbb{Z}_{(p)}$, in which case the assumption is simply that either p does not divide n or G has no element of order p.

Proof. The three cases are exactly parallel, so we only describe the last one.

The first step is to adapt the usual proof of Maschke's Theorem to prove that *if H is a finite group whose order is a unit in K, then every K-projective KH-module is KH-projective.* Choose a KH-homomorphism $\phi : W \to V$ from a free KH-module W onto V. Since V is projective as K-module, there is a K-homomorphism $\psi : V \to W$ such that the composite $\psi\phi$ is the identity automorphism of V. Check that the map defined by

$$\chi : V \to W, \qquad v \mapsto |H|^{-1} \sum_{h \in H} \big((vh)\psi\big)h^{-1}$$

is a KH-homomorphism such that the composite $\chi\phi$ is the identity automorphism of V. This proves that V is isomorphic to the KH-direct summand $V\chi$ of the free KH-module W, and so V is KH-projective.

The next step is to show that it suffices to prove the lemma for free KG-modules. Let V be a direct summand of a free KG-module W, and $\phi : W \to V$, $\chi : V \to W$ a pair of KG-homomorphisms whose composite $\chi\phi$ is the identity automorphism of V. These yield KG-homomorphisms $\mathcal{M}_n(W) \to \mathcal{M}_n(V)$ and $\mathcal{M}_n(V) \to \mathcal{M}_n(W)$ whose composite is the identity automorphism of $\mathcal{M}_n(V)$, showing that $\mathcal{M}_n(V)$ is isomorphic to a direct summand of $\mathcal{M}_n(W)$. Thus it suffices to prove that $\mathcal{M}_n(W)$ is KG-projective.

The remaining step is to deal with a free KG-module W. Let X be a free generating set of W as KG-module. The set XG $(=\{xg \mid x \in X,\ g \in G\})$ then freely generates W as K-module, and each orbit of G in XG is a regular orbit. We shall use that if H is any subgroup of G, then all the orbits of H in XG are also regular.

The free metabelian Lie algebra $\mathcal{M}(W)$ is a graded algebra, with W as homogeneous component of degree 1 and $\mathcal{M}_n(W)$ as homogeneous component of degree n. In terms of the Lie algebra generating set XG, there is also a finer grading which may be described as follows (cf. Lemma 4.1 in [3]). As K-module, $\mathcal{M}_n(W)$ is spanned by the 'monomials' which are Lie products with n factors, all from the set XG. Given any function, f say, from XG to the set of non-negative integers such that the sum of the values of f is n, there are only finitely many monomials in which each element xg of XG occurs precisely $f(xg)$ times among those n factors: the K-submodule they generate is the *multihomogeneous component* corresponding to f. It is easy to see that $\mathcal{M}_n(W)$ is the direct sum of these components and that G permutes these components among themselves. Further, the stabilizer in G of any such component is finite with order dividing n. (To see this, the key point is to notice that if a subgroup H stabilizes the component corresponding to f, then f must be constant on each orbit of H in XG. As the sum of the values of f is n, the set $\{xg \in XG \mid f(xg) > 0\}$ must be finite; as it is a union of regular orbits of H, the order $|H|$ is finite and n is $|H|$ times the sum of the values of f over any complete set of representatives of these orbits.) This shows that, as KG-module, $\mathcal{M}_n(W)$ is a (restricted) direct sum of modules induced from subgroups H whose orders divide n. If a prime q divides $|H|$, then G does have elements of order q and so (by assumption) q is a unit in K: thus $|H|$ itself is a unit in K. The KH-module to be induced to KG is a multihomogeneous component whose stabilizer is H; it is not only K-projective but even K-free, being freely generated by the left-normed basic Lie monomials it contains. Thus the above version of Maschke's Theorem shows that the multihomogeneous component is KH-projective, and then it follows that the induced KG-module is KG-projective. This completes the proof. □

3. On symmetric powers and metabelian Lie powers

We continue with K an arbitrary commutative ring with 1 and G any group. All tensor products will be over K. Let

$$0 \to A \to B \xrightarrow{\beta} C \to 0 \tag{3.1}$$

be a short exact sequence of K-free KG-modules, and identify A with its image in B. For $n > m \geqslant 0$, let $K_B^{n,m}$ denote the submodule of B^n spanned by the elements

$$a_1 \circ a_2 \circ \cdots \circ a_{m+1} \circ b_{m+2} \circ \cdots \circ b_n$$

where $a_1, a_2, \ldots, a_{m+1} \in A$, $b_{m+2}, \ldots, b_n \in B$. It will also be convenient to set $K_B^{n,-1} = B^n$. These submodules form a filtration

$$0 < A^n = K_B^{n,n-1} < K_B^{n,n-2} < \cdots < K_B^{n,0} < K_B^{n,-1} = B^n$$

which we shall refer to as the (A, C)-filtration of B^n.

Certain KG-homomorphisms $\pi_B^{n,m} : B^n \to B^m \otimes C^{n-m}$ will play an essential role in our examination of symmetric powers. These are obtained from the symmetrization homomorphism

$$\sigma_B^{n,m} : B^n \to B^m \otimes B^{n-m}$$

given by

$$b_1 \circ \cdots \circ b_n \mapsto \frac{1}{m!(n-m)!} \sum_{\omega} (b_{\omega(1)} \circ \cdots \circ b_{\omega(m)}) \otimes (b_{\omega(m+1)} \circ \cdots \circ b_{\omega(n)})$$

where the sum runs over all permutations ω of the indices $1, \dots, n$. (Note that no actual division is needed here: the $n!$ summands fall into $\binom{n}{m}$ sets, each consisting of $m!(n-m)!$ equal summands, and instead of division by $m!(n-m)!$ one might simply sum just one summand from each of these sets.) We define $\pi_B^{n,m}$ as the composite of $\sigma_B^{n,m}$ and the surjection

$$(1 \circ \cdots \circ 1) \otimes (\beta \circ \cdots \circ \beta) : B^m \otimes B^{n-m} \to B^m \otimes C^{n-m}$$

given by

$$(b_1 \circ \cdots \circ b_m) \otimes (b_{m+1} \circ \cdots \circ b_n) \mapsto (b_1 \circ \cdots \circ b_m) \otimes (b_{m+1}\beta \circ \cdots \circ b_n\beta).$$

It is obvious that if more than m terms of the sequence $b_1, \dots, b_n$ belong to A then each summand in the image of $b_1 \circ \cdots \circ b_n$ is 0: thus $K_B^{n,m}$ lies in the kernel of $\pi_B^{n,m}$. It is also easy to see the effect of $\pi_B^{n,m}$ on $b_1 \circ \cdots \circ b_n$ when precisely m terms of $b_1, \dots, b_n$ lie in A: then precisely $m!(n-m)!$ of the $n!$ summands are non-zero and these are all equal to each other, so the image is just

$$(b_{\omega(1)} \circ \cdots \circ b_{\omega(m)}) \otimes (b_{\omega(m+1)}\beta \circ \cdots \circ b_{\omega(n)}\beta)$$

where ω is any permutation such that $b_{\omega(1)}, \dots, b_{\omega(m)}$ are precisely the terms of $b_1, \dots, b_n$ that lie in A. Exploiting that C is K-free, this leads to an easy proof of the fact that the restriction of $\pi_B^{n,m}$ to $K_B^{n,m-1}$ has kernel $K_B^{n,m}$ and image $A^m \otimes C^{n-m}$, so it yields an isomorphism

$$K_B^{n,m-1}/K_B^{n,m} \to A^m \otimes C^{n-m}. \tag{3.2}$$

In particular, it follows that all filtration quotients $K_B^{n,m-1}/K_B^{n,m}$ are K-free.

It is not so easy to see the effect of $\pi_B^{n,m}$ on other filtration quotients, but careful examination can reveal a lot more: see [6, Section 3] and [10, Section 2]. To state some of the conclusions that we shall use here, note that the (A, C)-filtration of B^m induces a filtration

$$\begin{aligned} 0 < A^m \otimes C^{n-m} = K_B^{m,m-1} \otimes C^{n-m} < K_B^{m,m-2} \otimes C^{n-m} < \cdots \\ < K_B^{m,0} \otimes C^{n-m} < K_B^{m,-1} \otimes C^{n-m} = B^m \otimes C^{n-m} \end{aligned}$$

of the tensor product $B^m \otimes C^{n-m}$. The quotients of this filtration are also K-free, and of course $\pi_B^{m,l}$ yields isomorphisms

$$\big(K_B^{m,l-1} \otimes C^{n-m}\big)/\big(K_B^{m,l} \otimes C^{n-m}\big) \to A^l \otimes C^{m-l} \otimes C^{n-m}. \tag{3.3}$$

Lemma 3.1.

(i) *For $n > m > l \geqslant 0$, the homomorphism $\pi_B^{n,m}$ yields the first vertical map in a commutative diagram*

$$\begin{array}{ccc} K_B^{n,l-1}/K_B^{n,l} & \longrightarrow & A^l \otimes C^{n-l} \\ \downarrow & & \downarrow \\ (K_B^{m,l-1} \otimes C^{n-m})/(K_B^{m,l} \otimes C^{n-m}) & \longrightarrow & A^l \otimes C^{m-l} \otimes C^{n-m} \end{array}$$

where the horizontal maps are the isomorphisms (3.2) *with* $m = l$ *and* (3.3), *and the second vertical map is* $1 \otimes \sigma_C^{n-l,m-l}$.

(ii) *If* $n > m \geqslant 0$ *and the ring* K *has no additive torsion, then* $\ker \pi_B^{n,m} = K_B^{n,m}$.

Given any K-free KG-module A, it is well known (see [2, Corollary 3.2]) that the metabelian Lie power $\mathcal{M}_n(A)$ fits into a short exact sequence

$$0 \to \mathcal{M}_n(A) \to A \otimes A^{n-1} \to A^n \to 0 \tag{3.4}$$

of K-free KG-modules, with the maps given by

$$[a_1, a_2, a_3, \ldots, a_n] \mapsto a_1 \otimes (a_2 \circ a_3 \circ \cdots \circ a_n) - a_2 \otimes (a_1 \circ a_3 \circ \cdots \circ a_n)$$

and

$$a_1 \otimes (a_2 \circ \cdots \circ a_n) \mapsto a_1 \circ a_2 \circ \cdots \circ a_n$$

$(a_1, \ldots, a_n \in A)$, respectively. If n is invertible in K, then this sequence splits via the map $(1/n)\sigma_A^{n,1} : A^n \to A \otimes A^{n-1}$, giving a useful consequence.

Lemma 3.2. *If* n *is invertible in* K, *then there are isomorphisms*

$$A \otimes A^{n-1} \cong \mathcal{M}_n(A) \oplus A^n \quad \textit{and} \quad \operatorname{coker} \sigma_A^{n,1} \cong \mathcal{M}_n(A).$$

In another direction, one may use (3.4) to prove the following.

Lemma 3.3. *Any short exact sequence* (3.1) *of* K*-free* KG*-modules yields a short exact sequence*

$$0 \to \mathcal{M}_n(A) \to B \otimes A^{n-1} \to K_B^{n,n-2} \to 0.$$

Proof. Starting with any exact sequence (3.1), consider the map $\phi : B \otimes A^{n-1} \to B^n$ given by $b \otimes (a_1 \circ \cdots \circ a_{n-1}) \mapsto b \circ a_1 \circ \cdots \circ a_{n-1}$. This is clearly a KG-homomorphism, and its image in B^n is the submodule $K_B^{n,n-2}$. It maps $A \otimes A^{n-1}$ onto A^n, and upon restriction it yields the relevant map in (3.4): thus $(\ker \phi) \cap (A \otimes A^{n-1}) \cong \mathcal{M}_n(A)$. The lemma will therefore follow if we show that $\ker \phi \leqslant A \otimes A^{n-1}$. To this end, we use that (3.1) splits over K (because C is K-free): ignoring the action of G, we may think of B as a direct sum $A \oplus C$ of free K-modules. Let $\mathcal{A}$ be an ordered basis of A as free K-module, and let $\mathcal{C}$ be a basis of C as free K-module. Then the elements $a_1 \otimes (a_2 \circ \cdots \circ a_n)$ and $c \otimes (a_2 \circ \cdots \circ a_n)$ with $a_i \in \mathcal{A}$, $c \in \mathcal{C}$ and $a_2 \leqslant \cdots \leqslant a_n$ form a K-basis of $B \otimes A^{n-1}$, and the elements $a_1 \circ a_2 \circ \cdots \circ a_n$ and $c \circ a_2 \circ \cdots \circ a_n$ with $a_i \in \mathcal{A}$, $c \in \mathcal{C}$ and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$ form a K-basis of $K^{n,n-2}$. Since $(c \otimes (a_2 \circ \cdots \circ a_n))\phi = c \circ a_2 \circ \cdots \circ a_n$, the basis elements of $B \otimes A^{n-1}$ involving an entry from $\mathcal{C}$ are mapped one-to-one onto their counterparts in the basis of $K^{n,n-2}$. Hence $\ker \phi$ is contained in the span of the basis elements involving only entries from $\mathcal{A}$, that is in $A \otimes A^{n-1}$. $\square$

The next lemma will be a key ingredient in the proof of our main result.

Lemma 3.4. *Suppose that* K *has no additive torsion, and let* $n > m \geqslant 1$. *Then the restriction of the homomorphism* $\pi_B^{n,m} : B^n \to B^m \otimes C^{n-m}$ *to the submodule* $K_B^{n,m-2}$ *of* B^n *yields a four term exact sequence*

$$0 \to K_B^{n,m} \to K_B^{n,m-2} \to K_B^{m,m-2} \otimes C^{n-m} \to A^{m-1} \otimes \operatorname{coker} \sigma_C^{n-m+1,1} \to 0.$$

Proof. Lemma 3.1(ii) implies that $\pi_B^{n,m}$ maps the quotient $K_B^{n,m-2}/K_B^{n,m}$ injectively into $K_B^{m,m-2} \otimes C^{n-m}$. This yields the middle vertical map for a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^m \otimes C^{n-m} \arrow[r] \arrow[d] & K_B^{n,m-2}/K_B^{n,m} \arrow[r] \arrow[d] & A^{m-1} \otimes C^{n-m+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^m \otimes C^{n-m} \arrow[r] & K_B^{m,m-2} \otimes C^{n-m} \arrow[r] & A^{m-1} \otimes C \otimes C^{n-m} \arrow[r] & 0
\end{tikzcd}$$

where the horizontal exact sequences come from the (A, C)-filtrations (each using relevant instances of the isomorphisms (3.2), (3.3)). By Lemma 3.1(i), the vertical map on the left is the identity map and the vertical map on the right is $1 \otimes \sigma_C^{n-m+1,1}$. Thus the quotient of $K_B^{m,m-2} \otimes C^{n-m}$ by the image of $K_B^{n,m-2}$ is $A^{m-1} \otimes \operatorname{coker} \sigma_C^{n-m+1,1}$, and the result is proved. □

4. Symmetric and metabelian Lie powers of the augmentation ideal

From now on we work with the coefficient ring $K = \mathbb{Z}_{(p)}$ where p is an arbitrary but fixed prime, and we write R for the group ring $\mathbb{Z}_{(p)}G$. By Δ we denote the augmentation ideal of R, that is the kernel of the augmentation map $\varepsilon : R \to \mathbb{Z}_{(p)}$, and we are going to exploit the modules $K_R^{n,m}$ coming from the $(\Delta, \mathbb{Z}_{(p)})$-filtration of R^n that is determined by the augmentation sequence

$$0 \to \Delta \to R \xrightarrow{\varepsilon} \mathbb{Z}_{(p)} \to 0.$$

Since $R^m \otimes \mathbb{Z}_{(p)}^{n-m} = R^m$, the corresponding homomorphism $\pi_R^{n,m}$ discussed in Section 3 now goes simply from R^n to R^m.

The main result of this section is the following.

Lemma 4.1. *Let $n \geqslant 2$. Then*

(i) *Δ^n is projective as R-module whenever $n \not\equiv 0, 1 \bmod p$,*
(ii) *$\mathcal{M}_n(\Delta)$ is a projective R-module whenever $n \not\equiv 0, 2 \bmod p$.*

This an easy consequence of a technical result which we deduce first.

Lemma 4.2. *Let $n > m \geqslant 1$. Then $K_R^{n,m}$ is a projective R-module whenever n and m are not divisible by p.*

This lemma is essentially proved in [6, Lemma 8] except that the conclusion there is not that $K_R^{n,m}$ is projective but that the homology groups $H_k(G, K_R^{n,m})$ with $k \geqslant 1$ vanish. The latter holds for a much wider range of modules, namely for the modules $K_R^{np^t,mp^t}$ with $t \geqslant 0$, the case $t = 0$ being the one needed here. For the convenience of the reader we give the proof of Lemma 4.2, adapting the notation and argument of [6]. We shall use without proof special cases of the short exact sequences (9.c), (10) and (12) of [6, Lemma 3], re-stated as follows.

Lemma 4.3.

(i) *If $n > m \geqslant 1$, then there exists a short exact sequence*

$$0 \to K_R^{n,m} \to K_R^{n,m-1} \to K_R^{m,m-1} \to 0. \tag{4.1}$$

(ii) *If* $r \geqslant 1$ *and* $n > rp + 1$ *with* $n \not\equiv 0 \bmod p$, *then there is a short exact sequence*

$$0 \to K_R^{n,rp+1} \to K_R^{n,rp-1} \to K_R^{rp+1,rp-1} \to 0. \tag{4.2}$$

(iii) *If* $n > 1$ *with* $n \not\equiv 0 \bmod p$, *then there is a short exact sequence*

$$0 \to K_R^{n,1} \to R^n \to R \to 0. \tag{4.3}$$

We shall use without reference two obvious facts: one, if in an exact sequence all terms except the first one are known to be projective, then the first term is also projective; and two, the tensor product of a $\mathbb{Z}_{(p)}G$-projective module and a $\mathbb{Z}_{(p)}$-free $\mathbb{Z}_{(p)}G$-module is always $\mathbb{Z}_{(p)}G$-projective. Except for proclamations, we shall not keep repeating the common modulus, p, of our congruences.

Proof of Lemma 4.2. This will be done by induction on m. The case $m = 1$ follows from (4.3) and Lemma 2.1, so let $m > 1$. When $m \equiv 1$, say, $m = rp + 1$ with $r \geqslant 1$, the inductive hypothesis is that $K_R^{n,rp-1}$ is projective whenever $n > rp - 1$ and $n \not\equiv 0$. In particular, $K_R^{rp+1,rp-1}$ is also projective, so an appeal to (4.2) proves what we need. When $m \not\equiv 0, 1$, the inductive hypothesis is that $K_R^{n,m-1}$ is projective for all n with $n > m - 1$ and $n \not\equiv 0$, so in particular $K_R^{m,m-1}$ is also projective, and this time an appeal to (4.1) completes the inductive step. □

Proof of Lemma 4.1. The first part follows from Lemma 4.2 since $\Delta^n = K_R^{n,n-1}$. As to the second part, Lemma 3.3 applied to the augmentation sequence gives a short exact sequence

$$0 \to \mathcal{M}_n(\Delta) \to R \otimes \Delta^{n-1} \to K_R^{n,n-2} \to 0$$

in which the middle and right hand terms are projective R-modules: the former is projective because of the free tensor factor R and the latter is projective by Lemma 4.2. □

5. Proof of the Theorem

We need one more technical lemma. The relation module M fits into a short exact sequence

$$0 \to M \to P \to \Delta \to 0$$

where P is a free R-module (see e.g. [4, Chapter 6, §6]); this is usually referred to as the relation sequence. The critical point is that some of the terms and quotients of the (M, Δ)-filtrations of the symmetric powers P^n are projective.

Lemma 5.1. *If* $n > m \geqslant 0$ *and* $n \not\equiv 0 \bmod p$ *while* $n - m \not\equiv 1 \bmod p$, *then* $K_P^{n,m}$ *is projective.*

Proof. If (3.1) is the relation sequence, the isomorphism (3.2) yields the short exact sequence

$$0 \to K_P^{n,m} \to K_P^{n,m-1} \to M^m \otimes \Delta^{n-m} \to 0, \tag{5.1}$$

and the four term exact sequence of Lemma 3.4 turns into

$$0 \to K_P^{n,m} \to K_P^{n,m-2} \to K_P^{m,m-2} \otimes \Delta^{n-m} \to M^{m-1} \otimes \operatorname{coker} \sigma_\Delta^{n-m+1,1} \to 0. \tag{5.2}$$

The lemma will be proved by induction on m. If $m = 0$, (5.1) turns into

$$0 \to K_P^{n,0} \to P^n \to \Delta^n \to 0 \tag{5.3}$$

and our assumptions imply that $n \not\equiv 0, 1$. Then Δ^n is projective by Lemma 4.1(i) and P^n is projective by Lemma 2.1, so (5.3) gives that $K_P^{n,0}$ is projective. (Note that this case does not occur when $p = 2$.) If $m = 1$, (5.2) turns into

$$0 \to K_P^{n,1} \to P^n \to P \otimes \Delta^{n-1} \to \operatorname{coker} \sigma_\Delta^{n,1} \to 0 \tag{5.4}$$

and our assumptions imply that $n \not\equiv 0, 2$. Then Lemma 3.2 gives that $\operatorname{coker} \sigma_\Delta^{n,1} \cong \mathcal{M}_n(\Delta)$, and the latter is projective by Lemma 4.1(ii). Thus in (5.4) all terms to the right of $K_P^{n,1}$ are projective, and this implies that $K_P^{n,1}$ is projective as well.

For the inductive step, let $m > 1$. Suppose first that $n - m \not\equiv 0$. Then $n - (m-1) \not\equiv 1$, and hence $K_P^{n,m-1}$ is projective by the inductive hypothesis. Also, since $n - m \not\equiv 0, 1$, the symmetric power Δ^{n-m} is projective by Lemma 4.1(i), and hence $M^m \otimes \Delta^{n-m}$ is projective. Now the exact sequence (5.1) implies that $K_P^{n,m}$ is projective. It remains to deal with the case $n - m \equiv 0$. Then $n - (m-2) \not\equiv 1$ and $m - (m-2) \not\equiv 1$, and hence $K_P^{n,m-2}$ and $K_P^{m,m-2}$ are projective by the inductive hypothesis. Finally, since $n - m + 1 \equiv 1$, Lemma 3.2 gives that $\operatorname{coker} \sigma_\Delta^{n-m+1,1}$ is isomorphic to $\mathcal{M}_{n-m+1}(\Delta)$, and the latter is projective by Lemma 4.1(ii). Thus in (5.2) all terms to the right of $K_P^{n,m}$ are projective, and this implies that $K_P^{n,m}$ is also projective, as required to complete the inductive step. □

Proof of the Theorem. By Lemma 3.3, the relation sequence yields a short exact sequence

$$0 \to \mathcal{M}_n(M) \to P \otimes M^{n-1} \to K_P^{n,n-2} \to 0.$$

Here the middle term is a free module (because of the free tensor factor P), and the right hand term is projective by Lemma 5.1. Hence $\mathcal{M}_n(M)$ is projective. Now we turn to the Lie power. It is proved in [8, Section 3.1] that the Lie power $\mathcal{L}_n(M)$ has a finite filtration, called there the type series, whose quotients can be obtained from the metabelian Lie powers

$$\mathcal{M}_2(M), \mathcal{M}_3(M), \ldots, \mathcal{M}_n(M) \tag{5.5}$$

using the operations of taking metabelian Lie powers, symmetric powers and tensor products. Let $\mathcal{T}$ denote this class of modules. To prove the second part of the Theorem it suffices to show that all modules of degree not divisible by p in $\mathcal{T}$ are projective. We use induction on the number of operations required to obtain a module in $\mathcal{T}$ from the modules (5.5). The base of our induction is given by the already established result about metabelian Lie powers. For the inductive step, if $V \in \mathcal{T}$ and $U = \mathcal{M}_k(V)$ or $U = V^k$ are of degree not divisible by p, then neither k nor the degree of V are divisible by p. Hence V is projective by the inductive hypothesis and $\mathcal{M}_k(V)$ and V^k are projective by Lemma 2.1. If $U = V \otimes W$ with $V, W \in \mathcal{T}$, and the degree of U is not divisible by p, then either the degree of V or the degree of W is not divisible by p, and hence one of the tensor factors, V or W, is projective by the inductive hypothesis, so it follows that the tensor product $U = V \otimes W$ is projective as well. This completes the inductive step, and thereby the proof of the Theorem. □

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