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On Lie Powers of Regular Modules in Characteristic 2.

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ABSTRACT - We study Lie powers of regular modules for finite groups over a field of characteristic 2. First we prove two rather general reduction theorems, and then we apply them to Lie powers of the regular module for the Klein four group. For the latter, we solve the decomposition problem for the Lie power in degree 8, a module of dimension 8160. It has been known that of the infinitely many possible indecomposables, only four occur as direct summands in Lie powers of degree not divisible by 4, but that a fifth makes its appearance in the Lie power of degree 4. It is quite a surprise that no new indecomposables appear among the direct summands in degree 8.

1. Introduction.

Let G be a group, K a field of positive characteristic p, V a KG-module, let $L = L(V)$ denote the free Lie algebra on V and write $L = L_n(V)$ for the degree n homogeneous component of L. Then $L(V) = \bigoplus_{n=1}^{\infty} L_n(V)$ is a graded KG-module and the submodule $L_n(V)$ is termed the nth Lie power of V over K. In recent years, modular Lie powers have been studied in a number of papers. The eventual aim of these investigations is the solution of the decomposition problem, that is the identification of the indecomposable KG-modules occurring as direct summands in the Lie

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powers $L_n(V)$ and their respective Krull-Schmidt multiplicities. Most of the results obtained so far refer to the case where G is a group with cyclic p-Sylow subgroup (see [3], [4], [6] and references therein). However, very little is known about modular Lie powers for groups with non-cyclic p-Sylow subgroup. In this paper we investigate Lie powers of regular modules for arbitrary finite groups over fields of characteristic 2. In Section 3 we prove a rather general reduction theorem, which reduces the decomposition problem for Lie powers of regular modules to the decomposition problem for a specific free Lie algebra $L(S(X))$. In the special case where G has order 2, this Lie algebra turns out to be one dimensional, and results from [8], where the decomposition problem for the regular module for a group of order 2 was solved, occur as a special case. In general, however, the decomposition problem for $L(S(X))$ seems to be rather hard. The main disadvantage is that the action of G on $L(S(X))$ is not homogeneous. Nevertheless our reduction theorem provides a fascinating insight into the overall module structure of the free Lie algebra on a regular KG-module, and it is a direct generalization of conclusive results for the group of order 2. Moreover, in Sections 5 and 6 we exploit our reduction theorem to obtain information about the smallest non-trivial instance of a regular module for a group with non-cyclic 2-Sylow subgroup, namely the case where V is the regular module for the Klein four group. In particular, we solve the decomposition problem for the 8th Lie power $L_8(V)$, thus demonstrating that our reduction theorem provides access to the decomposition problem for Lie powers of small, yet previously inaccessible degrees. In Section 4 we prove another reduction theorem, which reduces the decomposition problem for the regular module of an arbitrary finite group to that of the augmentation ideal.

In this paper we restrict ourselves to modular Lie powers in characteristic 2 only. The reason for this (at a first glance strange) restriction is that our main technical tool, the so called restricted elimination in free restricted Lie algebras, takes a particularly simple form in the characteristic 2 case. We hope that our methods can be further developed and modified to provide deeper insight into the structure of modular Lie powers both in characteristic 2 and in arbitrary positive characteristic.

2. Notation and preliminaries.

Throughout this paper K is a field of characteristic 2. Our main technical tools in this paper are various elimination results for free Lie algebras and free restricted Lie algebras. Let $L = L(X)$ be the free Lie algebra on a set X over K. The Lazard Elimination Theorem (see Proposition 10 in § 2.9, Chapter 2, of [2]) reads as follows (here, and throughout this paper, we use the left normed convention for Lie brackets).

LEMMA 2.1. *Suppose that $X = X_1 \cup X_2$ is the disjoint union of its subsets X_1 and X_2. Then $L = L(X)$ is the direct sum of its free subalgebra $L(X_1)$ and the ideal $I(X_2)$ that is generated by X_2. Moreover, $I(X_2)$ is itself a free Lie algebra with free generating set*

$$X_2 \wr X_1 = \{[z, y_1, y_2, \dots, y_k];\ z \in X_2,\ y_1, y_2, \dots, y_k \in X_1,\ k \geqslant 0\}.$$

Thus Lazard Elimination yields a direct decomposition (over K)

$$L = L(X_1) \oplus L(X_2 \wr X_1)$$

which will be referred to as *elimination of the subalgebra* $L(X_1)$. In the special case where $X_1 = \{x\}$ is a singleton, this direct decomposition turns into

$$L = \langle x \rangle \oplus L(X|x) \tag{2.1}$$

where

$$X|x = (X \backslash \{x\}) \wr \{x\} = \{[y, x^k];\ y \in X \backslash \{x\},\ k \geqslant 0\},$$

$\langle x \rangle$ denotes the K-span of x in $L(X)$ and

$$[y, x^k] = [y, \underbrace{x, x, \dots, x}_{k}].$$

The direct decomposition (2.1) will be referred to as *elimination of the free generator* x. In this paper, we use exponential notation only in the sense of the last display, or for the «powering» operation in a restricted Lie algebra: when x is an element of a multiplicative group, we never write x^k to denote the kth power of x formed in that group. Now let $R = R(X)$ be the free restricted Lie algebra on X over K. Throughout this paper we will identify the free Lie algebra $L(X)$ with the Lie subalgebra that is generated by X in R. An immediate consequence of Lemma 2 is that under the above assumptions on X there are direct decompositions

$$R = R(X_1) \oplus R(X_2 \wr X_1), \qquad R = \langle x^{2^\alpha};\ \alpha \geqslant 0 \rangle \oplus R(X|x) \tag{2.2}$$

of the free restricted Lie algebra R. These will be used alongside a variation of elimination that is specific to free restricted Lie algebras.

LEMMA 2.2. *Let $x \in X$, and let J be the ideal of R that is generated by x^2 and $X \setminus \{x\}$. Then R is the direct sum of its subspace $\langle x \rangle$ and the ideal J. Moreover, J is itself a free restricted Lie algebra with free generating set*

$$X|_r x = \{x^2, z, [z, x];\ z \in X \setminus \{x\}\}.$$

For a proof of Lemma 2.2 we refer to the proof of Theorem 2.7.4 in [1]. This lemma yields a direct decomposition (over K)

$$R = \langle x \rangle \oplus R(X|_r x)$$

which will be referred to as *restricted elimination of the free generator* x. Note that the second decomposition in (2.2) can, in fact, be obtained by using restricted elimination repeatedly, that is by eliminating the free generators $x, x^2, x^{2^2}, \ldots$ successively in the obvious way. The first decomposition in (2.2) will be referred to as *elimination of the subalgebra* $R(X_1)$ and the second decomposition in (2.2) will be referred to as *full elimination of the free generator* x.

Another technical tool that is frequently used in this paper is that of replacing free generating sets by other more suitable free generating sets. Let $L(X)$ be as before and assume that X is the disjoint union

$$X = X_1 \cup X_2 \cup X_3 \cup \ldots$$

of its finite subsets $X_1, X_2, \ldots$. Let $L(<m)$ denote the subalgebra of $L(X)$ that is generated by $X_1, X_2, \ldots, X_{m-1}$. For each $n \geqslant 1$, let $\varphi_n \in GL(\langle X_n \rangle)$ be a linear automorphism of the space $\langle X_n \rangle$, and let $\varphi : X \to L(X)$ be a map that is, for all $x \in X_n$, $n = 1, 2, \ldots$, of the form

$$\varphi(x) = \varphi_n(x) + w_x \tag{2.3}$$

where $w_x \in L(<n)$. We need the following

LEMMA 2.3. *If $\varphi : X \to L(X)$ is a map of the form* (2.3), *then $\varphi(X)$ is a free generating set of $L(X)$.*

A proof of this simple fact can be found in [5], Section 2.3.

On several occasions later in the text we will use the decomposition of a free Lie algebra into the direct sum of multihomogeneous components. Let $X = \{x_1, \ldots, x_r\}$ and, for any string $(m_1, \ldots, m_r)$ of non-negative integers, let $L_{(m_1, \ldots, m_r)}$ denote the span of all Lie monomials of partial degree $m_1, \ldots, m_r$ in the free generators $x_1, \ldots, x_r$, respectively. The subspace $L_{(m_1, \ldots, m_r)}$ is termed the $(m_1, \ldots, m_r)$-multihomogeneous component of $L(X)$. Clearly, $L(X)$ is the direct sum (over K) of its multihomogeneous components, and each $L_n(X)$ is the direct sum of the multihomogeneous components $L_{(m_1, \dots, m_r)}$ with $m_1 + \dots + m_r = n$. If all entries of the defining string $(m_1, \dots, m_r)$ are equal to 0 or 1 we refer to $L_{(m_1, \dots, m_r)}$ as a multilinear component.

Finally, if G is a group and V is a KG-module, then the free Lie algebra on the module V is the free Lie algebra $L(V) = L(X)$ on X, where X is an arbitrary K-basis of the module V. In this context $L(V)$ and its homogeneous components $L_n(V) = L_n(X)$, the Lie powers of V, will be regarded as KG-modules with G-action induced by the G-action on $L_1(V) = V$. We write $R(V) = R(X)$ for the free restricted Lie algebra on V and $R_n(V) = R_n(X)$ for its homogeneous component of degree n, which will also be regarded as a KG-module in the obvious way. All KG-modules in this paper will be right modules.

3. A reduction theorem.

Let G be a finite group of order r and let $V = KG$ denote the regular module for G. We will assume that the elements of G are ordered in such a way that the identity element $e \in G$ is the smallest in this ordering. Consider the free Lie algebra $L = L(V) = L(X)$, where X is the basis of KG consisting of the elements of G. Our first observation is that the multihomogeneous components of $L(V)$ are permuted under the action of G. It is easily seen that the only multihomogeneous components $L_{(m_1, \dots, m_r)}$ with non-trivial stabilizer H in G are the ones for which the partial degrees in the defining string are constant on the left cosets of H in G. In particular, the total degree of those multihomogeneous components is divisible by the order of H. An immediate consequence of this observation is the following easy

FACT 1. *The Lie powers $L_n(V)$ of the regular KG-module V are free whenever* $(n, |G|) = 1$.

Here $(n, |G|)$ denotes the highest common factor of n and $|G|$. Towards our first reduction theorem, consider the free restricted Lie algebra $R(V) = R(X)$. Consecutive restricted elimination of all the elements of the free generating set X, according to their order and starting with the smallest, gives a direct decomposition

$$R(X) = \langle X \rangle \oplus R(Y) \tag{3.1}$$

where Y consists of the elements

(3.2) $[x_1, x_2, \dots, x_k]$ with $x_1, \dots, x_k \in X,\ x_1 > x_2 < \dots < x_k,\ k \geq 2$

and the elements

(3.3) $[x_1^2, x_2, \dots, x_k]$ with $x_1, \dots, x_k \in X,\ x_1 < x_2 < \dots < x_k,\ k \geq 1.$

Clearly, (3.1) is a direct decomposition of KG-modules. Note that the elements (3.2) are left normed basic commutators in X of two types. On the one hand, these elements include all multilinear left normed basic commutators of degree k with $2 \leq k \leq r$ (that is, the left normed basic commutators in (3.2) in which each element of X occurs at most once). The remaining elements in (3.2) are the left normed basic commutators of degree k with $3 \leq k \leq r+1$ in which one element of X occurs exactly twice (namely as the first entry x_1 and once more), and all the other elements of X occur with multiplicity at most 1. Now we rewrite Y as the disjoint union

$$Y = Y_1 \cup Y_2$$

where Y_1 consists of all multilinear left normed basic commutators in X of degrees $2, 3, \dots, r$, and Y_2 consists of the remaining elements from (3.2) and (3.3). We write $Y_1^{(n)}$ and $Y_2^{(n)}$ for the sets consisting of all elements of degree n in Y_1 and Y_2, respectively. Note that $Y_2^{(2)} = X^2 = \{x^2; x \in X\}$. Let $L'(Y)$ denote the derived algebra of $L(Y)$.

LEMMA 3.1. *Modulo $L'(Y)$, the elements of $Y_2^{(n)}$ with $n = 2, 3, \dots, r+1$ and the elements of $Y_1^{(r-1)}$ span free KG-modules. Moreover, each of these free KG-modules has a free generating set consisting of Lie monomials from $Y_2^{(n)} (n = 2, 3, \dots, r+1)$ and $Y_1^{(r-1)}$, respectively.*

PROOF. Let $Y_2^{(n, e)}$ denote the set of all elements (3.3) of degree n with $x_1 = e$. If $u = [e^2, x_2, \dots, x_{n-2}] \in Y_2^{(n, e)} (2 \leq n \leq r+1)$ and $g \in G$, then

$$ug = [g^2, x_2 g, \dots, x_{n-2} g].$$

Using the fact that modulo $L'(Y)$ commutators of the form $[u_0, u_1, u_2, \dots, u_k]$ and $[u_1^2, u_2, \dots, u_k]$ with $u_0, u_1, \dots, u_k \in X$ are symmetric in the entries $u_2, \dots, u_k$, it is easily seen that, modulo $L'(Y)$, ug is congruent to a unique element of $Y_2^{(n)}$, and conversely, every element of $Y_2^{(n)}$ is congruent to an element of the form ug for some $u \in Y_2^{(n, e)}$ and

$g \in G$. This is trivial for $n=2$ where $Y_2^{(2)}=X^2$. Now let $n>2$, and consider the element ug. Because of the symmetry mentioned above, we may assume (working modulo $L'(Y)$) that $x_2 g < \ldots < x_{n-2} g$. If $g < x_2 g$, then the element ug is congruent to an element of the form (3.3), and if $g > x_2 g$, we get (by using that symmetry again)

$$\begin{aligned} ug &\equiv [g^2, x_2 g, \ldots, x_{n-2} g] \\ &\equiv [x_2 g, g, g, \ldots, x_{n-2} g] \\ &\equiv [g, x_2 g, \ldots, g, \ldots, x_{n-2} g] \end{aligned}$$

where the latter is an element of the form (3.2) from $Y_2^{(n)}$. Hence $Y_2^{(n)}$ spans a free KG-module in the quotient $L(Y)/L'(V)$, and the elements of $Y_2^{(n,\, e)}$ generate this module freely. This proves the assertion about $Y_2^{(n)}$. Now consider the set $Y_1^{(r-1)}$. It consist of all left normed multilinear basic commutators of degree $r-1$ with entries in X. Each element of $Y_1^{(r-1)}$ involves exactly $r-1$, that is, all but one, of the free generators from X. For $x \in X$, let $Y_1^{(r-1)}(x)$ denote the subset of $Y_1^{(r-1)}$ consisting of all elements not involving x. Then

$$Y_1^{(r-1)} = \bigcup_{x \in X} Y_1^{(r-1)}(x) \quad \text{and} \quad \langle Y_1^{(r-1)} \rangle = \bigoplus_{x \in X} \langle Y_1^{(r-1)}(x) \rangle.$$

It is easily seen that modulo $L'(Y)$ the action of G on the set of subspaces $\langle Y_1^{(r-1)}(x) \rangle$ $(x \in X)$ is given by

$$\langle Y_1^{(r-1)}(x) \rangle g = \langle Y_1^{(r-1)}(xg) \rangle.$$

Consequently, G acts regularly on the set of these subspaces. This implies the assertion about $Y_1^{(r-1)}$. Any of the r subsets $\langle Y_1^{(r-1)}(x) \rangle (x \in X)$ may be taken as a free generating set of the free KG-submodule $\langle Y_1^{(r-1)} \rangle$ of $L(Y)/L'(Y)$. ■

Now let $Y^{(n)}$ denote the set of all elements of degree n in Y, and write the latter as a disjoint union

$$Y = Y^{(2)} \cup Y^{(3)} \cup \ldots \cup Y^{(r)} \cup Y^{(r+1)}.$$

Here $Y^{(n)} = Y_1^{(n)} \cup Y_2^{(n)}$ for $n = 2, \ldots, r$ and $Y^{(r+1)} = Y_2^{(r+1)}$. By Lemma 3.1, we can find sets $W_2^{(n)}$ $(n = 3, \ldots, r+1)$ and $W_1^{(r-1)}$ of Lie monomials in $Y_2^{(n)}$ and $Y_1^{(r-1)}$, respectively, such that the sets

$$W_2^{(n)} G = \{ug;\, u \in W_2^{(n)},\, g \in G\}$$

and

$$W_1^{(r-1)}G = \{ug;\ u \in W_1^{(r-1)},\ g \in G\}$$

are linearly independent, and, moreover, modulo $L'(Y)$ we have

$$\langle W_2^{(n)}G\rangle = \langle Y_2^{(n)}\rangle \bmod L'(Y) \quad (n = 3, \dots, r+1)$$

and

$$\langle W_1^{(r-1)}G\rangle = \langle Y_1^{(r-1)}\rangle \bmod L'(Y).$$

It follows easily that every element ug of $W_2^{(n)}G$ $(n = 3, \dots, r+1)$ can be written as

$$ug = \varphi_n(y) + w_y$$

where $y \in Y_2^{(n)}$, φ_n is a linear automorphism of $\langle Y_2^{(n)}\rangle$, and $w_y \in L'(Y)$. Since both ug and y are elements of degree n, so is w_y, and hence

$$w_y \in L'(Y) \cap L_n(X) \subseteq L(Y^{(2)} \cup \dots \cup Y^{(n-1)}).$$

Similarly, every element ug of $W_1^{(r-1)}G$ can be written as

$$ug = \psi_{r-1}(y) + w_y$$

where $y \in Y_1^{(r-1)}$, ψ_{r-1} is a linear automorphism of $\langle Y_1^{(r-1)}\rangle$, and

$$w_y \in L(Y^{(2)} \cup \dots \cup Y^{(r-2)}).$$

Now a straightforward application of Lemma 2.3 gives that the elements of $Y_2^{(n)}$ with $3 \leq n \leq r+1$ and the elements of $Y_1^{(r-1)}$ in the free generating set Y can be replaced by the elements of the set WG where

$$W = W_2^{(3)} \cup \dots \cup W_2^{(r)} \cup W_2^{(r+1)} \cup W_1^{(r-1)}.$$

By construction, the set WG consists of monomials and G acts freely on WG. We denote the resulting new free generating set of $R(Y)$ by Z. Let $S(X) = Y_1 \backslash Y_1^{(r-1)}$. Thus $S(X)$ is the set of all left normed multilinear basic commutators of degree $2, 3, \dots, r-2$ and degree r in X (that is, all those of degree at most r, excluding the degrees 1 and $r-1$). Clearly,

$$Z = S(X) \cup X^2 \cup WG,$$

and G acts freely on both X^2 and WG. Next we observe that the elements of $S(X)$ generate a G-invariant subalgebra in $R(Z)$. This is because of the obvious fact that if $u = [x_1, \dots, x_n]$ is a multilinear left normed basic commutator with entries $x_1, \dots, x_n \in X$ then, for any $g \in G$, $ug = [x_1 g, \dots, x_n g]$ involves n distinct elements $x_1 g, \dots, x_n g \in X$, and when written as a linear combination of Hall basic commutators, each of those basic commutators is multilinear with entries $x_1 g, \dots, x_n g \in X$, and hence it is either multilinear left-normed of degree n or a Lie product of multilinear left normed basic commutators of degree at least 2 and at most $n-2$ with disjoint entry sets. We may now summarize our discussion so far as follows.

ELIMINATION STEP 1. *There exists a set W of Lie monomials in* $L_3 \oplus L_4 \oplus \dots \oplus L_{r+1}$ *such that*

$$R = \langle X \rangle \oplus R(Z) \tag{3.4}$$

where

$$Z = S(X) \cup X^2 \cup WG.$$

Moreover, $S(X)$ generates a G-invariant subalgebra of $R(Z)$ and G acts freely on both X^2 and WG.

Elimination of the subalgebra generated by $S(X)$ from $R(Z)$ yields a direct decomposition

$$R(Z) = R(S(X)) \oplus R(U) \tag{3.5}$$

where

$$U = \{[a, b_1, \dots, b_k];\ a \in X^2 \cup WG,\ b_1, \dots, b_k \in S(X),\ k \geqslant 0\}.$$

It is not hard to see that the span of U in $R(U)$ is G-invariant. Moreover, there is an isomorphism

$$\langle U \rangle \cong \langle X^2 \cup WG \rangle \otimes T(S(X))$$

where $T(S(X)) = \bigoplus_{n=0}^{\infty} T_n(S(X))$ denotes the tensor algebra on $S(X)$, regarded as a graded KG-module in the obvious way. Since $\langle X^2 \cup WG \rangle$ is a free KG-module, so is $\langle U \rangle$. In fact, the elements of U with $a \in \{e^2\} \cup W$ form a free generating set for $\langle U \rangle$ as a free KG-module. Evidently, with the sole exception of e^2 this free generating set consists of Lie monomials in X. A straightforward application of Lemma 2 gives that the free generating set U of $R(U)$ can be replaced by

$$U_1 = \{[a, b_1, \dots, b_k]\, g;\ a \in \{e^2\} \cup W,\ b_1, \dots, b_k \in S(X),\ g \in G,\ k \geqslant 0\}.$$

Note that G acts freely on U_1. Observe also that U_1 contains X^2 as a subset, and the elements of the latter generate a free restricted Lie subalgebra in $R(U_1)$. Moreover, $U_1 \setminus X^2$ consists of Lie monomials in X. Elimination of the subalgebra generated by X^2 from $R(U_1)$ gives a direct decomposition

$$R(U) = R(X^2) \oplus R(U_2) \tag{3.6}$$

where

$$U_2 = \{[a, b_1, b_2, \dots, b_k];\ a \in U_1 \setminus X^2,\ b_1, \dots, b_k \in X^2,\ k \geqslant 0\}.$$

It is evident that G acts freely on the set U_2, and it is also clear that U_2 consists of Lie monomials. By combining (3.6) with (3.5) and (3.4) we obtain the following.

ELIMINATION STEP 2. *There exists a set U_2 of Lie monomials in $L(X)$ such that*

$$R(X) = \langle X \rangle \oplus R(S(X)) \oplus R(X^2) \oplus R(U_2), \tag{3.7}$$

and G acts freely on U_2.

Since $R(X) \cong R(X^2)$, we can now apply (3.7) to $R(X^2)$. The results is the direct decomposition

$$R(X^2) = \langle X^2 \rangle \oplus R(S(X)\varphi) \oplus R(X^4) \oplus R(U_2\varphi)$$

where φ is the endomorphism of $R(X)$ determined by $x \mapsto x^2$ for all $x \in X$. This direct decomposition of $R(X^2)$ involves $R(X^4)$ as a direct summand, to which (3.7) can be applied once more. Iterative application of (3.7) eventually gives in the limit the direct decomposition of $R(X)$ that is given in Elimination Step 3 below. In the statement we adopt the convention that φ^0 is the identity map.

ELIMINATION STEP 3. *There exists a set U_2 of Lie monomials in $L(X)$ such that*

$$R(X) = \Big\langle \bigcup_{i \geqslant 0} X^{2^i} \Big\rangle \oplus \bigoplus_{i \geqslant 0} R(S(X)\,\varphi^i) \oplus \bigoplus_{i \geqslant 0} R(U_2\varphi^i). \tag{3.8}$$

Moreover, G acts freely on the set U_2, and hence on the sets $U_2\varphi^i$, and *all the sets $U_2\varphi^i$ and $S(X)\varphi^i$ $(i \geqslant 0)$ are free generating sets for the free restricted Lie algebras they generate.*

Now consider the free Lie algebra $L(X)$. Since the free generating sets $U_2\varphi^i$ and $S(X)\varphi^i$ $(\alpha \geqslant 0)$ consist of Lie elements, the following is an easy consequence of (3.8).

ELIMINATION STEP 4. *For the free Lie algebra $L(X)$ there is a direct decomposition*

$$L(X) = \langle X\rangle \oplus \bigoplus_{i\geqslant 0} L(S(X)\varphi^i) \oplus \bigoplus_{i\geqslant 0} L(U_2\varphi^i) \tag{3.9}$$

where $S(X)$, U_2 and φ are as in Elimination Step 3.

This is the crucial step in the proof of our reduction theorem. It provides a direct decomposition of $L(V) = L(X)$ into the direct sum of $V = \langle X\rangle$, an infinite series of isomorphic G-invariant free Lie algebras $L(S(X)\varphi^i)$ $(i \geqslant 0)$, and another infinite series of isomorphic G-invariant free Lie algebras $L(U_2\varphi^i)$. Since G acts freely on the sets $U_2\varphi^i$, the latter are free Lie algebras on free KG-modules. Our aim is to decompose the whole of $L(X)$ into a direct sum of terms of the form

$$\langle X_j\rangle \oplus \bigoplus_{i\geqslant 0} L(S(X_j)\varphi_j^i), \tag{3.10}$$

that is, terms similar to the first two direct summands on the right hand side in (3.9). To be more precise, we introduce some more notation.

DEFINITION. Let X_j be a set of Lie monomials in $R(X)$ such that G acts regularly on X_j and X_j is a free generating set for the subalgebra it generates in $R(X)$. Furthermore, suppose we are given an arbitrary but fixed bijective map $X \to X_j$ that agrees with the G-action on X and X_j. Then we write $S(X_j)$ for the image of $S(X)$ under the endomorphism of $R(X)$ that is determined by the given G-map $X \to X_j$, and we write φ_j for the endomorphism of $R(X_j)$ given by $x \mapsto x^2$ for all $x \in X_j$.

Roughly speaking, $S(X_j)$ and φ_j are for X_j defined in exactly the same way as $S(X)$ and φ for X. In what follows, when writing $S(X_j)$, we will always assume that an appropriate choice for the map $X \to X_j$ has been made, and the existence of such a map will be obvious from the context. With these definitions in place and with Elimination Step 4 at our disposal, the above stated aim is not hard to achieve.

LEMMA 3.2. *In $L(X)$ there exist an infinite sequence $u_1(=e)$, $u_2, u_3, \ldots$ of Lie monomials with* $\deg u_1 \leqslant \deg u_2 \leqslant \deg u_3 \leqslant \ldots$ *and subsets $Y_{k,s}$ (where $k = 1, 2, 3, \ldots$ and s runs over some countable index set I_k) such that, for each $k \geqslant 1$,*

(i) *the sets $Y_{k,s}$ with $s \in I_k$ consist of Lie monomials of degree greater than or equal to* $\deg u_k$,

(ii) *G acts regularly on the set $X_k = \{u_k g;\, g \in G\}$,*

(iii) *G acts freely on the sets $Y_{k,s}$ for all $s \in I_k$, and*

(iv) *$L(X)$ decomposes into the direct sum*

$$L(X) = \bigoplus_{j=1}^{k} (\langle X_j \rangle \oplus \bigoplus_{i \geqslant 0} L(S(X_j)\,\varphi_j^i)) \oplus \bigoplus_{s \in I_k} L(Y_{k,s})$$

where each of direct summands $L(S(X_j)\,\varphi_j^i)$ is freely generated by the set $S(X_j)\,\varphi_j^i$ and each of the direct summands $L(Y_{k,s})$ is freely generated by $Y_{k,s}$.

PROOF. The required sequence and the associated sets $Y_{k,s}$ are constructed by induction. We start by setting $u_1 = e$. Then $X_1 = X$, and Elimination Step 4 gives that (i)-(iv) hold for $k = 1$ with the sets $U_2\varphi^i$ in the place of the $Y_{1,s}$. Now suppose that we have a finite sequence $u_1, \ldots, u_k$ such that (i)-(iv) hold. Let u_{k+1} be an element of smallest possible degree in $\bigcup_{s \in I_k} Y_{k,s}$; say, $u_{k+1} \in Y_{k,t}$ for $t \in I_k$. Then elimination of the Lie subalgebra generated by $X_{k+1} = \{u_{k+1} g;\, g \in G\}$ from $L(Y_{k,t})$ gives a direct decomposition

$$L(Y_{k,t}) = L(X_{k+1}) \oplus L(\widetilde{X}_{k+1})$$

where

$$\widetilde{X}_{k+1} = (Y_{k,t} \setminus X_{k+1}) \wr X_{k+1}.$$

It is easily seen that $\widetilde{X}_{k+1}$ consists of Lie monomials of degree greater than or equal to $\deg u_{k+1}$ and that G acts freely on $\widetilde{X}_{k+1}$. The inductive step now follows by applying Elimination Step 4 to $L(X_{k+1})$ and by taking the sets U_2 (more precisely, the canonical image of $U_2 \subseteq L(X)$ in $L(X_{k+1})$), $\widetilde{X}_{k+1}$ and $Y_{k,s}$ with $s \in I_k \setminus \{t\}$ as the sets $Y_{k+1,s}$. ■

REMARK. It may be seen from the proof that our construction of the sequence $u_1, u_2, u_3, \ldots$ is effective in that it allows us to compute its elements explicitly up to any given degree n. Indeed, although the construction involves infinite collections of infinite sets $Y_{k,s}$ and various choices to be made, it is easily seen that all of these sets contain only finitely many elements of any given degree, and that all elements of degree larger than n can be ignored if we want the u_i only up to degree n.

Since each homogeneous component of $L(X)$ has finite dimension, Lemma 3.2 (iv) gives in the limit ($k \to \infty$) the desired decomposition of $L(X)$ into a direct sum of terms of the form (3.10).

THEOREM 1. *In $L(V) = L(X)$ there exists a sequence $u_1 (= e)$, $u_2, u_3, \ldots$ of Lie monomials such that the sets $X_j = \{u_j g; g \in G\}$ span regular KG-submodules of $L(X)$, and $L(X)$ decomposes into the direct sum*

$$L(X) = \bigoplus_{j=1}^{\infty} (\langle X_j \rangle \oplus \bigoplus_{i=0}^{\infty} L(S(X_j)\, \varphi_j^i))$$

where each of direct summands $L(S(X_j)\, \varphi_j^i)$ is freely generated by the set $S(X_j)\, \varphi_j^i$.

Since the free Lie algebras $L(S(X_j)\, \varphi_j^i)$ are for all $i \geqslant 0$, $j \geqslant 1$ isomorphic to $L(S(X))$ (as graded KG-modules), this theorem reduces the decomposition problem for $L(V)$ to the decomposition problem of $L(S(X))$. Moreover, the theorem reveals a peculiar feature of the free Lie algebra $L(V)$ as a KG-module: it contains an infinite series $\langle X_1 \rangle, \langle X_2 \rangle, \ldots$ of regular submodules, and each of those gives rise to an infinite series of direct summands $L(S(X_j)\, \varphi_j^i)$ $(i = 0, 1, 2, \ldots)$ which are free Lie algebras isomorphic to $L(S(X))$. Furthermore, the members of such a series are obtained from the first member, $L(S(X_j))$, via the powering maps φ_j. This bears a striking analogy to results in [8], [9] and [4] where similar features of free Lie algebras on modules for groups of order p (over fields of characteristic p) are exhibited. In fact, in the case where $p = 2$, our theorem is a straight generalization of these results. Indeed, let G be the group of order 2 with generator g, V the regular KG-module with basis $X = \{e, g\}$. Then the set $S(X)$ consists of a single element, $[g, e]$, and hence the free Lie algebra $L(S(X))$ is one-dimensional. Since $[g, e]$ is fixed under the action of G, all the free Lie algebras $L(S(X_j)\, \varphi_j^i)$ in the theorem are just one-dimensional trivial KG-modules. Thus the theorem turns into the following result which was first proved in [8].

COROLLARY. *Let G be the group of order 2 with generator g, V the regular module for V and $L(V)$ the free Lie algebra on V. Then there exists* *a sequence of monomials* $u_1, u_2, u_3, \ldots$ *in* $L(V)$ *such that the elements* $u_j, u_j g$ $(j = 1, 2, \ldots)$ *together with the elements*

$$[u_j^{2^i} g, u_j^{2^i}] \quad (j = 1, 2, 3, \ldots, \ i = 0, 1, 2, \ldots)$$

form a basis of $L(V)$.

In fact, our corollary is slightly stronger than the original result from [8] since it ensures that the sequence $u_1, u_2, \ldots$, and hence the whole basis of $L(V)$, consists of monomials. It should be pointed out, however, that since the publication of [8], the main result of that paper and a number of variations and far reaching generalizations thereof have been obtained by different methods (see [11], [9], [4], [5], [13]). In particular, a monomial basis similar to the one in the corollary has been obtained in [13].

REMARK. The above corollary extends to the case where V is an arbitrary free KG-module of countable rank (see Theorem 1 in [13] for the finite rank case and the proof of Corollary 1 in [5] for how to extend it to the case of countably infinite rank). We will make use of that in Section 5.

4. Another reduction theorem.

Let G, V and $X = \{e, a, b, \ldots, c\}$ be as before, and consider the free restricted Lie algebra $R = R(X)$. Restricted elimination of the identity element e gives a direct decomposition

$$R(X) = \langle e \rangle \oplus R(Y)$$

where

$$Y = \{e^2, a, b, \ldots, c, [a, e], [b, e], \ldots, [c, e]\}.$$

Now consider $L(Y)$, the free Lie subalgebra generated by Y in $R(X)$. Since $L(Y)$ contains all left normed basic commutators of degree $\geqslant 2$ in X, we have that $L(Y)$ contains the derived algebra $L'(X) = L_2(X) \oplus \oplus L_3(X) \oplus \ldots$. On the other hand, since all elements of Y except e^2 are Lie elements, we have $L(Y) \cap R_n(X) \subseteq L_n(X)$ for all $n \geqslant 3$. Hence

$$L(Y) \cap R_n(X) = L_n(X) \quad \text{for all } n \geqslant 3. \tag{4.1}$$

Now let

$$\widetilde{X} = \{e, a + e, b + e, \ldots, c + e\}.$$

Clearly, $R(X) = R(\widetilde{X})$ and $L(X) = L(\widetilde{X})$. Note that the elements $a+e$, $b+e, \ldots, c+e$ span an $(r-1)$-dimensional KG-submodule Δ of the regular module V. If V is identified with KG, this submodule is the augmentation ideal of KG. Now restricted elimination of e gives a direct decomposition

$$R(X) = R(\widetilde{X}) = \langle e \rangle \oplus R(\widetilde{Y})$$

where

$$\widetilde{Y} = \{e^2, a+e, b+e, \ldots, c+e, [a, e], [b, e], \ldots, [c, e]\}.$$

Since

$$a^2 = [a, e] + (a+e)^2 + e^2,$$

we may replace the free generating set $\widetilde{Y}$ by

$$\widehat{Y} = \{e^2, a^2, b^2, \ldots, c^2, a+e, b+e, \ldots, c+e\}.$$

Thus

$$R(X) = \langle e \rangle \oplus R(\widehat{Y}).$$

The advantage of the free generating set $\widehat{Y}$ is that it spans a KG-submodule of $R(X)$. Indeed, the elements $e^2, a^2, b^2, \ldots, c^2$ span a regular module and the elements $a+e, b+e, \ldots, c+e$ span Δ. Now consider the free Lie algebra $L(\widehat{Y})$. Obviously, $L(\widehat{Y}) \cap R_n(X) \subseteq L_n(X)$ for all $n \geqslant 3$. On the other hand, since both Y and $\widehat{Y}$ consist of $r-1$ elements of degree 1 and r elements of degree 2, it follows that

$$\dim(L(\widehat{Y}) \cap R_n(X)) = \dim(L(Y) \cap R_n(X)) \quad \text{for all } n \geqslant 2,$$

and then (4.1) implies that $L(\widehat{Y}) \cap R_n(X)) = L_n(X)$ for all $n \geqslant 3$. Thus we have proved the following

THEOREM 2. *In the free restricted Lie algebra $R(V) = R(X)$ of the regular KG-module V with $X = \{e, a, b, \ldots, c\}$, let*

$$\widehat{Y} = \{e^2, a^2, b^2, \ldots, c^2, a+e, b+e, \ldots, c+e\}.$$

Then the Lie subalgebra $L(\widehat{Y})$ is freely generated by $\widehat{Y}$, and

$$L(\widehat{Y}) \cap R_n(X) = L_n(X) \quad \textit{for all } n \geqslant 3.$$

Moreover,

$$\langle e^2, a^2, b^2, \dots, c^2 \rangle \cong V \quad \textit{and} \quad \langle a+e, b+e, \dots, c+e \rangle \cong \Delta,$$

so that

$$L(\widehat{Y}) \cong L(\Delta \oplus V).$$

This theorem reduces the decomposition problem for the Lie powers of the regular module V to the decomposition problem for the Lie powers of Δ. However, since we are not able to solve either of those we will not elaborate on that. Instead, we record some consequences of the theorem which are of independent interest. Since $L_n(a+e, b+e, \dots, c+e)$ is a direct summand of $L(\widehat{Y})$, it is also a direct summand of $L(\widehat{Y}) \cap R_n(X)$, and since the latter coincides with $L_n(X)$ for all $n \geqslant 3$, we have the following

COROLLARY. *For all $n \geqslant 3$, $L_n(\Delta)$ is a direct summand of $L_n(V)$.*

Combining this with Fact 1 in Section 3 gives

COROLLARY. *The Lie powers $L_n(\Delta)$ are free KG-modules for all $n \geqslant 3$ such that $(n, |G|) = 1$.*

In the concluding two sections we study the Lie powers of the regular module for the Klein four group.

5. The Klein four group: preliminary discussion.

For the rest of this paper $G = \{e, x, y, z\}$ is the Klein four group, and $V = \langle X \rangle$ with $X = \{e, x, y, z\}$ is the regular module for G. The indecomposable KG-modules have been fully classified by Conlon [10]. In this paper, however, we need only a few of them, namely, the regular module V, the 3-dimensional indecomposable module Δ (the augmentation ideal in KG), its dual Δ^* (which is isomorphic to the quotient $KG/(e+x+y+z)KG$), the trivial module K and the three 2-dimensional induced modules

$$V_x = K \uparrow_{\langle x \rangle}^G, \quad V_y = K \uparrow_{\langle y \rangle}^G, \quad \text{and} \quad V_z = K \uparrow_{\langle z \rangle}^G.$$

By Fact 3 in Section 3, $L_n(V)$ is a free KG-module for all odd n.

Further results have been obtained by Michos [12]. He solved the decomposition problem for $L_{2q}(V)$ for odd q, and for $L_4(V)$. It turned out that the only indecomposables occurring in $L_{2q}(V)$ are the regular module and the induced modules V_x, V_y, V_z, while $L_4(V) \cong 2\Delta^* \oplus V_x \oplus V_y \oplus \oplus V_z \oplus 12V$. Michos' approach was entirely based on studying the restriction of $L_n(V)$ to the cyclic subgroups of G and, for $n = 4$, on explicit calculations.

In this paper we pursue a different approach. By Theorem 1, the decomposition problem for the Lie powers of the regular module V reduces to the decomposition problem for the free Lie algebra $L(S(X))$, and it is this Lie algebra we will focus on now. Assuming that X is ordered by $e < < x < y < z$, we have

$$S(X) = \{[x,e],[z,y],[y,e],[z,x],[z,e],[y,x],[x,e,y,z],[y,e,x,z],[z,e,y,z]\}.$$

In order to simplify notation, we set

$$\mathsf{x} = [x, e], \quad \mathsf{y} = [y, e], \quad \mathsf{z} = [z, e],$$

$$\bar{\mathsf{x}} = [z, y], \quad \bar{\mathsf{y}} = [z, x], \quad \bar{\mathsf{z}} = [y, x],$$

$$\mathsf{a}' = [[x, e],[y, z]] = [\bar{\mathsf{x}}, \mathsf{x}],$$

$$\mathsf{b}' = [[y, e],[z, x]] = [\bar{\mathsf{y}}, \mathsf{y}],$$

$$\mathsf{c}' = [[z, e],[y, x]] = [\bar{\mathsf{z}}, \mathsf{z}],$$

and

$$\mathsf{a} = [x, e, y, z],$$
$$\mathsf{b} = [y, e, x, z] + [[y, e],[z, x]],$$
$$\mathsf{c} = [z, e, x, y].$$

Then we have

$$L(S(X)) = L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}, \mathsf{a}, \mathsf{b}, \mathsf{c}).$$

Obviously,

$$\mathsf{x}x = \mathsf{x}, \quad \mathsf{y}x = \bar{\mathsf{y}}, \quad \mathsf{z}x = \bar{\mathsf{z}},$$

$$\bar{\mathsf{x}}x = \bar{\mathsf{x}}, \quad \mathsf{y}y = \mathsf{y}, \quad \mathsf{z}y = \bar{\mathsf{z}},$$

$$\mathsf{x}y = \bar{\mathsf{x}}, \quad \bar{\mathsf{y}}y = \bar{\mathsf{y}}, \quad \mathsf{z}z = \mathsf{z},$$

$$\mathsf{x}z = \bar{\mathsf{x}}, \quad \mathsf{y}z = \bar{\mathsf{y}}, \quad \bar{\mathsf{z}}z = \bar{\mathsf{z}},$$

and consequently

$$\langle \mathsf{x}, \bar{\mathsf{x}} \rangle \cong V_x, \qquad \langle \mathsf{y}, \bar{\mathsf{y}} \rangle \cong V_y, \qquad \langle \mathsf{z}, \bar{\mathsf{z}} \rangle \cong V_z.$$

In particular, we have

$$L_2(V) = \langle \mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}} \rangle \cong V_x \oplus V_y \oplus V_z.$$

As to the action of G on the remaining elements of $S(X)$, it is plain that a', b' and c' are fixed by G, and one easily calculates that the action on a, b and c is given by

$$(5.1) \qquad \begin{array}{lll} \mathsf{a}x = \mathsf{a} + \mathsf{a}' & \mathsf{b}x = \mathsf{a} + \mathsf{c} + \mathsf{a}' + \mathsf{b}' & \mathsf{c}x = \mathsf{a} + \mathsf{b} + \mathsf{b}' \\ \mathsf{a}y = \mathsf{b} + \mathsf{c} + \mathsf{c}' & \mathsf{b}y = \mathsf{b} + \mathsf{b}' & \mathsf{c}y = \mathsf{a} + \mathsf{b} + \mathsf{b}' + \mathsf{c}' \\ \mathsf{a}z = \mathsf{b} + \mathsf{c} + \mathsf{c}' + \mathsf{a}' & \mathsf{b}z = \mathsf{a} + \mathsf{c} + \mathsf{a}' & \mathsf{c}z = \mathsf{c} + \mathsf{c}'. \end{array}$$

The latter shows that the action of G on $L(S(X))$ is not homogeneous. This is a major disadvantage which complicates matters tremendously.

We now examine the restrictions of $L(S(X))$ to the cyclic subgroups of G. To this end we first notice that

$$L(S(X)) = L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}, \mathsf{a}, \mathsf{b}, \mathsf{b}x),$$

in other words, that the element c in the original free generating set of $L(S(X))$ can be replaced by $\mathsf{b}x$. This follows immediately from (5.1). Now full elimination of x and $\bar{\mathsf{x}}$ gives a direct decomposition

$$L(S(X)) = \langle \mathsf{x}, \bar{\mathsf{x}} \rangle \oplus L(Y)$$

where

$$Y = \{[\bar{\mathsf{x}}, \mathsf{x}^k, \bar{\mathsf{x}}^m];\ k \geq 1,\ m \geq 0\} \cup$$
$$\cup \{[u, \mathsf{x}^k, \bar{\mathsf{x}}^m];\ u \in \{\mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}, \mathsf{a}, \mathsf{b}, \mathsf{b}x\},\ k \geq 0,\ m \geq 0\}$$

It is easily seen that $\langle Y \rangle$ is a free $\langle x \rangle$-module. Indeed, since x and $\bar{\mathsf{x}}$ are fixed by x, the elements $[u, \mathsf{x}^k, \bar{\mathsf{x}}^m]$ with $u \in \{\mathsf{y}, \bar{\mathsf{y}}\}$, $u \in \{\mathsf{z}, \bar{\mathsf{z}}\}$ and $u \in \{\mathsf{b}, \mathsf{b}x\}$ are interchanged by x, and for the remaining elements we have (using (5.1))

$$[\mathsf{a}, \mathsf{x}^k, \bar{\mathsf{x}}^m]\, x = [\mathsf{a}, \mathsf{x}^k, \bar{\mathsf{x}}^m] + [\bar{\mathsf{x}}, \mathsf{x}^{k+1}, \bar{\mathsf{x}}^m], \quad [\bar{\mathsf{x}}, \mathsf{x}^{k+1}, \bar{\mathsf{x}}^m]\, x = [\bar{\mathsf{x}}, \mathsf{x}^{k+1}, \bar{\mathsf{x}}^m].$$

Hence these come also in pairs which generate regular $\langle x \rangle$-modules. By the corollary to Theorem 1, and the remark thereafter, $L(Y)$ has an $\langle x \rangle$-invariant basis consisting of pairs of elements which are swapped by x

and elements which are fixed by x. Moreover, all of the fixed elements have even degree with respect to Y. Since the free generators in Y have even degree with respect to X, it follows that the degrees of the fixed elements with respect to X are divisible by 4. This implies that for all $n>2$ which are not divisible by 4, the intersection $L(S(X)) \cap L_n(X)$ is a free $\langle x\rangle$-module. The same holds of course for the restrictions to the other cyclic subgroups of G. We record this result in the following

LEMMA 5.1. *For all $n>2$ such that $n \not\equiv 0 \bmod 4$, the restrictions of $L(S(X)) \cap L_n(X)$ to the cyclic subgroups of G are free modules for those cyclic groups.*

Observe that the elements $\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}$ generate a G-invariant free Lie subalgebra of $L(S(X))$ with

$$L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}) \cong L(V_x \oplus V_y \oplus V_z). \tag{5.2}$$

Set $V_1 = \langle \mathsf{x}, \bar{\mathsf{x}}\rangle$, $V_2 = \langle \mathsf{y}, \bar{\mathsf{y}}\rangle$, $V_3 = \langle \mathsf{z}, \bar{\mathsf{z}}\rangle$. By Lemma 3 of [7], the free Lie algebra (5.2) has a direct decomposition

$$L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}) = L(\mathsf{x}, \bar{\mathsf{x}}) \oplus L(\mathsf{y}, \bar{\mathsf{y}}) \oplus L(\mathsf{z}, \bar{\mathsf{z}}) \oplus L(U_2 \oplus U_3 \oplus \dots) \tag{5.3}$$

where, for all $n \geqslant 2$, U_n is the direct sum of the subspaces

$$[V_{i_1}, V_{i_2}, \dots, V_{i_n}] \quad (i_1 > i_2 \leqslant \dots \leqslant i_n),$$

and, moreover, all these U_n are free KG-submodules of $L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}})$. We will use this fact in the concluding section.

Before we continue our examination of $L(S(X))$, consider the module Δ. Set $\tilde{\mathsf{a}} = x + e$, $\tilde{\mathsf{b}} = y + e$, $\tilde{\mathsf{c}} = z + e$. Then $\Delta = \langle \tilde{\mathsf{a}}, \tilde{\mathsf{b}}, \tilde{\mathsf{c}}\rangle$. The G-action on Δ is given by

$$\begin{array}{lll} \tilde{\mathsf{a}}x = \tilde{\mathsf{a}} & \tilde{\mathsf{b}}x = \tilde{\mathsf{a}} + \tilde{\mathsf{c}} & \tilde{\mathsf{c}}x = \tilde{\mathsf{a}} + \tilde{\mathsf{b}} \\ \tilde{\mathsf{a}}y = \tilde{\mathsf{b}} + \tilde{\mathsf{c}} & \tilde{\mathsf{b}}y = \tilde{\mathsf{b}} & \tilde{\mathsf{c}}y = \tilde{\mathsf{a}} + \tilde{\mathsf{b}} \\ \tilde{\mathsf{a}}z = \tilde{\mathsf{b}} + \tilde{\mathsf{c}} & \tilde{\mathsf{b}}z = \tilde{\mathsf{a}} + \tilde{\mathsf{c}} & \tilde{\mathsf{c}}z = \tilde{\mathsf{c}}. \end{array}$$

By comparing this with (5.1) it is easily seen that in $L(S(X))$

$$\langle \mathsf{a}, \mathsf{b}, \mathsf{c}\rangle \cong \Delta \bmod L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}).$$

Elimination of $L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}})$ from $L(S(X))$ gives a direct decomposition

$$L(S(X)) = L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}) \oplus L(W) \tag{5.4}$$

where

$$W = \{[u, v_1, \dots, v_k];\ u \in \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}, v_1, \dots, v_k \in \{\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}\},\ k \geqslant 0\}.$$

Notice that (5.4) is merely a direct decomposition over K. It is *not* a direct decomposition of KG-modules, since $\langle W \rangle$ is not G-invariant. However, modulo the G-invariant subalgebra $L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}})$ the span of W is G-invariant, and its structure as a KG-module is particularly simple. Let $W^{(m)}$ $(m = 0, 1, 2, \dots)$ denote the set of elements in W with $k = m$.

LEMMA 5.2. *For each $m \geqslant 0$, consider the map*

$$[u, v_1, \dots, v_m] \mapsto \tilde{u} \otimes v_1 \otimes \dots \otimes v_m$$

where $u \in \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ and $v_1, \dots, v_m \in \{\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}\}$. Modulo the submodule $L(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}})$, this may be extended to an isomorphism

$$\langle W^{(m)} \rangle \to \Delta \otimes T^m(V_x \oplus V_y \oplus V_z).$$

PROOF. This follows immediately from (5.1). ■

Clearly,

$$T^m(V_x \oplus V_y \oplus V_z) \cong T^m(V_x) \oplus T^m(V_y) \oplus T^m(V_z) \oplus Q_m \tag{5.5}$$

where Q_m is a free KG-module, and

$$T^m(V_x) \cong V_x \oplus \dots \oplus V_x \qquad (2^{m-1} \text{ copies}).$$

In particular, the dimension of the free KG-module Q_m is given by $\dim Q_m = 2^m(3^m - 3)$.

LEMMA 5.3. *Let $\Delta = \langle \tilde{\mathsf{a}}, \tilde{\mathsf{b}}, \tilde{\mathsf{c}} \rangle$ and $V_x = \langle \mathsf{x}, \bar{\mathsf{x}} \rangle$. Then there is an isomorphism of KG-modules*

$$\Delta \otimes V_x \cong V_x \oplus V.$$

Moreover, the elements

$$\tilde{\mathsf{a}} \otimes \mathsf{x}, \qquad (\tilde{\mathsf{b}} + \tilde{\mathsf{c}}) \otimes \bar{\mathsf{x}}$$

span a direct summand isomorphic to V_x *in* $\varDelta \otimes V_x$.

PROOF. It is easily verified by direct calculation that

$$\varDelta \otimes V_x = \langle \tilde{\mathsf{a}} \otimes \mathsf{x}, (\tilde{\mathsf{b}} + \tilde{\mathsf{c}}) \otimes \bar{\mathsf{x}} \rangle \oplus \langle (\tilde{\mathsf{b}} \otimes \mathsf{x})\, KG \rangle,$$

and that the direct summands on the right hand side are isomorphic to V_x and V, respectively. This proves the lemma. ■

Finally, we record the fact that

(5.6) $$L_2(\varDelta) = [\mathsf{a}, \mathsf{b}]\, KG \quad \text{and} \quad L_2(\varDelta) \cong \varDelta^*.$$

This is easily verified by direct calculation.

6. The Klein four group: results.

Now we exploit the results of the previous section and Theorem 1 to obtain information about the Lie powers $L_n(V)$ of the regular module for the Klein four group G. Since these are free for all odd n (Fact 1), we restrict ourselves to the case where n is even. First we examine the case where $n = 2q$ with q odd. Let

(6.1) $$u_1,\ u_2,\ u_3,\ \ldots$$

be the sequence from Theorem 1, and $X_1 = X, X_2, X_3, \ldots$ the corresponding sequence of subsets. Note that the free generating sets $S(X_j)\,\varphi_j^i$ consist of elements of degree $2^{i+1}\deg u_j$ and $2^{i+2}\deg u_j$. Hence the direct summands $L(S(X_j)\,\varphi_j^i$ with $i > 0$ do not contribute to the homogeneous components $L_{2q}(V)$ with q odd. Now let q be an arbitrary but fixed odd natural number. Then we have, by Theorem 1,

(6.2) $$L_{2q}(V) = \bigoplus_{j \in I} \langle X_j \rangle \oplus \bigoplus_{j \in J} (L(S(X_j)) \cap L_{2q}(V))$$

where I is the set of all positive integers k such that $\deg u_k = 2q$ and J is the set of all positive integers m such that $\deg u_m$ divides q.

Let $m \in J$. If $\deg u_m = q$, we have obviously

(6.3) $$L(S(X_m)) \cap L_{2q}(V) = L_2(X_m) \cong V_x \oplus V_y \oplus V_z.$$

Now let $m \in J$ with $d = \deg u_m \neq q$. Then $q = dk$ where both d and k

are odd and $k > 1$. We then have

$$L(S(X_m)) \cap L_{2q}(V) = L(S(X_m)) \cap L_{2k}(X_m).$$

By Lemma 5.2, this module is free on restriction to the cyclic subgroups of G. Now consider the multihomogeneous components $L_{(a,\, b,\, c,\, d)}(X)$ with $a+b+c+d = 2q$ and a, b, c, d indicating the partial degree in e, x, y, z, respectively. Let F denote the set of all 4-tuples (a, b, c, d) with $a+b+c+d = 2q$ such that that the corresponding multihomogeneous components are freely permuted by the action of G. Then the direct sum of these multihomogeneous components is a free KG-module, and so is its direct summand

$$L(S(X_m)) \cap \bigoplus_{(a,\, b,\, c,\, d) \in F} L_{(a,\, b,\, c,\, d)}(X).$$

It is easily seen that the remaining multihomogeneous components of total degree $2q$ are determined by 4-tuples of the form

$$(a, a, b, b), (b, b, a, a), (a, b, a, b), (b, a, b, a), (a, b, b, a), (b, a, a, b)$$

where $a + b = q$. Obviously, the direct sums

$$\begin{aligned} &L_{(a,\, a,\, b,\, b)}(X) \oplus L_{(b,\, b,\, a,\, a)}(X), \\ &L_{(a,\, b,\, a,\, b)}(X) \oplus L_{(b,\, a,\, b,\, a)}(X), \\ &L_{(a,\, b,\, b,\, a)}(X) \oplus L_{(b,\, a,\, a,\, b)}(X) \end{aligned} \tag{6.4}$$

are direct summands of the KG-module $L_{2q}(X)$, and hence so are their intersections with $L(S(X_m))$. Consider the intersection

$$L(S(X_m)) \cap (L_{(a,\, a,\, b,\, b)}(X) \oplus L_{(b,\, b,\, a,\, a)}(X)) =$$
$$= (L(S(X_m)) \cap (L_{(a,\, a,\, b,\, b)}(X)) \oplus (L(S(X_m)) \cap L_{(b,\, b,\, a,\, a)}(X)).$$

Clearly, the two direct summands on the right hand side are permuted by the action of y and z. But by Lemma 5.1, each of these direct summands is a free $\langle x \rangle$-module. It follows that the direct sum of the two is a free KG-module. The same holds, by a similar argument, for the other two modules in (6.4). Hence the direct summands $L(S(X_m)) \cap L_{2q}(V)$ in (6.2) are free KG-modules for all $m \in J$ except when $\deg u_m = q$, where (6.3) applies. Finally, note that the number of elements u_m with $\deg u_m = q$ in the sequence (6.1) is $\frac{1}{4} \dim L_q(X)$ (since the corresponding X_m span the free KG-module $L_q(X)$). Now we can summarize our discussion as follows.

THEOREM 3 (Michos [12]). *Let V be the regular module for the Klein four group G. Then, for all odd $q \geqslant 1$,*

$$L_{2q}(V) \cong s(V_x \oplus V_y \oplus V_z) \oplus tV$$

where

$$s = \frac{1}{4q} \sum_{d|q} \mu(d) 4^{q/d} \quad \textit{and} \quad t = \frac{1}{8q} \sum_{d|q} \mu(d)(4^{2q/d} - 4 \cdot 4^{q/d}).$$

Our next task is to identify the module structure of $L(S(X)) \cap L_n(V)$ for $n = 4$ and $n = 8$. Clearly,

$$L(S(X)) \cap L_4(V) = \langle \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{a}', \mathsf{b}', \mathsf{c}' \rangle \oplus \langle A \rangle$$

where

$$A = \{[u, v];\ u, v \in \{\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}\}\} \setminus \{\mathsf{a}', \mathsf{b}', \mathsf{c}'\}.$$

It is easily seen that

$$\langle A \rangle \cong 3V,$$

and it is easily verified by direct calculation using (5.1) that

$$\langle \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{a}', \mathsf{b}', \mathsf{c}' \rangle = \mathsf{a}KG \oplus \mathsf{b}KG,$$

and that

$$\mathsf{a}KG \cong \mathsf{b}KG \cong \Delta^*.$$

Hence

$$L(S(X)) \cap L_4(V) \cong 2\Delta^* \oplus 3V. \tag{6.5}$$

Now consider the intersection $L(S(X)) \cap L_8(V)$. In view of (5.4) we have that

$$\begin{aligned} L(S(X)) \cap L_8(V) = {} & L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}) \oplus \\ & \oplus \langle [u, v_1, v_2];\ u \in \{\mathsf{a}, \mathsf{b}, \mathsf{c}\},\ v_1, v_2 \in \{\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}\} \rangle \\ & \oplus \langle [\mathsf{a}, \mathsf{b}], [\mathsf{a}, \mathsf{c}], [\mathsf{b}, \mathsf{c}] \rangle. \end{aligned} \tag{6.6}$$

This again is merely a direct sum of vector spaces and not a direct sum of KG-modules. However, the first direct summand on the right hand side of (6.6) is a KG-submodule of $L(S(X)) \cap L_8(V)$, and so is the direct sum of the first and the second, U say. This gives rise to a filtration

(6.7) $$L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}) \subset U \subset L(S(X)) \cap L_8(V).$$

It is easily seen that the top quotient of this filtration is $([\mathsf{a}, \mathsf{b}] + U)KG$ and is isomorphic to $L_2(\Delta)$, whence by (5.6) we have

(6.8) $$(L(S(X)) \cap L_8(V))/U \cong \Delta^*.$$

We claim that the top quotient (6.8) splits off. To see this, consider the element

$$w = [\mathsf{a}, \mathsf{b}] + [\mathsf{a}, \mathsf{z}, \bar{\mathsf{z}}] + [\mathsf{b}, \mathsf{x}, \bar{\mathsf{x}}] + [\mathsf{c}, \mathsf{y}, \bar{\mathsf{y}}].$$

Clearly, $w \in [\mathsf{a}, \mathsf{b}] + U$, so $wKG + U = L(S(X)) \cap L_8(V)$ and wKG has dimension at least 3. Using (5.1), one gets

$$\begin{aligned} w(e+x+y+z) = {} & [\mathsf{a}, \mathsf{b}] + [\mathsf{a}, \mathsf{z}, \bar{\mathsf{z}}] + [\mathsf{b}, \mathsf{x}, \bar{\mathsf{x}}] + [\mathsf{c}, \mathsf{y}, \bar{\mathsf{y}}] \\ & + [\mathsf{a}+\mathsf{a}', \mathsf{a}+\mathsf{c}+\mathsf{b}'+\mathsf{a}'] + [\mathsf{a}+\mathsf{a}', \bar{\mathsf{z}}, \mathsf{z}] \\ & + [\mathsf{a}+\mathsf{c}+\mathsf{a}'+\mathsf{b}', \mathsf{x}, \bar{\mathsf{x}}] + [\mathsf{a}+\mathsf{b}+\mathsf{b}', \bar{\mathsf{y}}, \mathsf{y}] \\ & + [\mathsf{b}+\mathsf{c}+\mathsf{c}', \mathsf{b}+\mathsf{b}'] + [\mathsf{b}+\mathsf{c}+\mathsf{c}', \bar{\mathsf{z}}, \mathsf{z}] \\ & + [\mathsf{b}+\mathsf{b}', \bar{\mathsf{x}}, \mathsf{x}] + [\mathsf{a}+\mathsf{b}+\mathsf{b}'+\mathsf{c}', \mathsf{y}, \bar{\mathsf{y}}] \\ & + [\mathsf{b}+\mathsf{c}+\mathsf{a}'+\mathsf{c}', \mathsf{a}+\mathsf{c}+\mathsf{a}'] + [\mathsf{b}+\mathsf{c}+\mathsf{a}'+\mathsf{c}', \mathsf{z}, \bar{\mathsf{z}}] \\ & + [\mathsf{a}+\mathsf{c}+\mathsf{a}', \bar{\mathsf{x}}, \mathsf{x}] + [\mathsf{c}+\mathsf{c}', \bar{\mathsf{y}}, \mathsf{y}] \\ = {} & 0, \end{aligned}$$

so wKG has dimension at most 3. Consequently, we have a direct decomposition

(6.9) $$L(S(X)) \cap L_8(V) = U \oplus wKG \quad \text{with } wKG \cong \Delta^*.$$

Now consider the quotient $U/L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}})$. By Lemma 5.2, there is an isomorphism

$$U/L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}) \cong \Delta \otimes T_2(V_x \oplus V_y \oplus V_z),$$

and by (5.5) this gives

$$U/L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}) \cong$$

$$\cong (\Delta \otimes T_2(V_x)) \oplus (\Delta \otimes T_2(V_y)) \oplus (\Delta \otimes T_2(V_z)) \oplus (\Delta \otimes Q_2)$$

where Q_2 is a free KG-module of dimension 24. Since Q_2 is a free KG-module, so is the tensor product $\Delta \otimes Q_2$. For the tensor square $T_2(V_x)$ we have obviously

$$T_2(V_x) = \langle \mathsf{x}\otimes\mathsf{x}, \bar{\mathsf{x}}\otimes\bar{\mathsf{x}}\rangle \oplus \langle \mathsf{x}\otimes\bar{\mathsf{x}}, \bar{\mathsf{x}}\otimes\mathsf{x}\rangle \cong V_x \oplus V_x,$$

and there are similar decompositions for $T_2(V_y)$ and $T_2(V_z)$. Now Lemma 5.3 together with Lemma 5.2 gives that

$$U/L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}) \cong 2V_x \oplus 2V_y \oplus 2V_z \oplus P \tag{6.10}$$

where P is a free KG-module. Moreover, the elements

$$[\mathsf{a}, \mathsf{x}, \mathsf{x}],\ [\mathsf{b}+\mathsf{c}, \bar{\mathsf{x}}, \bar{\mathsf{x}}] \quad\text{and}\quad [\mathsf{a}, \mathsf{x}, \bar{\mathsf{x}}],\ [\mathsf{b}+\mathsf{c}, \bar{\mathsf{x}}, \mathsf{x}] \tag{6.11}$$

span two direct summands isomorphic to V_x in $U/L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}})$, and the elements obtained from (6.11) via the substitutions

$$\mathsf{x}\mapsto\mathsf{y},\ \bar{\mathsf{x}}\mapsto\bar{\mathsf{y}} \quad\text{and}\quad \mathsf{x}\mapsto\mathsf{z},\ \bar{\mathsf{x}}\mapsto\bar{\mathsf{z}}$$

span two direct summands isomorphic to V_y and V_z, respectively. Of course the direct summand P of $U/L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}})$ splits off from U, and it therefore remains to examine the elements (6.11) (and the corresponding elements obtained under the above substitutions). Before we turn to those elements, we examine $L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}})$, the module at the bottom of the filtration (6.7). In view of (5.3) one has

$$L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}) = L_4(\mathsf{x}, \bar{\mathsf{x}}) \oplus L_4(\mathsf{y}, \bar{\mathsf{y}}) \oplus L_4(\mathsf{z}, \bar{\mathsf{z}}) \oplus L_2(U_2) \oplus U_4 \tag{6.12}$$

where U_2 and U_4 are free KG-modules. Moreover, it is easily seen that U_2 is a free KG-module of rank 3, and hence

$$L_2(U_2) \cong 3(V_x \oplus V_y \oplus V_z) \oplus P_1 \tag{6.13}$$

where P_1 is a free KG-module. Finally,

$$L_4(\mathsf{x}, \bar{\mathsf{x}}) = \langle [\bar{\mathsf{x}}, \mathsf{x}, \mathsf{x}, \mathsf{x}], [\bar{\mathsf{x}}, \mathsf{x}, \bar{\mathsf{x}}, \bar{\mathsf{x}}]\rangle \oplus \langle [\bar{\mathsf{x}}^2, \mathsf{x}^2]\rangle, \tag{6.14}$$

and similarly for $L_4(\mathsf{y}, \bar{\mathsf{y}})$ and $L_4(\mathsf{z}, \bar{\mathsf{z}})$. Now return to the elements (6.11). We first examine the first two of them. Note that

$$[\mathsf{b}+\mathsf{c}, \bar{\mathsf{x}}, \bar{\mathsf{x}}] \equiv [\mathsf{b}+\mathsf{c}+\mathsf{c}', \bar{\mathsf{x}}, \bar{\mathsf{x}}] \bmod L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}).$$

Using (5.1) we find that

$$\begin{aligned}
[\mathsf{a}, \mathsf{x}, \mathsf{x}]\,x &= [\mathsf{a}+\mathsf{a}', \mathsf{x}, \mathsf{x}] && = [\mathsf{a}, \mathsf{x}, \mathsf{x}] + [\bar{\mathsf{x}}, \mathsf{x}, \mathsf{x}, \mathsf{x}]\\
[\mathsf{a}, \mathsf{x}, \mathsf{x}]\,y &= [\mathsf{b}+\mathsf{c}+\mathsf{c}', \bar{\mathsf{x}}, \bar{\mathsf{x}}] &&\\
[\mathsf{a}, \mathsf{x}, \mathsf{x}]\,z &= [\mathsf{b}+\mathsf{c}+\mathsf{c}'+\mathsf{a}', \bar{\mathsf{x}}, \bar{\mathsf{x}}] && = [\mathsf{b}+\mathsf{c}+\mathsf{c}', \bar{\mathsf{x}}, \bar{\mathsf{x}}] + [\bar{\mathsf{x}}, \mathsf{x}, \bar{\mathsf{x}}, \bar{\mathsf{x}}].
\end{aligned}$$

This implies that the elements

(6.15) $$[\mathsf{a}, \mathsf{x}, \mathsf{x}],\ [\mathsf{b}+\mathsf{c}+\mathsf{c}', \bar{\mathsf{x}}, \bar{\mathsf{x}}],\ [\bar{\mathsf{x}}, \mathsf{x}, \mathsf{x}, \mathsf{x}],\ [\bar{\mathsf{x}}, \mathsf{x}, \bar{\mathsf{x}}, \bar{\mathsf{x}}]$$

span a regular KG-submodule of U, and, clearly, the same holds for their counterparts under the substitutions described above. Now consider the remaining two elements in (6.11). First note that

$$[\mathsf{b}+\mathsf{c}, \bar{\mathsf{x}}, \mathsf{x}] \equiv [\mathsf{b}+\mathsf{c}+\mathsf{c}', \bar{\mathsf{x}}, \mathsf{x}] \bmod L_4(\mathsf{x}, \bar{\mathsf{x}}, \mathsf{y}, \bar{\mathsf{y}}, \mathsf{z}, \bar{\mathsf{z}}).$$

Using (5.1) again, we find that

$$\begin{aligned} [\mathsf{a}, \mathsf{x}, \bar{\mathsf{x}}]\, x &= [\mathsf{a}+\mathsf{a}', \mathsf{x}, \bar{\mathsf{x}}] && = [\mathsf{a}, \mathsf{x}, \bar{\mathsf{x}}] + [\bar{\mathsf{x}}, \mathsf{x}^2] \\ [\mathsf{a}, \mathsf{x}, \bar{\mathsf{x}}]\, y &= [\mathsf{b}+\mathsf{c}+\mathsf{c}', \bar{\mathsf{x}}, \bar{\mathsf{x}}] && \\ [\mathsf{a}, \mathsf{x}, \bar{\mathsf{x}}]\, z &= [\mathsf{b}+\mathsf{c}+\mathsf{c}'+\mathsf{a}', \bar{\mathsf{x}}, \mathsf{x}] && = [\mathsf{b}+\mathsf{c}+\mathsf{c}', \bar{\mathsf{x}}, \mathsf{x}] + [\bar{\mathsf{x}}, \mathsf{x}^2]. \end{aligned}$$

This gives that the elements

(6.16) $$[\mathsf{a}, \mathsf{x}, \bar{\mathsf{x}}],\quad [\mathsf{b}+\mathsf{c}+\mathsf{c}', \bar{\mathsf{x}}, \mathsf{x}],\quad [\bar{\mathsf{x}}^2, \mathsf{x}^2]$$

span a 3-dimensional KG-submodule of U, and it is plain that this submodule is isomorphic to Δ^*. As before, the same holds for their counterparts under the substitutions described above. It follows that U is a direct sum of three isomorphic copies of Δ^* (coming from the elements (6.15)), three isomorphic copies of $V_x \oplus V_y \oplus V_z$ (coming from $L_2(U_2)$, see (6.12) and (6.13)) and a free KG-module coming from the free direct summand P in (6.10), the free direct summand P_1 in (6.13) and the free direct summand U_4 in (6.12). By combining this with (6.9), which provides an additional direct summand of $L(S(X)) \cap L_8(V)$, which is isomorphic to Δ^*, we obtain a complete solution of the decomposition problem for this module.

LEMMA 6.1. *There is an isomorphism*

$$L(S(X)) \cap L_8(V) \cong 4\Delta^* \oplus 3(V_x \oplus V_y \oplus V_z) \oplus F$$

where F is a free KG-module.

Armed with this lemma, (6.5), and the fact that the second Lie power of a free module of rank m is a direct sum of a free KG-module and m isomorphic copies of $V_x \oplus V_y \oplus V_z$, complete decompositions of the Lie powers $L_4(V)$ and $L_8(V)$ can now be easily obtained from Theorem 1.

THEOREM 4. *Let V be the regular module for the Klein four group G. Then*

$$L_4(V) \cong 2\Delta^* \oplus V_x \oplus V_y \oplus V_z \oplus 12V$$

and

$$L_8(V) \cong 6\Delta^* \oplus 13(V_x \oplus V_y \oplus V_z) \oplus 2016V.$$

PROOF. As we have pointed out in the Remark after the proof of Lemma 3.2, the construction in Section 3 provides us with an explicit recipe for constructing the sequence $u_1, u_2, u_3, \dots$. A straightforward analysis of the elimination procedure leading to this sequence shows that in the case where V is the regular module for the Klein four group, this sequence contains precisely one element of degree 1, namely $u_1 = e$, no elements of degree 2, and nine elements of degree 4. In fact, the elements of degree 4 emerge already after the execution of Elimination Steps 1 and 2. Three of them,

$$[e^2, x, y], \ [e^2, x, z] \ \text{and} \ [e^2, y, z]$$

(elements of the form (3.3)) are created in the Elimination Step 1, and six more,

$$[e^2, \mathsf{x}], \ [e^2, \bar{\mathsf{x}}], \ [e^2, \mathsf{y}], \ [e^2, \bar{\mathsf{y}}], \ [e^2, \mathsf{z}], \ [e^2, \bar{\mathsf{z}}]$$

are added when the subalgebra $R(S(X))$ is eliminated from $R(Z)$ in Elimination Step 3. It is easily seen that no more elements of degree 1, 2 or 4 emerge in the whole elimination procedure. For each positive integer m, set $N^{(m)} = \{j; \deg u_j = m\}$. Then we have by Theorem 1,

$$L_4(X) = (L(S(X)) \cap L_4(X)) \oplus (L(S(X^2) \cap L_4(X)) \oplus \bigoplus_{j \in N^{(4)}} \langle X_j \rangle.$$

Since $L(S(X)) \cap L_4(X) \cong 2\Delta^* \oplus 3V$ by (6.5),

$$L(S(X^2) \cap L_4(X) = L(S(X^2) \cap L_2(X^2) \cong V_x \oplus V_y \oplus V_z,$$

and $|N^{(4)}| = 9$ this proves the assertion about $L_4(X)$. For $L_8(X)$, Theorem 1 gives

$$
\begin{aligned}
L_8(X) = {} & (L(S(X)) \cap L_8(X)) \\
& \oplus (L(S(X^2)) \cap L_8(X)) \\
& \oplus (L(S(X^4)) \cap L_8(X)) \\
& \oplus \bigoplus_{j \in N^{(4)}} (L(S(X_j)) \cap L_8(X)) \\
& \oplus \bigoplus_{j \in N^{(8)}} \langle X_j \rangle .
\end{aligned} \tag{6.17}
$$

The first direct summand on the right hand side, $(L(S(X)) \cap L_8(X))$, has been identified in Lemma 6.1. For the second and third we have

$$L(S(X^2)) \cap L_8(X) = L(S(X^2)) \cap L_4(X^2) \cong 2\Delta^* \oplus 3V$$

by (6.5) and

$$L(S(X^4)) \cap L_8(X) = L(S(X^2)) \cap L_2(X^4) \cong V_x \oplus V_y \oplus V_z .$$

Similarly, one has

$$L(S(X_j)) \cap L_8(X) = L(S(X_j)) \cap L_2(X_j) \cong V_x \oplus V_y \oplus V_z$$

for all $j \in N^{(4)}$. Since the terms under the last big direct sum sign in (6.17) are regular KG-modules, we have now identified all non-regular direct summands in $L_8(X)$. The multiplicity of the regular module V in $L_8(X)$ is then easily calculated from the corresponding dimensions. ■

As we have pointed out earlier, the result for $L_4(V)$ is due to Michos [12]. Our result for $L_8(V)$ disproves a conjecture from [12] about the decomposition of Lie powers of 2-power degree.

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