Torsionfree Varieties of Metabelian Groups

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Abstract. It is proved here that the free groups of the variety $\mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}^2$ are torsionfree. As usual, $\mathfrak{A}$ denotes the variety of abelian groups, $\mathfrak{A}_s$ the variety of abelian groups of exponent dividing s, and $\mathfrak{N}_c$ the variety of nilpotent groups of class at most c.

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We call a variety of groups ***torsionfree*** if its free groups are torsionfree. Apart from this, we follow the notation and terminology of Hanna Neumann's book [4]. Recent work by Samuel M. Vovsi and the first author [3] on the growth of varieties of groups depends (among other things) on results of J. R. J. Groves [2]. In turn, these make use of ~~the~~ classification of the torsionfree varieties of metabelian groups, Theorem our/ 6.1.2 in R. A. Bryce [1]. All but one step of the proof of this classification was given in Appendix I of [1], but our 'forthcoming' paper which would have contained the missing step was never written. In view of the renewed interest, it seems desirable to place the missing step on record.

Theorem (6.1.2 in [1]). *The varieties of groups $\mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}^2$ $(c, s \geq 1)$ are torsionfree and join-irreducible. Every torsionfree proper subvariety of $\mathfrak{A}^2$ can be uniquely expressed as an irredundant join of some of these torsionfree join-irreducibles.*

All joins mentioned here are the joins of finitely many join-irreducibles. The uniqueness claim is a particularly important part of the Theorem. It implies (by very simple and general lattice-theoretic considerations) that one join of join-irreducibles, $\bigvee_i \mathfrak{U}_i$, is contained in another, $\bigvee_j \mathfrak{V}_j$, if and only if to each i there is a j such that $\mathfrak{U}_i \subseteq \mathfrak{V}_j$. In particular, a join of join-irreducibles is irredundant if and only if its components are pairwise incomparable. It is also well-known that comparability is easy to settle here: $\mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}^2 \subseteq \mathfrak{N}_{c'}\mathfrak{A}_{s'} \wedge \mathfrak{A}^2$ if and only if s divides s' and $c \leq c'$. It is with these points in mind that one speaks of a 'classification'. We conclude the paper with a short argument which justifies these points independently: instead of appealing to the uniqueness claim, it actually implies that claim.

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The outstanding step in the proof of the theorem itself is to show the first of its claims, namely that each $\mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}^2$ is torsionfree. The key point of the proof is the following.

Lemma. *Let G be a free group of the variety $\mathfrak{N}_c \wedge \mathfrak{A}^2$ freely generated by $g_1, \dots, g_n$, let m be an integer with $1 \leq m < n$, and denote by H the subgroup of G generated by the commutator subgroup G' and the elements $g_{m+1}, \dots, g_n$. Then the commutator factor group H/H' is torsionfree.*

Proof. All we really need to show is that G'/H' is torsionfree. We shall be working with basic commutators with reference to the given free generating set of G. As is well known (see 36.32 in [4]), the left-normed basic commutators of weight at least 2 and at most c form a basis of G' as free abelian group. Let X denote the union of this set with $\{ g_j \mid m < j \leq n \}$. Clearly, H is just the subgroup generated by X. Let Y be the set of those left normed basic commutators of weight at least 2 whose last entry has subscript larger than m: thus a typical element y of Y has the form $[g_{i_1}, \dots, g_{i_t}]$ with $t \geq 2$ and $i_t > m$. Here $[g_{i_1}, \dots, g_{i_{t-1}}]$ lies in X: by definition when $t > 2$, and when $t = 2$ then because y is basic. Thus $Y \subset H'$.

We claim that the subgroup $\langle Y \rangle$ generated by Y is normal in H. Of course Y is centralized by G', so what we have to show to prove this claim is that if $y \in Y$ and $j > m$ then $[y, g_j] \in \langle Y \rangle$. If $i_t \leq j$ then $[y, g_j]$ is basic as written and lies in Y. If $i_2 \leq j < i_t$ then $[y, g_j]$ is not basic as written but is still equal to an element of Y, for in a metabelian group the order of the entries of a left-normed commutator is irrelevant beyond the first two places (see 34.51 in [4]). Finally, if $j < i_2$ then $i_1 > i_2 > j > m$ and $m < j < i_2 \leq \cdots \leq i_t$ because y is basic, so all $t+1$ entries of $[y, g_j]$ belong to $\{ g_j \mid m < j \leq n \}$. In this case of course $[y, g_j]$ can be written as a product of basic commutators of weight at least $t+1$, all entries of all the basic commutators involved coming from $\{ g_j \mid m < j \leq n \}$. Then all these basic commutators lie in Y and so $[y, g_j]$ lies in $\langle Y \rangle$.

Next we claim that if $u, v \in X$ then $[u, v] \in \langle Y \rangle$. If $u, v \in G'$ then $[u, v] = 1$ while if $u, v \notin G'$ then either $[u, v]$ or $[u, v]^{-1}$ lies in Y, so it suffices to deal with the case of $u = [g_{i_1}, \dots, g_{i_t}]$ with $t \geq 2$, $v = g_j$ with $j > m$. If now $i_t \leq j$ then $[u, v]$ is basic as written and so lies in Y; otherwise $i_t > j > m$ so $u \in Y$ and then $[u, v] \in \langle Y \rangle$ follows by the previous paragraph.

We have proved that the subset Y of H' generates a normal subgroup in H which contains the commutator of each pair of elements from the generating set X of H: thus $\langle Y \rangle = H'$. Since Y is a subset of a basis of the free abelian group G', it follows that G'/H' is also free abelian. □

Proof of the first claim of the Theorem. It suffices to deal with noncyclic relatively free groups of finite rank. Let F be a noncyclic absolutely free group of rank m, and let A be the verbal subgroup $\mathfrak{A}_s(G)$. By Schreier's Theorem, A is also absolutely free of finite rank and its rank is greater than m: denote this rank by n. Set $N = A''\mathfrak{N}_c(A)$ and

$G = A/N$: then G is an $(\mathfrak{N}_c \wedge \mathfrak{A}^2)$-free group of rank n. The quotient A/F' is free abelian of rank m (because it is a subgroup of finite index in the free abelian group F/F' of rank m), so A/A' splits over F'/A', and F'/A' is free abelian of rank $n-m$. Choose a basis for A/F', and choose a preimage $\{g_1, \ldots, g_m\}$ for that basis in A/N. Similarly, choose $g_{m+1}, \ldots, g_n$ in F'/N so that the image of $\{g_{m+1}, \ldots, g_n\}$ modulo A'/N is a basis for F'/A'. Then $\{g_1, \ldots, g_n\}$ generates G modulo G'. As is well known (see 31.25 in [4]), this implies that $\{g_1, \ldots, g_n\}$ generates G itself, and (see 32.1, ~~41.4~~, and 41.33 in [4]) generates it freely. We may therefore set $H = F'/N$, note 41.44 that $H' = F''N/N$, and apply the Lemma to conclude that $F'/F''N$ is torsionfree. Of course then $F/F''N$ is also torsionfree. Since $F''N = F''\mathfrak{N}_c(A)$ by the definition of N and since $\mathfrak{N}_c(A) = \mathfrak{N}_c\mathfrak{A}_s(F)$, we have that $F''N = (\mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}^2)(F)$. This completes the proof of the Theorem. □

As promised in the introduction, we close with a simple proof of the fact that

$$\mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}^2 \subseteq \bigvee_i (\mathfrak{N}_{c(i)}\mathfrak{A}_{s(i)} \wedge \mathfrak{A}^2) \tag{1}$$

cannot hold unless for some i we have $s | s(i)$ and $c \leq c(i)$.

Suppose that (1) holds.

By Dirichlet's Theorem, there are infinitely many primes p such that $s|(p-1)$. For such a p, in the holomorph of a group of order p one can find an element g of order p and an element h of order s. The subgroup $\langle g, h\rangle$ is in $\mathfrak{A}_p\mathfrak{A}_s$ and hence also in $\mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}^2$. Consider the left-normed commutator

$$[x, y^{t(1)}, \ldots, y^{t(n)}]$$

where $n = \sum(c(i)+1)$ and for each i the sequence $t(1), \ldots, t(n)$ has $c(i)+1$ terms equal to $s(i)$. This commutator is a law in each of the $\mathfrak{N}_{c(i)}\mathfrak{A}_{s(i)}$, but setting $x = g$, $y = h$ shows that it is not a law in $\langle g, h\rangle$ unless some $s(i)$ is divisible by s. This proves that there is at least one i with $s|s(i)$.

Suppose now that $c(i) < c$ whenever $s|s(i)$: we shall show that this leads to a contradiction. Let s' denote the least common multiple of the $s(i)$ that are divisible by s. The inclusion (1) remains valid if we replace all the corresponding join components on the right hand side by $\mathfrak{N}_{c-1}\mathfrak{A}_{s'}$. Instead of changing notation, we assume without loss of generality that s divides $s(1)$ but does not divide any of the other $s(i)$, and that $c(1) < c$.

Choose p large — say, so that also $p > c + \sum(c(i) + s(i))$. Let m be an integer such that, in the holomorph considered above, the conjugate g^h is the same as the power g^m. The wreath product W of two groups of order p is nilpotent of class p (which is larger than c), and it has an automorphism of order s which acts trivially on the top group and mth poweringly on the base group. Let $P = W/\mathfrak{N}_c(W)$; let g be the image in P of a generator of one of the coordinate subgroups of W, and k the image of a generator of the top group. It follows that P has an automorphism which sends g to g^m and fixes k. Let G be the semidirect product of P by $\langle h\rangle$, with h acting on P as the automorphism just described. The normal closure of g is abelian (of order p^c),

and the factor group over that is cyclic (of order ps), so $G \in \mathfrak{N}_c\mathfrak{A}_s \wedge \mathfrak{A}^2$. It is easy to see that a nontrivial element of the normal closure of g and a nontrivial element of $\langle h \rangle$ can never commute. Consider the left-normed commutator

$$[x_1^{s(1)}, \dots, x_c^{s(1)}, y^{t(1)}, \dots, y^{t(n)}]$$

where this time $n = \sum_{i \neq 1}(c(i) + 1)$ and for each $i \neq 1$ the sequence $t(1), \dots, t(n)$ has $c(i)+1$ terms equal to $s(i)$. This commutator is also a law in each of the $\mathfrak{N}_{c(i)}\mathfrak{A}_{s(i)}$, but setting

$$x_1 = g, \; x_2 = \cdots = x_c = k, \; y = h$$

shows that it is not a law in G. The desired contradiction has been reached and the proof is complete. □

References

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