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# Tensor Factorizations of Group Algebras and Modules

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Here a group algebra is always the group algebra of a finite group over a commutative field. We consider connections between three kinds of factorizations: writing the group as a direct product of subgroups; writing the group algebra as a tensor product of subalgebras; and writing the regular module (the group algebra viewed as a module over itself) as a tensor product of modules. In the principal result the field has prime characteristic, the group order is a power of this prime, and the group is abelian. If in these circumstances the regular module is isomorphic to a tensor product of two modules, then the group has a direct decomposition with one (direct factor) subgroup acting regularly on one of the (tensor factor) modules and the other subgroup acting regularly on the other module. Moreover, the module varieties of the tensor factors must be linear subspaces of the vector space which is the variety of the trivial module, and the two subspaces must form a direct decomposition of that space. 

## 1. INTRODUCTION

Taking the tensor product of two algebras over a field is an operation often used for creating new algebras from old. It is also well known that the factorization of an algebra as a tensor product is seldom unique. A most familiar example is the isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{M}$ where $\mathbb{R}$ and $\mathbb{C}$ are the systems of real and complex numbers, $\mathbb{H}$ is the ring of real

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quaternions, and $\mathbb{M}$ is the algebra of 2-by-2 real matrices. The example demonstrates that, in addition to there being no Krull–Schmidt type theorem for tensor factorizations, there is also no cancellation theorem. We can see further, in Example 1.1 below, that the situation is just as bad even if the algebras are assumed to be commutative.

In spite of the negative evidence, there seems to be some hope for uniqueness of tensor factorizations for some local algebras. In this paper we concentrate on group algebras. The main result of Section 2 shows that if $\mathbb{F}G$ is the group algebra of an abelian $p$-group $G$ over a field $\mathbb{F}$ of characteristic $p$, then tensor factorizations of $\mathbb{F}G$ are associated with direct factorizations of $G$. In particular the tensor factors are isomorphic to group algebras of direct factors of $G$.

Beginning with Section 3, we turn our attention to tensor factorizations of the regular $\mathbb{F}G$-module, $\mathbb{F}G$, as a tensor product of smaller $\mathbb{F}G$-modules. No uniqueness can possibly be expected here beyond a very few special cases. For example if $G = \langle x, y \rangle$ is elementary abelian of order $p^2$ and $\mathbb{F}$ has characteristic $p$, then as $\mathbb{F}G$-modules, $\mathbb{F}G \cong U \otimes_{\mathbb{F}} V_\alpha$ where $U = \mathbb{F}G/(x-1)$ and $V_\alpha = \mathbb{F}G/((y-1) + \alpha(x-1))$, regardless of the choice of $\alpha \in \mathbb{F}$. On the other hand, in Section 3 we show that whenever $\mathbb{F}G = U \otimes_{\mathbb{F}} V$ and $G$ is abelian, the modules $U$ and $V$ admit multiplications which make them into $\mathbb{F}$-algebras in such a way that the isomorphism is simultaneously an algebra and a module isomorphism.

More can be said in terms of the algebraic varieties associated to the modules (in the sense discussed in [1, 5]). If $\mathbb{F}G$ is the tensor product of two modules $U$ and $V$, then there are severe restrictions on the varieties of the modules $U$ and $V$. For example, a (rather more general) result in [4] yields that if $G$ is an elementary abelian $p$-group and $\mathbb{F}$ is an algebraically closed field of characteristic $p$ then the varieties of $U$ and $V$ must have degree 1. In Section 4, we extend this to arbitrary finite abelian $p$-groups. Specifically we show that if $\mathbb{F}G \cong U \otimes_{\mathbb{F}} V$ then the varieties of $U$ and $V$ must be linear subspaces of the variety of the trivial module. (The technique of intersection multiplicities used in [4] is not available when the group is not elementary abelian. Our methods here are totally elementary and do not seem strong enough to extend the more general results of [4].) In Section 5, we briefly discuss some of the questions for nonabelian groups.

We end this introduction with the promised example of tensor factorizations of commutative algebras.

1.1. EXAMPLE. Consider a Galois extension $\mathbb{E}|\mathbb{F}$ whose Galois group contains a direct product of two cyclic groups, one of order 4 and one of order 2: say, $G = H \times K$ with $|H| = 4$ and $|K| = 2$. Write $\overline{G}$ for the subfield of $\mathbb{E}$ consisting of the elements fixed by each element of $G$. Each

$\overline{G}$-subalgebra of $\mathbb{E}$ is a subfield containing $\overline{G}$, so the lattice of these subalgebras is dual to the subgroup lattice of $G$, with $\overline{G}$-dimensions matching indices in $G$. One can immediately see from this that

$$\mathbb{E} = \overline{H} \otimes_{\overline{G}} \overline{K},$$

an internal tensor product. Next, let $h$ be a generator of $H$, let $L$ denote the (unique) direct complement of $H$ in $G$ different from $K$, and let $e$ be an element of $\overline{KL}$ outside $\overline{G}$. Since $\{1, e\}$ is a $\overline{G}$-basis for $\overline{KL}$ and $h \notin KL$, we must have $eh \neq e$. Since $\overline{K}$ and $\overline{L}$ generate $\mathbb{E}$, there is a natural homomorphism $x \otimes y \mapsto xy$ from their external tensor product $\overline{K} \otimes_{\overline{G}} \overline{L}$ onto $\mathbb{E}$, with $(e \otimes 1) - (1 \otimes e)$ in its kernel. The automorphism $x \otimes y \mapsto xh \otimes y$ of $\overline{K} \otimes_{\overline{G}} \overline{L}$ followed by that natural homomorphism is then another homomorphism of $\overline{K} \otimes_{\overline{G}} \overline{L}$ onto $\mathbb{E}$, and this one has a different kernel (because it maps $(e \otimes 1) - (1 \otimes e)$ to $eh - e$, not to 0). The sum of the two kernels must be the whole tensor product (for the cokernels, being fields, have no proper nonzero ideals), so counting $\overline{G}$-dimensions shows that the intersection of the two kernels is 0. Thus

$$\overline{K} \otimes_{\overline{G}} \overline{L} \cong \mathbb{E} \oplus \mathbb{E} \cong \left(\overline{H} \otimes_{\overline{G}} \overline{K}\right) \oplus \left(\overline{H} \otimes_{\overline{G}} \overline{K}\right) \cong \left(\overline{H} \oplus \overline{H}\right) \otimes_{\overline{G}} \overline{K}.$$

*The tensor factor $\overline{K}$ cannot be cancelled here*, for the field $\overline{L}$ is certainly not isomorphic to the proper direct sum $\overline{H} \oplus \overline{H}$. The same example may also be written as

$$\mathbb{E} \oplus \mathbb{E} \cong \overline{K} \otimes_{\overline{G}} \overline{L} \cong \left(\overline{G} \oplus \overline{G}\right) \otimes_{\overline{G}} \overline{H} \otimes_{\overline{G}} \overline{K}.$$

None of the five tensor factors here has a proper tensor factorization: for each proper subalgebra of $\overline{K}$ or $\overline{L}$ is a subfield and is therefore contained in $\overline{HK}$, so neither $\overline{K}$ nor $\overline{L}$ can be generated by proper subalgebras, while each of the other three tensor factors has prime dimension. Thus *there is a commutative algebra which has two unrefinable tensor factorizations, one with two factors and the other with three.*

## 2. FACTORIZATIONS OF GROUP ALGEBRAS OF $p$-GROUPS

In this paper, all algebras will be finite-dimensional associative algebras over a (commutative) field $\mathbb{F}$. We assume that each algebra has a multiplicative identity element and that these identity elements are respected by all algebra homomorphisms. Subalgebras must have the same identity element as the whole algebra. When the ground field $\mathbb{F}$ is understood, the tensor product symbol $\otimes$ will mean $\otimes_{\mathbb{F}}$. By the external tensor product of two algebras $B$ and $C$ we mean the vector space $B \otimes C$ made into an algebra by defining $(b \otimes c)(b' \otimes c') = bb' \otimes cc'$. In this section we shall be concerned mostly with internal tensor products. That is, if $B$ and $C$ are subalgebras of an algebra $A$ and if the algebra homomorphism from the external tensor product $B \otimes C$ to $A$, which maps $b \otimes c$ to $bc$, is an isomorphism, then we say that $A$ is the internal tensor product of $B$ and $C$. In this case we write $A = B \otimes C$.

Several things about internal tensor products are worth noting. First, $A$ cannot be the internal tensor product of $B$ and $C$ unless each element of $B$ commutes with each element of $C$, the subalgebras $B$ and $C$ together generate the algebra $A$, and $\dim A = (\dim B)(\dim C)$. Conversely, if the subalgebras $B$ and $C$ of $A$ satisfy these three conditions, then it follows that $A$ is their internal tensor product. It may happen that $A = B \otimes C = B \otimes C'$ for two different subalgebras $C$ and $C'$, and we saw in the Introduction that in general this does not imply that $C$ and $C'$ are isomorphic. However, if $B$ is an augmented algebra, such as a group algebra, with an algebra homomorphism $\varepsilon\colon B \to \mathbb{F}$, then $A = B \otimes C = B \otimes C'$ does imply $C \cong C'$. (To prove this, consider the ideal, $J$ say, generated in $A$ by the kernel of $\varepsilon$: it is easy to see that $J = (\ker \varepsilon) \otimes C = (\ker \varepsilon) \otimes C'$ and hence $C = \mathbb{F} \otimes C \cong A/J \cong \mathbb{F} \otimes C' = C'$.)

The following is the main theorem of this section.

2.1. THEOREM. *Let $p$ be a prime, $\mathbb{F}$ a field of characteristic $p$, and $G$ a finite abelian $p$-group. If $B$ and $C$ are subalgebras of $\mathbb{F}G$ such that $\mathbb{F}G = B \otimes C$, then there exist subgroups $H$ and $K$ in $G$ such that $G = H \times K$, $B \cong \mathbb{F}H$, $C \cong \mathbb{F}K$, and, as internal tensor products,*

$$\mathbb{F}G = B \otimes \mathbb{F}K = \mathbb{F}H \otimes C. \tag{1}$$

This theorem is a special case of a more general result.

2.2. THEOREM. *Let $p$, $\mathbb{F}$, and $G$ be as in Theorem 2.1. To each tensor factorization $\mathbb{F}G = \bigotimes_{i=1}^{n} A_i$ of $\mathbb{F}G$, there is a direct decomposition $G = \prod_{i=1}^{n} G_i$ such that $A_i \cong \mathbb{F}G_i$ and*

$$\mathbb{F}G = A_i \otimes \left(\bigotimes_{j \neq i} \mathbb{F}G_j\right) \tag{$1^+$}$$

*for all $i$.*

For the proof, we have to prepare some elementary linear algebra. Let $\mathbb{F}$ be any field, and $\mathbb{E}$ any subfield of $\mathbb{F}$. For each $\mathbb{E}$-space $U$, write the $\mathbb{F}$-space $U \otimes_{\mathbb{E}} \mathbb{F}$ simply as $U^{\mathbb{F}}$.

2.3. LEMMA. *Let $U$ be an $\mathbb{E}$-space.*

(a) *If $R$ and $S$ are $\mathbb{F}$-subspaces in $U^{\mathbb{F}}$ with* $\dim R = \dim S$, *then there is an $\mathbb{E}$-subspace $T$ in $U$ such that $T^{\mathbb{F}}$ is a common complement to $R$ and $S$ in $U^{\mathbb{F}}$.*

(b) *If $U^{\mathbb{F}} = X \oplus Y$, then $U$ has a direct decomposition $U = V \oplus W$ such that $U^{\mathbb{F}} = X \oplus W^{\mathbb{F}} = V^{\mathbb{F}} \oplus Y$.*

*Proof.* (a) Consider $U$ an $\mathbb{E}$-subspace in $U^{\mathbb{F}}$. If the common codimension of $R$ and $S$ in $U^{\mathbb{F}}$ is 0, then $T = 0$ will do. This provides the initial step for a proof by induction on that common codimension. For the inductive step, suppose the common codimension is positive. Then the intersections $R \cap U$ and $S \cap U$ are proper subspaces in $U$, and no vector space can be the set-union of just two proper subspaces. Thus there is an element, $u$ say, which is in $U$ but neither in $R$ nor in $S$. The inductive hypothesis applies with $\mathbb{F}u \oplus R$ and $\mathbb{F}u \oplus S$ in place of $R$ and $S$, and so there is a subspace, $T_0$ say, in $U$ such that $T_0^{\mathbb{F}}$ is a common complement to $\mathbb{F}u \oplus R$ and $\mathbb{F}u \oplus S$. It is easy to see that $T = \mathbb{E}u \oplus T_0$ will do.

(b) Apply (a) twice. First, with $R = S = Y$. The $T$ so obtained will be our $V$. Second, with $R = V^{\mathbb{F}}$ and $S = X$, and the $T$ now obtained will serve as $W$. ∎

In fact, we shall need a "filtered" version of this result.

2.4. LEMMA. *Let $0 = U_0 \leq \cdots \leq U_j \leq \cdots \leq U_n = U$ be a chain of $\mathbb{E}$-spaces.*

(a) *If $R$ and $S$ are $\mathbb{F}$-subspaces in $U^{\mathbb{F}}$ with* $\dim(R \cap U_j^{\mathbb{F}}) = \dim(S \cap U_j^{\mathbb{F}})$ *whenever $0 \leq j \leq n$, then there is an $\mathbb{E}$-subspace $T$ in $U$ such that each $(T \cap U_j)^{\mathbb{F}}$ is a common complement to $R \cap U_j^{\mathbb{F}}$ and $S \cap U_j^{\mathbb{F}}$ in $U_j^{\mathbb{F}}$.*

(b) *If $U^{\mathbb{F}} = X \oplus Y$ such that each $U_j^{\mathbb{F}}$ is the sum of its intersections with $X$ and $Y$, then $U$ has a direct decomposition $U = V \oplus W$ such that each $U_j$ is the sum of its intersections with $V$ and $W$ and $U^{\mathbb{F}} = X \oplus W^{\mathbb{F}} = V^{\mathbb{F}} \oplus Y$; indeed, each $U_j^{\mathbb{F}}$ is the sum of its intersections with $X$ and $W^{\mathbb{F}}$, and also the sum of its intersections with $V^{\mathbb{F}}$ and $Y$.*

*Proof.* In adapting the proof of Lemma 2.3, the only substantive change one has to make is in the inductive step of the proof of (a). Once the common codimension of $R$ and $S$ in $U^{\mathbb{F}}$ is positive, there is a unique $m$ such that $U_m \not\leq R \cup S$ but $U_{m-1} \leq R \cap S$. Choose $u$ as any element of this $U_m$ which is neither in $R$ nor in $S$. The rest of the adaptation is routine and left to the reader. ∎

We turn to groups next. Recall some notation. For a multiplicative finite abelian $p$-group $G$ and a nonnegative integer $j$, let

$$\Omega_j G = \{g \mid g \in G,\ g^{p^j} = 1\} \qquad \text{and} \qquad \mho G = \{g^p \mid g \in G\}.$$

It will be convenient here to view $G/\mho G$ as a vector space $\overline{G}$ over the field $\mathbb{F}_p$ of $p$ elements, and to adopt a "bar convention." For each element $g$ and subgroup $H$ in $G$, let $\bar{g}$ and $\overline{H}$ denote the image under the natural homomorphism $G \to \overline{G}$. Note that $\overline{H}$ is not $H/\Omega H$ unless $H = G$. Replace \Omega by \mho

Suppose that $G = H \times K$; then $\mho G = \mho H \times \mho K$ and so $\overline{G} = \overline{H} \oplus \overline{K}$. Similarly, $\Omega_j G = \Omega_j H \times \Omega_j K$, and indeed $\overline{\Omega_j G} = \overline{\Omega_j H} \oplus \overline{\Omega_j K}$. It follows that $\dim \overline{\Omega_j K}$ is the number of direct factors of order at most $p^j$ in any decomposition of $K$ as direct product of cyclic groups, and the order of $K$ can be recovered from the dimensions:

$$|K| = \prod_{j \geq 1} p^{j(\dim \overline{\Omega_j K} - \dim \overline{\Omega_{j-1} K})}. \tag{2}$$

With reference to the direct decomposition $G = 1 \times G$, this specializes to

$$|G| = \prod_{j \geq 1} p^{j(\dim \overline{\Omega_j G} - \dim \overline{\Omega_{j-1} G})}. \tag{2'}$$

Consider any basis for $\overline{G}$ with the property that each $\overline{\Omega_j G}$ is spanned by the basis elements it contains. For each basis element choose a preimage in $G$, ensuring that if a basis element lies in $\overline{\Omega_j G}$ then its preimage lies in $\Omega_j G$. These preimages together generate $G$, because $\mho G$ is the Frattini subgroup, and so $G$ is a homomorphic image of the external direct product of the cyclic subgroups generated by the individual preimages. On the other hand, the order of that direct product is at most the order of $G$, so this homomorphism must be an isomorphism and $G$ must be the internal direct product of those cyclic subgroups. In short, one may put this conclusion as follows: each basis of $\overline{G}$ matching the filtration of $\overline{G}$ provided by the $\overline{\Omega_j G}$ comes from some direct decomposition of $G$ with cyclic direct factors. We shall need only a weak consequence of this.

2.5. LEMMA. *Suppose that $\overline{G} = V \oplus W$ and each $\overline{\Omega_j G}$ is the direct sum of its intersections with $V$ and $W$. Then $G$ has a direct decomposition $G = H \times K$ such that $V = \overline{H}$ and $W = \overline{K}$.*

*Proof.* Choose bases for $V$ and $W$ such that each $\overline{\Omega_j G} \cap V$ and each $\overline{\Omega_j G} \cap W$ is spanned by the basis elements it contains. Corresponding to the union of these bases, obtain a direct decomposition of $G$ with cyclic direct factors as above. Then $H$ and $K$ may be chosen as products of direct factors of that decomposition. ∎

Now let us imitate the construction in the context of commutative algebras over fields $\mathbb{F}$ of characteristic $p$. For such an algebra $A$, set

$$\Omega_j A = \{a | a \in \text{rad } A,\, a^{p^j} = 0\} \qquad \text{and} \qquad \mho A = (\text{rad } A)^2.$$

No misunderstanding should arise from extending our "bar convention": for each subset $X$ of $A$, write $\overline{X}$ for the image of $X$ under the natural map $A \twoheadrightarrow A/\mho A$. In particular,

$$\overline{\text{rad } A} = (\text{rad } A)/\mho A \qquad \text{and} \qquad \overline{\Omega_k A} = (\Omega_k A + \mho A)/\mho A,$$

but, for a subalgebra $B$, $\overline{\text{rad } B}$ is usually not $(\text{rad } B)/\mho B$.

An easy induction on dim $A$ (applying the inductive hypothesis to the quotient of $A$ modulo the last nonzero power of rad $A$) readily shows that if $\overline{X}$ spans $\overline{A}$ then the subalgebra generated by $X$ is $A$ itself.

Suppose that codim rad $A = 1$: identifying $\mathbb{F}$ with the unique 1-dimensional subalgebra of $A$, we may express this by writing $A = \mathbb{F} \oplus \text{rad } A$, a vector space direct sum. Suppose also that $A = B \otimes C$. Since the radical of a commutative algebra is the set of its nilpotent elements, then $(\text{rad } A) \cap B = \text{rad } B$, $B = \mathbb{F} \oplus \text{rad } B$, and similarly, $C = \mathbb{F} \oplus \text{rad } C$. It follows that

$$\text{rad } A = (\text{rad } B) \oplus (\text{rad } B)(\text{rad } C) \oplus (\text{rad } C),$$

and therefore

$$\mho A = \mho B \oplus (\text{rad } B)(\text{rad } C) \oplus \mho C.$$

Of course also $(\Omega_j A) \cap B = \Omega_j B$, and (as $a \mapsto a^p$ is a ring endomorphism)

$$\Omega_j A \subseteq \Omega_j B \oplus (\text{rad } B)(\text{rad } C) \oplus \Omega_j C,$$

whence

$$\Omega_j A + \mho A = (\Omega_j B + \mho B) \oplus (\text{rad } B)(\text{rad } C) \oplus (\Omega_j C + \mho C).$$

Routine steps now lead to the first two sentences of the following.

2.6. LEMMA. *If* codim rad $A = 1$ *and* $A = B \otimes C$, *then* $\overline{\text{rad } A} = \overline{\text{rad } B} \oplus \overline{\text{rad } C}$ *and* $\overline{\Omega_j A} = \overline{\Omega_j B} \oplus \overline{\Omega_j C}$. *Each* $\overline{\Omega_j A}$ *is the sum of its intersections with* $\overline{\text{rad } B}$ *and* $\overline{\text{rad } C}$. *Moreover,*

$$\dim B \leq \prod_{j \geq 1} p^{j(\dim \overline{\Omega_j B} - \dim \overline{\Omega_{j-1} B})}. \tag{3}$$

*Proof.* It is easy to see that $\mho A \cap B = \mho B$ and so $(\operatorname{rad} B)/\mho B \cong \overline{\operatorname{rad} B}$. Thus if $X$ is a subset of rad $B$ such that $\bar{X}$ spans $\overline{\operatorname{rad} B}$ then the image of $X$ under $B \to B/\mho B$ spans $(\operatorname{rad} B)/\mho B$; since $B = \mathbb{F} \oplus \operatorname{rad} B$ and $\mho B$ is omissible in $B$, the subalgebra generated by $X$ must then be $B$ itself. Choose a basis for $\overline{\operatorname{rad} B}$ such that each $\overline{\Omega_j B}$ is generated by the basis elements it contains. For each basis element choose a preimage in rad $B$, ensuring that if a basis element lies in $\overline{\Omega_j B}$ then its preimage lies in $\Omega_j B$. Because of the above argument these preimages together generate $B$ as an algebra. Hence $B$ is a homomorphic image of the external tensor product of the subalgebras generated by the individual preimages. Since the dimension of the subalgebra generated by any single element of $\Omega_j B$ is obviously at most $p^j$, this proves the last sentence. ∎

The hypothesis codim rad $A = 1$ is certainly satisfied when $A$ is the group algebra $\mathbb{F}G$ of a finite abelian $p$-group $G$. As is well known, $\bar{G} \to \overline{\operatorname{rad} A}$, $\bar{g} \mapsto \overline{g - 1}$ is an isomorphism when $\mathbb{F} = \mathbb{F}_p$, and in any case it leads to an $\mathbb{F}$-isomorphism $\alpha\colon \bar{G} \otimes_{\mathbb{F}_p} \mathbb{F} \to \overline{\operatorname{rad} A}$. We shall need that this leads from the filtration of $\bar{G}$ provided by the $\overline{\Omega_j G}$ precisely to the filtration of $\overline{\operatorname{rad} A}$ provided by the $\overline{\Omega_j A}$. To ease typography, for any $\mathbb{F}_p$-space $U$ let us write $U \otimes_{\mathbb{F}_p} \mathbb{F}$ simply as $U^{\mathbb{F}}$.

2.7. LEMMA. *The isomorphism $\alpha$ maps $\overline{\Omega_j G}^{\mathbb{F}}$ onto $\overline{\Omega_j A}$. Moreover, if $G = H \times K$ then $\overline{\Omega_j H}^{\mathbb{F}} \alpha = \overline{\Omega_j(\mathbb{F}H)}$ and $\overline{\Omega_j K}^{\mathbb{F}} \alpha = \overline{\Omega_j(\mathbb{F}K)}$. In particular, $\bar{H}^{\mathbb{F}}\alpha = \overline{\operatorname{rad} \mathbb{F}H}$ and $\bar{K}^{\mathbb{F}}\alpha = \overline{\operatorname{rad} \mathbb{F}K}$.*

*Proof.* Since $g^{p^j} = 1$ implies $(g-1)^{p^j} = 0$, it is clear that $\alpha$ maps $\overline{\Omega_j G}^{\mathbb{F}}$ into $\overline{\Omega_j A}$. Hence

$$\dim \overline{\Omega_j G} \le \dim \overline{\Omega_j A} \qquad \text{whenever } j \ge 0.$$

Beware that the first dimension here is taken over $\mathbb{F}_p$, and the second over $\mathbb{F}$. Comparing (2′) and (3), we get that

$$\sum_{k \ge 1} j\left(\dim \overline{\Omega_j G} - \dim \overline{\Omega_{j-1} G}\right) \le \sum_{k \ge 1} j\left(\dim \overline{\Omega_j A} - \dim \overline{\Omega_{j-1} A}\right).$$

Of course $\overline{\Omega_0 G} = \overline{\Omega_0 A} = 0$ while $\dim \overline{\Omega_j G} = \dim \bar{G} = \dim \overline{\operatorname{rad} A} = \dim \overline{\Omega_j A}$ whenever $j$ is large enough, and so it follows that $\dim \overline{\Omega_j G} = \dim \overline{\Omega_j A}$ whenever $j \ge 0$. This proves the first claim.

Suppose that $G = H \times K$; then $\overline{\Omega_j G} = \overline{\Omega_j H} \oplus \overline{\Omega_j K}$. Apply Lemma 2.5 with $B = \mathbb{F}H$ and $C = \mathbb{F}K$ to conclude that $\overline{\Omega_j A} = \overline{\Omega_j(\mathbb{F}H)} \oplus \overline{\Omega_j(\mathbb{F}K)}$. If $h \in \Omega_j H$ then $h - 1 \in \Omega_j(\mathbb{F}H)$, so $\overline{\Omega_j H}^{\mathbb{F}} \alpha \subseteq \overline{\Omega_j(\mathbb{F}H)}$, and similarly $\overline{\Omega_j K}^{\mathbb{F}} \alpha \subseteq \overline{\Omega_j(\mathbb{F}K)}$. In view of the first claim, both inclusions must be equalities. When $j$ is large enough, $H = \Omega_j H$ and $\operatorname{rad} \mathbb{F}H = \Omega_j(\mathbb{F}H)$, and the final statements also follow. ∎

*Proof of Theorem* 2.1. For an application of Lemma 2.4, set $\mathbb{E} = \mathbb{F}_p$ and $U = \overline{G}$ with $U_j = \overline{\Omega_j G}$. Write $\mathbb{F}G = A$ and identify $U^{\mathbb{F}}$ with $\overline{\operatorname{rad} A}$ along the isomorphism $\alpha$ of Lemma 2.6. Then $U_j^{\mathbb{F}}$ becomes $\overline{\Omega_j A}$. By Lemma 2.5, the hypotheses of Lemma 2.4 will be satisfied by $X = \overline{\operatorname{rad} B}$, $Y = \overline{\operatorname{rad} C}$, and we get $\overline{G} = V \approx W$ with

$$\overline{\operatorname{rad} A} = \overline{\operatorname{rad} B} \oplus W^{\mathbb{F}} = V^{\mathbb{F}} \oplus \overline{\operatorname{rad} C}.$$

Lemma 2.4 also yields that $\overline{G} = V \oplus W$ satisfies the hypotheses of Lemma 2.5 and we get $G = H \times K$ with $V = \overline{H}$ and $W = \overline{K}$.

We complete the proof of the theorem by showing that the subgroups $H, K$ so obtained satisfy (1). Because of the symmetry of the situation, it will in fact suffice to show that $A = B \otimes \mathbb{F}K$.

In view of the second half of Lemma 2.7, the last displayed equation is the same as

$$\overline{\operatorname{rad} A} = \overline{\operatorname{rad} B} \oplus \overline{\operatorname{rad} \mathbb{F}K} = \overline{\operatorname{rad} \mathbb{F}H} \oplus \overline{\operatorname{rad} C}.$$

As $A = \mathbb{F} \oplus \operatorname{rad} A$ and $\mho A$ is omissible, it follows that $B$ and $\mathbb{F}K$ together generate $A$, so we shall be done if we show that $\dim A \geq (\dim B)(\dim \mathbb{F}K)$, or equivalently that

$$|G| \geq (\dim B)|K|.$$

It was precisely to this end that Lemma 2.4 stated even more than what we have used so far. It also yields that each $\overline{\Omega_j A}$ is the (direct) sum of its intersections with $\overline{\operatorname{rad} B}$ and $\overline{\operatorname{rad} \mathbb{F}K}$. From Lemma 2.6 one readily sees that the first of these intersections is $\overline{\Omega_j B}$. Similarly, from Lemma 2.6, applied with $A = \mathbb{F}H \otimes \mathbb{F}K$ in place of $A = B \otimes C$, the second intersection is seen to be $\overline{\Omega_j(\mathbb{F}K)}$. By Lemma 2.7, this has the same dimension as $\overline{\Omega_j K}$. It follows that

$$\dim \overline{\Omega_j A} = \dim \overline{\Omega_j B} + \dim \overline{\Omega_j K}.$$

This, (2), (2′), and (3) together imply that $|G| \geq (\dim B)|K|$, as required. ∎

The argument has also proved the following.

2.8. COROLLARY. *If the algebra $A$ in Lemma 2.5 is a group algebra, then the inequality* (3) *is an equality.*

We need only note that if $A = \mathbb{F}G$ then $G$ must be abelian, and if also $\operatorname{codim} \operatorname{rad} A = 1$ then $|G|$ must be a power of $p$.

*Proof of Theorem* 2.2. Let $\mathbb{F}G = \bigotimes_{i=1}^{n} A_i$. We shall proceed by induction on $n$. We have to prove that there is a direct factorization $G = \prod_{i=1}^{n} G_i$ such that $(1^+)$ holds whenever $1 \leq i \leq n$. The case $n = 1$ is a tautology, so let $n > 1$. Apply Theorem 2.1 with $B = \bigotimes_{i=1}^{n-1} A_i$ and $C = A_n$, to obtain a direct decomposition $G = H \times K$ with $\mathbb{F}G = B \otimes \mathbb{F}K = \mathbb{F}H \otimes C$. Set $G_n = K$. If $n = 2$, we set $G_1 = H$, and we are done. Suppose that $n > 2$, and let $J$ denote the ideal of $\mathbb{F}G$ generated by $\operatorname{rad} \mathbb{F}K$. So $\mathbb{F}G$ has two "semidirect" decompositions (that is, vector space direct decompositions with one summand a subalgebra and the other an ideal): $\mathbb{F}G = B \oplus J = \mathbb{F}H \oplus J$. It follows that $A_i \oplus J = [\mathbb{F}H \cap (A_i + J)] \oplus J$, so $[\mathbb{F}H \cap (A_i + J)] \cong A_i$ and $\mathbb{F}H = \bigotimes_{i=1}^{n-1} [\mathbb{F}H \cap (A_i + J)]$. Apply the inductive hypothesis to obtain a matching direct decomposition $H = \prod_{i=1}^{n-1} G_i$. With this definition of the $G_i$, the $i = n$ case of $(1^+)$ is simply $\mathbb{F}G = \mathbb{F}H \otimes C$, so we need only consider $i < n$. Fix such an $i$ and set

$$D = \bigotimes_{\substack{j \neq i \\ j < n}} \mathbb{F}G_j:$$

then we have that

$$\mathbb{F}G = [\mathbb{F}H \cap (A_i + J)] \otimes D \otimes \mathbb{F}G_n. \tag{4}$$

Our remaining task is to show that (4) remains valid when the first tensor factor is replaced by $A_i$. Since $\mathbb{F}H \cap (A_i + J)$ and $A_i$ have the same dimension, this will follow if we can show that $A_i$ and $D \otimes \mathbb{F}G_n$ together generate $\mathbb{F}G$. By (4) and Lemma 2.5 we know that

$$\overline{\operatorname{rad} \mathbb{F}G} = \overline{\operatorname{rad}[\mathbb{F}H \cap (A_i + J)]} \oplus \overline{\operatorname{rad} D} \oplus \overline{\operatorname{rad} \mathbb{F}G_n}, \tag{5}$$

and what we need will follow if $\overline{\operatorname{rad} A_i}$ and $\overline{\operatorname{rad} D} \oplus \overline{\operatorname{rad} \mathbb{F}G_n}$ together generate $\overline{\operatorname{rad} \mathbb{F}G}$.

For a proof of the latter claim, let us return to $A_i \oplus J = [\mathbb{F}H \cap (A_i + J)] \oplus J$. Since $J$ is a nilpotent ideal, this yields that

$$\begin{aligned}(\operatorname{rad} A_i) \oplus J = \operatorname{rad}[A_j \oplus J] &= \operatorname{rad}([\mathbb{F}H \cap (A_i + J)] \oplus J) \\ &= (\operatorname{rad}[\mathbb{F}H \cap (A_i + J)]) \oplus J,\end{aligned}$$

whence

$$\overline{\operatorname{rad} A_i} + \bar{J} = \overline{\operatorname{rad}[\mathbb{F}H \cap (A_i + J)]} + \bar{J}.$$

As $\mho(\mathbb{F}G) + \operatorname{rad} \mathbb{F}G_n$ is an ideal, by the definition of $J$ this ideal is also $\mho(\mathbb{F}G) + J$. So $\bar{J} = \overline{\operatorname{rad} \mathbb{F}G_n}$. Thus (5) and the last displayed equation together prove that $\overline{\operatorname{rad} A_i}$ and $\overline{\operatorname{rad} D} \oplus \overline{\operatorname{rad} \mathbb{F}G_n}$ generate $\overline{\operatorname{rad} \mathbb{F}G}$, as required. ∎

## 3. TENSOR FACTORIZATIONS OF THE REGULAR MODULE

In this section, the main result concerns tensor factorizations of the regular module for a group algebra $\mathbb{F}G$, with $\mathbb{F}$ any field (of any characteristic) and $G$ any finite abelian group. The discussion is kept as general as we can manage without getting involved in unnecessary work.

In the context of modules there are no internal tensor products, at least not in any sense that would be analogous to internal tensor products of algebras. Also the tensor factors need not appear as submodules (or quotient modules) of the tensor product. Thus, in the first instance, a tensor factorization of a module can only mean an isomorphism to an externally constructed tensor product. On the other hand, one does speak of "outer" tensor products of modules: if $X$ and $Y$ are modules for the algebras $B$ and $C$, respectively, then the algebra $B \otimes C$ acts componentwise on the vector space tensor product $X \otimes Y$. By way of distinction, sometimes outer tensor products are written as $X \# Y$. It will be convenient to adopt that convention here, and to use similar notation for outer tensor products of representations as well. In particular, if $A$ is an algebra and $\rho$ and $\sigma$ are representations of $A$, the representation $\rho \# \sigma$ of $A \otimes A$ is always defined. To define a representation $\rho \otimes \sigma$ of $A$, one needs a "diagonal" map $A \to A \otimes A$ as well (provided by $g \mapsto g \otimes g$ when $A = \mathbb{F}G$), and then $\rho \otimes \sigma$ is the composite of that with $\rho \# \sigma$. A less familiar observation is that if $A = B \otimes C$ is a tensor factorization of any algebra, then by restriction $\rho$ and $\sigma$ give rise to representations $\rho_B$ and $\sigma_C$, and $\rho_B \# \sigma_C$ is a representation of $A$. If it happens (as it does when $A$ is a group algebra) that $\rho \otimes \sigma$ is also defined, then $\rho_B \# \sigma_C$ and $\rho \otimes \sigma$ provide actions of $A$ on the same tensor product space, but in general the two actions on the one space are very different. Indeed, one cannot even expect them to be equivalent representations. In giving examples of two such representations being not just equivalent but actually equal, the main result of this section draws attention to something rather rare.

3.1. THEOREM. *Let $\mathbb{F}$ be a field, $G$ a finite abelian group, and $U, V$ modules for $\mathbb{F}G$, with $\rho: \mathbb{F}G \to \mathrm{End}_{\mathbb{F}} U$, $\sigma: \mathbb{F}G \to \mathrm{End}_{\mathbb{F}} V$ the representations afforded by them. If $\rho \otimes \sigma$ is equivalent to the regular representation, then there is a tensor factorization $\mathbb{F}G = B \otimes C$ and representations $\beta: B \to \mathrm{End}_{\mathbb{F}} U$, $\gamma: C \to \mathrm{End}_{\mathbb{F}} V$ equivalent to the regular representation of $B$ and $C$, respectively, such that $\mathbb{F}G\rho = B\beta$, $\mathbb{F}G\sigma = C\gamma$, and $\rho \otimes \sigma = \beta \# \gamma$. If moreover each of $U$ and $V$ has a nonzero quotient on which $G$ acts trivially, then $\rho = \beta \# \varepsilon_C$ and $\sigma = \varepsilon_B \# \gamma$ where $\varepsilon_B: B \to \mathbb{F}$ and $\varepsilon_C: C \to \mathbb{F}$ denote the restrictions to $B$ and $C$ of the augmentation map $\varepsilon: \mathbb{F}G \to \mathbb{F}$, which takes every element $g$ of $G$ to 1.*

Restricting the last two equations to $B$ and $C$ then yields the following.

3.2. COROLLARY. *If $\rho$ and $\sigma$ are representations of the group algebra $\mathbb{F}G$ of a finite abelian group such that the modules affording these representations have nonzero $G$-trivial quotients, and if $\rho \otimes \sigma$ is equivalent to the regular representation, then $\mathbb{F}G$ has a tensor factorization $\mathbb{F}G = B \otimes C$ such that $\rho \otimes \sigma = \rho_B \# \sigma_C$, the restrictions $\rho_B$ and $\sigma_C$ are equivalent to the regular representations of $B$ and $C$, respectively, and $B\rho = \mathbb{F}G\rho$, $C\sigma = \mathbb{F}G\sigma$, while $B\sigma, C\rho$ consist of scalars.*

For the proof of the theorem, we need to consider *monogenic* modules: that is, modules which can be generated by a single element. A module is monogenic if and only if it is a homomorphic image of the regular module. A module is isomorphic to the regular module if and only if it is monogenic and its dimension equals that of the algebra in question. It follows from Nakayama's Lemma that a module is monogenic if and only if its largest semisimple quotient is monogenic.

3.3. LEMMA. *If $U$ and $V$ are modules for a commutative algebra $A$ and $U \otimes V$ is monogenic, then both $U$ and $V$ are monogenic.*

*Proof.* The largest semisimple quotient of any commutative algebra $A$ is a direct sum of fields, so the largest semisimple quotient of the regular module for $A$ is multiplicity-free (that is, a direct sum of pairwise nonisomorphic simple modules). It follows that an $A$-module $U$ is monogenic if and only if the largest semisimple quotient of $U$ is multiplicity-free; equivalently, if and only if no nonzero quotient of $U$ can be written as a direct sum of two isomorphic modules. Thus if $U$ is not monogenic, then it has a nonzero quotient, $U/U_0$ say, which is a direct sum $U/U_0 \cong U_1 \oplus U_2$ with $U_1 \cong U_2$. If now $V$ is any other nonzero $A$-module, then

$$(U \otimes V)/(U_0 \otimes V) \cong (U/U_0) \otimes V \cong (U_1 \otimes V) \oplus (U_2 \otimes V)$$
$$\text{with } U_1 \otimes V \cong U_2 \otimes V$$

shows that $U \otimes V$ cannot be monogenic. ∎

Note that if $U$ is a 2-dimensional $G$-trivial $\mathbb{F}G$-module and $V$ is an absolutely simple $FG$-module with $\dim V > 1$, then $U \otimes V$ is monogenic but $U$ is not: so the commutativity hypothesis cannot be omitted from Lemma 3.3.

*Proof of Theorem* 3.1. Since $G$ is abelian, $\mathbb{F}G\rho \subseteq \mathrm{End}_{\mathbb{F}G} U$ and $\mathbb{F}G\sigma \subseteq \mathrm{End}_{\mathbb{F}G} V$, and therefore

$$\begin{aligned}\mathbb{F}G(\rho \otimes \sigma) &\subseteq \mathbb{F}G\rho \otimes \mathbb{F}G\sigma \\ &\subseteq (\mathrm{End}_{\mathbb{F}G} U) \otimes (\mathrm{End}_{\mathbb{F}G} V) \\ &\subseteq \mathrm{End}_{\mathbb{F}G\otimes\mathbb{F}G}\left(U \# V\right) \\ &\subseteq \mathrm{End}_{\mathbb{F}G}(U \otimes V).\end{aligned}$$

Suppose now that $\rho \otimes \sigma$ is equivalent to the regular representation. Then $\ker(\rho \otimes \sigma) = 0$ and $\mathrm{End}_{\mathbb{F}G}(U \otimes V) \cong \mathbb{F}G$, so $\dim \mathrm{End}_{\mathbb{F}G}(U \otimes V) = \dim(\mathbb{F}G(\rho \otimes \sigma))$. Thus all the inclusions above must in fact be equalities. In particular, it follows that $\mathbb{F}G(\rho \otimes \sigma) = \mathbb{F}G\rho \otimes \mathbb{F}G\sigma$. With $B$ and $C$ chosen as the inverse images of $\mathbb{F}G\rho$ and $\mathbb{F}G\sigma$ under $\rho \otimes \sigma$, we obviously have $\mathbb{F}G = B \otimes C$, and there exist representations $\beta$: $B \to \mathrm{End}_{\mathbb{F}} U$ and $\gamma$: $C \to \mathrm{End}_{\mathbb{F}} V$ such that $\mathbb{F}G\rho = B\beta$, $\mathbb{F}G\sigma = C\gamma$, and $\rho \otimes \sigma = \beta \# \gamma$.

By Lemma 3.3, $U$ is a monogenic $\mathbb{F}G$-module, and so it is also a monogenic $\mathbb{F}G\rho$-module. It follows from $\mathbb{F}G\rho = B\beta$ that the $B$-module defined on $U$ by $\beta$ is also monogenic, whence $\dim U \leq \dim B$. Similarly, $\dim V \leq \dim C$. With

$$(\dim U)(\dim V) = |G| = (\dim B)(\dim C),$$

these inequalities imply that $\dim U = \dim B$ and $\dim V = \dim C$. Therefore the monogenic $B$-module defined on $U$ by $\beta$ must be isomorphic to the regular $B$-module; similarly, the $C$-module defined on $V$ by $\gamma$ is also regular.

Now suppose also that $U$ has a proper submodule, $U_0$ say, such that each element of $G$ acts trivially on $U/U_0$. We may as well choose $U_0$ to have codimension 1. As $\mathbb{F}G$-module, $(U/U_0) \otimes V$ may then be identified with $V$, and so each element of $\mathbb{F}G$ acts on $(U/U_0) \otimes V$ via $\sigma$. On the other hand, $U_0$ is also a submodule in the $B$-module defined on $U$ by $\beta$ (because $B\beta = \mathbb{F}G\rho$) and, with $\beta_1$ denoting the corresponding representation of $B$ on $U/U_0$, the action of $\mathbb{F}G$ on $(U/U_0) \otimes V$ may also be described as $\beta_1 \# \gamma$. Thus $\sigma = \beta_1 \# \gamma$, whence it is immediate that $B\sigma$ consists of scalars and $\sigma_C = \gamma$.

Finally, assume that $V$, too, has a nonzero $G$-trivial quotient, and argue similarly that then $\rho_B = \beta$. Since $G\rho$ acts trivially on $U/U_0$, the action of $B$ on $U/U_0$ obtained from $\rho_B$ is that given by $\varepsilon_B$, while $\beta_1$ was defined as the action of $B$ on $U/U_0$ obtained from $\beta$. Thus $\beta_1 = \varepsilon_B$ and hence $\sigma = \varepsilon_B \# \gamma$. Similarly, $\rho = \beta \# \varepsilon_C$, and the proof of the theorem is complete. ∎

A different way of expressing much the same result is the following.

3.4. THEOREM. *Let $\mathbb{F}G$ be the group algebra of a finite abelian group and let $U, V$ be $\mathbb{F}G$-modules such that each has a nonzero $G$-trivial quotient and $U \otimes V$ is isomorphic to the regular module. Then there are algebra structures on $U$ and $V$ and a map from $\mathbb{F}G$ to $U \otimes V$ which is both an algebra isomorphism and an $\mathbb{F}G$-module isomorphism.*

Towards the proof of this we shall need that, under the present hypotheses, if $u$ and $v$ generate $U$ and $V$ (as $\mathbb{F}G$-modules), then $u \otimes v$ generates $U \otimes V$ (as $\mathbb{F}G$-module). This would be obvious, without any hypotheses, if we were dealing with an outer tensor product. We could claim it here because we know from Corollary 3.2 that $U \otimes V$ may be viewed as an outer tensor product $U_B \# V_C$, and that all the action of $\mathbb{F}G$ on $U$ and $V$ comes from the action of the relevant subalgebras $B$ and $C$, respectively, so $u$ and $v$ generate $U_B$ and $V_C$ as well. The following general lemma gives it more directly.

3.5. LEMMA. *If $U$ and $V$ are modules over the group algebra $\mathbb{F}G$ of a finite abelian group, if $U \otimes V$ is monogenic, and if $u, v$ generate $U, V$, respectively, then $u \otimes v$ generates $U \otimes V$.*

*Proof.* *Step* 1. The claim holds if $\mathbb{F}$ is a splitting field for $G$ and $U, V$ are semisimple. For then $U = \bigoplus U_i$, $V = \bigoplus V_j$, and $U \otimes V = \bigoplus \bigoplus (U_i \otimes V_j)$, with the $U_i \otimes V_j$ all of dimension 1 and pairwise nonisomorphic because $U \otimes V$ is monogenic. Since $u$ generates $U$, we have $u = \Sigma u_i$ with $0 \neq u_i \in U_i$ for all $i$; similarly, $v = \Sigma v_j$; and consequently $u \otimes v = \Sigma\Sigma(u_i \otimes v_j)$ with all $u_i \otimes v_j$ nonzero. So $u \otimes v$ generates $U \otimes V$ as required.

*Step* 2. The claim holds over any $\mathbb{F}$ as long as $U$ and $V$ are semisimple. To see this, let $\mathbb{E}$ be the algebraic closure of $\mathbb{F}$ and write $U^{\mathbb{E}} = U \otimes_{\mathbb{F}} \mathbb{E}$, as usual, viewing $U$ a subset of $U^{\mathbb{E}}$. The $\mathbb{E}G$-modules $U^{\mathbb{E}}, V^{\mathbb{E}}$ are semisimple and generated by $u, v$, respectively, and their tensor product $U^{\mathbb{E}} \otimes_{\mathbb{E}} V^{\mathbb{E}}$ is monogenic because it is isomorphic to $(U \otimes_{\mathbb{F}} V)^{\mathbb{E}}$, so by Step 1 we know that $u \otimes_{\mathbb{E}} v$ generates $U^{\mathbb{E}} \otimes_{\mathbb{E}} V^{\mathbb{E}}$. This could not be the case if the $\mathbb{F}G$-submodule of $U \otimes_{\mathbb{F}} V$ generated by $u \otimes_{\mathbb{F}} v$ were smaller than $U \otimes V$, for the natural isomorphism between $U^{\mathbb{E}} \otimes_{\mathbb{E}} V^{\mathbb{E}}$ and $(U \otimes_{\mathbb{F}} V)^{\mathbb{E}}$ matches $u \otimes_{\mathbb{E}} v$ to $u \otimes_{\mathbb{F}} v$.

*Step* 3. The claim holds in full generality. For this, note that semisimplicity is only an issue if the characteristic of $\mathbb{F}$ is a prime, $p$ say, and then an $\mathbb{F}G$-module is semisimple if and only if all $p$-elements of $G$ act trivially on it: thus it is readily seen (using $(u \otimes v)(g - 1) = u(g - 1) \otimes vg + u \otimes v(g - 1)$, for example) that

$$(U \otimes V)/\mathrm{rad}(U \otimes V) \cong (U/\mathrm{rad}\, U) \otimes (V/\mathrm{rad}\, V). \tag{6}$$

By our hypotheses, $u + \operatorname{rad} U$ and $v + \operatorname{rad} V$ generate $U/\operatorname{rad} U$ and $V/\operatorname{rad} V$, respectively. Using (6), we know also that $(U/\operatorname{rad} U) \otimes (V/\operatorname{rad} V)$ is monogenic. So by Step 2 the latter module is generated by $(u + \operatorname{rad} U) \otimes (v + \operatorname{rad} V)$. In the isomorphism (6), $(u + \operatorname{rad} U) \otimes (v + \operatorname{rad} V)$ corresponds to $(u \otimes v) + \operatorname{rad}(U \otimes V)$, and this element generates $(U \otimes V)/\operatorname{rad}(U \otimes V)$. Then an appeal to Nakayama's Lemma completes the argument. ∎

Note that when $U$ and $V$ are 2-dimensional simple modules for the quaternion group of order 8 over the field of three elements, $U \otimes V$ is monogenic but no element of the form $u \otimes v$ can generate it: so the commutativity hypothesis cannot be omitted from Lemma 3.5.

*Proof of Theorem* 3.4. Let us write $\mathbb{F}$ for the 1-dimensional $G$-trivial $\mathbb{F}G$-module, as usual. By assumption, there exist surjective module homomorphisms $\kappa\colon U \to \mathbb{F}$ and $\lambda\colon V \to \mathbb{F}$, and a module isomorphism $\mu\colon \mathbb{F}G \to U \otimes V$. When we identify $V$ with $\mathbb{F} \otimes V$, the composite of $\mu$ and $\kappa \otimes 1\colon U \otimes V \to \mathbb{F} \otimes V$ becomes a module homomorphism, $\kappa'$ say, from $\mathbb{F}G$ onto $V$; in a similar sense, $\mu$ followed by $1 \otimes \lambda\colon U \otimes V \to U \otimes \mathbb{F}$ gives a surjective module homomorphism $\lambda'\colon \mathbb{F}G \to U$. Since $\mathbb{F}G$ is commutative, the kernels of $\lambda'$ and $\kappa'$ are two-sided ideals, and so algebra structures may be defined on $U$ and $V$ along $\lambda'$ and $\kappa'$, that is, so that $\lambda'$ and $\kappa'$ become (surjective) algebra homomorphisms as well. Note that the identity elements of the algebras $U$ and $V$ are the images of the identity element of $\mathbb{F}G$ under surjective $\mathbb{F}G$-module homomorphisms. Hence they generate $U$ and $V$ as $\mathbb{F}G$-modules. It follows by Lemma 3.5 that the identity element of the algebra $U \otimes V$ generates $U \otimes V$ as $\mathbb{F}G$-module. Consider the composite of the diagonal map $\mathbb{F}G \to \mathbb{F}G \otimes \mathbb{F}G$ with $\lambda' \otimes \kappa'\colon \mathbb{F}G \otimes \mathbb{F}G \to U \otimes V$. This is an algebra homomorphism, so it maps the identity element of $\mathbb{F}G$ to the identity element of $U \otimes V$. It is also an $\mathbb{F}G$-module homomorphism, and one of the elements of its image generates its codomain as $\mathbb{F}G$-module, and it must be surjective. By dimension comparison therefore, it is a bijection. ∎

We close this section with a brief discussion of the question whether in Corollary 3.2 one could relax the requirement that the modules, $U$ and $V$, which afford $\rho$ and $\sigma$, have nonzero $G$-trivial quotients. Of course, if $U \otimes V$ is regular, then it has precisely one nonzero $G$-trivial quotient, so the following result is applicable and provides some encouragement for a while. In this, we again write $\mathbb{F}$ for the 1-dimensional $\mathbb{F}G$-module on which $G$ acts trivially.

3.5. Lemma. *If $G$ is abelian, if $U, V$ are $\mathbb{F}G$-modules, and if*

$$\dim \operatorname{Hom}_{\mathbb{F}G}(U \otimes V, \mathbb{F}) = 1,$$

*then $U$ and $V$ have (unique, 1-dimensional) quotients $U/U_0$ and $V/V_0$ such that $(U/U_0) \otimes (V/V_0) \cong \mathbb{F}$.*

*Proof.* First suppose that $U$ and $V$ are simple. As $\mathrm{Hom}_{\mathbb{F}G}(U \otimes V, \mathbb{F})$ is isomorphic to $\mathrm{Hom}_{\mathbb{F}G}(U, V^*)$ where $V^*$ stands for the $\mathbb{F}G$-module $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$, in this case we must have that $U \cong V^*$ and $\mathrm{End}_{\mathbb{F}G}\, U = \mathbb{F}$. Since $G$ is abelian, the linear transformations representing $G$ on $U$ all lie in $\mathrm{End}_{\mathbb{F}G}\, U$, and so now they are all scalars. Hence the simple $U$ itself has dimension 1, and $U \otimes V \cong \mathbb{F}$ as required.

In general, $\mathrm{Hom}_{\mathbb{F}G}(U \otimes V, \mathbb{F}) \cong \mathrm{Hom}_{\mathbb{F}G}((U \otimes V)/\mathrm{rad}(U \otimes V), \mathbb{F})$, so (6) gives that, if $\dim \mathrm{Hom}_{\mathbb{F}G}(U \otimes V, \mathbb{F}) = 1$, then $U$ and $V$ have semisimple quotients $U/U_0$ and $V/V_0$ such that $\dim \mathrm{Hom}_{\mathbb{F}G}((U/U_0) \otimes (V/V_0), \mathbb{F}) = 1$. Of course then they also have simple quotients with this property and our claim has been reduced to the case we considered first. ∎

3.7. EXAMPLE. Let $G$ be a cyclic group of order 4 generated by $g$, and $\mathbb{F}$ a field which contains an element, $i$, of multiplicative order 4. Let $U$ be the 2-dimensional $\mathbb{F}G$-module on which the eigenvalues of $g$ are $-1$ and $i$, and let $V$ be similarly defined with reference to $i$ and $-i$ instead. All four possible eigenvalues for $g$ occur, each just once, on $U \otimes V$, so this tensor product is isomorphic to the regular module. It is clear from the proof of the first part of Theorem 3.1 that $\rho \otimes \sigma = \beta \# \gamma$ can only hold with the one choice of $B, C, \beta, \gamma$ given there. With that choice in the present case, $C$ contains $e_1 + e_i$ where $e_1$ and $e_i$ are the primitive idempotents of $\mathbb{F}G$ such that $e_1 g = e_1$ and $e_i g = ie_i$. The idempotent $e_1 + e_i$ acts on $U$ neither as 0 nor as 1 (for it annihilates the eigenvector of $g$ with eigenvalue $-1$ but fixes that with eigenvalue $i$), and therefore $(e_1 + e_i)\rho$ cannot be a scalar. Thus by allowing the hypothesis of Corollary 3.2 concerning nonzero $G$-trivial quotients to fail, we find that one of the conclusions fails as well: $C\rho$ does not consist of scalars.

## 4. THE VARIETY OF A TENSOR FACTOR OF THE REGULAR MODULE

We now combine some of the results of the previous sections to obtain information about the varieties of the tensor factors of the regular module for a finite abelian $p$-group over a field of characteristic $p$. In the case of an elementary abelian $p$-group, the results of this section could be derived from the theorem on degrees of varieties in [4]. However, for a general abelian $p$-group, the intersection multiplicities technique for analyzing degrees does not seem to be available. Hence it is necessary to fall back on methods which are similar to those used in [3]. The results, accordingly, are weaker than those obtained in [4]. Our first theorem follows directly from the work of the previous sections.

4.1. THEOREM. *Let p be a prime,* $\mathbb{F}$ *any field of characteristic p, and G any finite abelian p-group. If U and V are* $\mathbb{F}G$*-modules such that G acts regularly on* $U \otimes V$*, then G has a direct decomposition such that one direct factor acts regularly on U while the other acts regularly on V.*

*Proof.* By Lemma 3.3 and Corollary 3.2, both $U$ and $V$ are monogenic, and the algebra $\mathbb{F}G$ has a tensor factorization $\mathbb{F}G = B \otimes C$ such that $\dim B = \dim U$ and $\dim C = \dim V$ while both the action of $B$ on $V$ and the action of $C$ on $U$ are by scalars only. By Theorem 2.1, $G$ has a direct decomposition $G = H \times K$ such that $\mathbb{F}G = B \otimes \mathbb{F}K = \mathbb{F}H \otimes C$. Since $\mathbb{F}G = \mathbb{F}H \otimes C$ and $C$ acts on $U$ by scalars, all the action of $\mathbb{F}G$ on $U$ comes from the action of $\mathbb{F}H$: thus $U_H$ is also monogenic. On the other hand, $\dim U_H = \dim B = |H|$, so $U_H$ must in fact be regular. The regularity of $V_K$ is proved similarly. ∎

Note that we have no reason to expect that $H$ would act by scalars on $V$, or that $K$ would act by scalars on $U$.

Next we need some facts about the actions of group algebras on free modules.

4.2. LEMMA (For example, see (2.2) of [3]). *Let* $S = \langle s_1, \ldots, s_m \rangle$ *be an elementary abelian group of order* $p^m$*. An* $\mathbb{F}S$*-module W is free if and only if*

$$\dim\left( W \prod_{i=1}^{m} (s_i - 1)^{p-1} \right) = (\dim W)/p^m.$$

Let $H$ be a finite abelian $p$-group and $S$ the socle of $H$ (that is, the unique largest elementary abelian $p$-subgroup of $H$). Write $H$ as a direct product of nontrivial cyclic subgroups, $H = \langle h_1 \rangle \times \cdots \times \langle h_m \rangle$, and let $S = \langle x_1 \rangle \times \cdots \times \langle s_m \rangle$ be the corresponding direct decomposition of $S$. To be more specific, for $i = 1, \ldots, m$, let $s_i = g_i^{p^{k(i)}}$ with $k(i)$ chosen so that $s_i \neq 1$ but $s_i^p = 1$. Let $\mathbb{F}$ be an arbitrary field of characteristic $p$. In the group algebra $\mathbb{F}H$, set $x_i = h_i - 1$ and $y_i = s_i - 1$, noting that $x_i^{p^{k(i)}} = y_i$ and $y_i^p = 0$. Recall that the elements $\prod_{i=1}^{m} x_i^{\mu(i)}$ with $0 \leq m(i) < p^{k(i)+1}$ for $i = 1, \ldots, m$ form a basis for $\mathbb{F}S$, and that $\operatorname{rad} \mathbb{F}H$ is the ideal generated by $\{x_1, \ldots, x_m\}$. Also, $\operatorname{rad} \mathbb{F}S$ is the ideal of $\mathbb{F}S$ generated by $\{y_1, \ldots, y_m\}$, and it is the vector space direct sum of $(\operatorname{rad} \mathbb{F}S)^2$ with the $m$-dimensional subspace spanned by $\{y_1, \ldots, y_m\}$.

4.3. LEMMA. *For* $a \in \mathbb{F}H$*, we have* $a^p = 0$ *if and only if a can be written as* $a = \sum_i b_i y_i$ *with suitable* $b_1, \ldots, b_m$ *in* $\mathbb{F}H$.

*Proof.* The "if" claim follows from the fact that the $y_i^p$ are all 0 and the $p$th powering is a ring endomorphism of $\mathbb{F}H$. For the proof of the "only if" claim, suppose that $a^p = 0$. Write $a$ in terms of the basis of $\mathbb{F}H$ mentioned above, as

$$a = \sum_{\mu} f_{\mu} \prod_{i} x_i^{\mu(i)} \tag{7}$$

with $f_\mu \in \mathbb{F}$ for $\mu = (\mu(1), \ldots, \mu(m))$. Then $0 = a^p = \sum f_\mu^p \prod_{i=1}^m x_i^{p\mu(i)}$, and here the range of summation may as well be restricted to the $\mu$ with $m(i) < p^{k(i)}$ for $i = 1, \ldots, m$. It follows that $f_\mu = 0$ for all such $\mu$, and so in (7) the sum need only be taken over the $\mu$ to which there is a $j$ such that $\mu(j) \geq p^{k(j)}$. The summand corresponding to such a $\mu$ is $b_j y_j$ where

$$b_j = f_\mu x_j^{\mu(j) - p^{k(j)}} \prod_{i \neq j} x_i^{\mu(i)} \in \mathbb{F}H.$$

This completes the proof. ∎

4.4. LEMMA. *Let $V$ be a free $\mathbb{F}H$-module and $a = \sum_i b_i y_i$ with $b_1, \ldots, b_m \in \mathbb{F}H$. If not all of the $b_i$ lie in* rad $\mathbb{F}H$, *then $V_{\langle 1+a \rangle}$ is free.*

*Proof.* Without loss of generality we may assume that $b_1 \notin \operatorname{rad} \mathbb{F}H$. Note that

$$(ay_2 \cdots y_m)^{p-1} = b_1^{p-1} y_1^{p-1} y_2^{p-1} \cdots y_m^{p-1} \tag{8}$$

(simply because $y_i^p = 0$ for all $i$). Combine the actions of $\langle 1 + a \rangle$ and $\langle s_2, \ldots, s_m \rangle$ on $V$ into an action of their external direct product $E = \langle 1 + a \rangle \times \langle s_2, \ldots, s_m \rangle$, so $V$ becomes an $\mathbb{F}E$-module.

Of course, $V_S$ is free, so we may apply the "only if" part Lemma 4.2 with $W = V_S$. In view of (8), the "if" part of that lemma may then be applied with $E$ and $V$ playing the roles of $S$ and $W$. The conclusion is that $V$ as $\mathbb{F}E$-module is free. From this, it follows that $V_{\langle 1+a \rangle}$ is also free. ∎

4.5. LEMMA. *If $V$ is a free $\mathbb{F}H$-module and $a = \sum_i b_i y_i$ with $b_1, \ldots, b_m \in$* rad $\mathbb{F}H$, *then $V_{\langle 1+a \rangle}$ is not free.*

*Proof.* We may assume that $V \cong \mathbb{F}H$ as an $\mathbb{F}H$-module. By Lemma 4.3, the group $\langle 1 + a \rangle$ has order $p$. Thus by Lemma 4.2, what we have to prove is that $\dim Va^{p-1} \neq (\dim V)/p$, that is, that $\dim \mathbb{F}Ha^{p-1} \neq |H|/p$. Denote the algebraic closure of $\mathbb{F}$ by $\mathbb{E}$. As an $\mathbb{E}$-space, $\mathbb{E} \otimes_{\mathbb{F}} (\mathbb{F}Ha^{p-1})$ is isomorphic to $\mathbb{E}Ha^{p-1}$, so it suffices to show that $\dim_{\mathbb{E}} \mathbb{E}Ha^{p-1} \neq |H|/p$.

Let $G$ be the external direct product $H \times \langle 1 + a \rangle$ and combine the natural actions of the two direct factors into an action of $G$ on $\mathbb{E}H$. Notice that $\mathbb{E}H$ is not a free $\mathbb{E}G$-module since $\dim \mathbb{E}H < |G|$. The unique maximal elementary abelian subgroup of $G$ is the external direct product $S \times \langle 1 + a \rangle$. By Chouinard's Theorem (5.2.4 in [1]; see [2] for an elementary proof of the relevant special case), the fact that $\mathbb{E}H$ is not free as $\mathbb{E}G$-module implies that $\mathbb{E}H$ cannot be free as $\mathbb{E}(S \times \langle 1 + a \rangle)$-module. By Dade's Lemma (5.8.4 in [1]), it follows that there are scalars $e_1, \dots, e_{m+1} \in \mathbb{E}$ such that the unit $1 + e_1 y_1 + \cdots + e_m y_m + e_{m+1} a$ does not act freely on $\mathbb{E}H$. But this unit acts as $1 + \sum_{i=1}^{m}(e_i + e_{m+1} b_i) y_i$, so by the previous lemma each $e_i + e_{m+1} b_i$ must lie in $\operatorname{rad} \mathbb{E}H$. As $a_i \in \operatorname{rad} \mathbb{F}H \subseteq \operatorname{rad} \mathbb{E}H$, this means that the scalars $e_1, \dots, e_m$ must vanish. Of course then $e_{m+1} \neq 0$, and we have that $1 + e_{m+1} a$ does not act freely on $\mathbb{E}H$. By Lemma 4.2, it follows that $\dim_{\mathbb{E}}(\mathbb{E}He_{m+1}^{p-1}a^{p-1}) \neq |H|/p$. As $e_{m+1}$ is a nonzero scalar, this is equivalent to what we had to prove. ∎

For the rest of this section, assume that $\mathbb{F}$ is algebraically closed, $G$ is a finite abelian $p$-group with socle $R$, and set $J = \operatorname{rad} \mathbb{F}R$. It is known that if $0 \neq a \in J^2$ then $\langle 1 + a \rangle$ cannot act freely on any nonzero $\mathbb{F}R$-module (see (6.1) in [3]). Moreover, if $V$ is any $\mathbb{F}R$-module and $a, b \in J$ with $0 \neq a \equiv b \bmod J^2$, then $V_{\langle 1+a \rangle}$ is free if and only if $V_{\langle 1+b \rangle}$ is free (see (6.2) in [3]). Thus it makes sense to define, for any $\mathbb{F}G$-module $U$, the variety $\mathrm{V}_G(U)$ of $U$ by

$$\mathrm{V}_G(U) = \left\{ a + J^2 \in J/J^2 | U_{\langle 1+a \rangle} \text{ is not free} \right\} \cup \{0\}.$$

An obvious example is that $\mathrm{V}_G(\mathbb{F}) = J/J^2$ (where, as before, $\mathbb{F}$ stands for the 1-dimensional trivial $\mathbb{F}G$-module). The experienced reader will notice that what we have defined here as $\mathrm{V}_G(U)$ is usually called the rank variety of $U_R$. However, because $G$ is abelian, $\mathrm{V}_G(U)$ (and its embedding in $J/J^2$) is isogenous to the usual cohomological variety (and its embedding in the variety of the trivial module). In particular, $\mathrm{V}_G(U)$ is a homogeneous affine subvariety of $J/J^2$. (See [1] or [5].)

As usual, when we say that a finite abelian $p$-group has rank $m$ we mean that it is a direct product of $m$ nontrivial cyclic groups.

4.6. THEOREM. *Let $p$ be a prime, $\mathbb{F}$ an algebraically closed field of characteristic $p$, and $G$ a finite abelian $p$-group. Denote the socle of $G$ by $R$ and the radical of $\mathbb{F}R$ by $J$. If $H$ is a direct factor of rank $m$ in $G$ and $U$ is an $\mathbb{F}G$-module such that $U_H$ is isomorphic to the regular $\mathbb{F}H$-module $\mathbb{F}H$, then $\mathrm{V}_G(U)$ is a linear subspace of codimension $m$ in $J/J^2$.*

*Proof.* Denote the rank of $G$ by $n$. Extending the notation used in our lemmas, write $H = \langle h_1 \rangle \times \cdots \times \langle h_m \rangle$ and $G = H \times \langle h_{m+1} \rangle \times \cdots \times \langle h_n \rangle$. Let $R = \langle s_1 \rangle \times \cdots \times \langle s_n \rangle$ be the corresponding direct decomposition of $R$, and $y_k = s_k - 1$ for $k = 1, \ldots, n$. We shall use that $J$ is the vector space direct sum of $J^2$ with the $n$-dimensional subspace spanned by $\{y_1, \ldots, y_n\}$.

Because $H$ acts regularly on $U$, there exist $u_0$ in $U$ such that $U = u_0 \mathbb{F}H$. Given such a $u_0$, one can choose, for $j = m+1, \ldots, n$, an $a_j$ in $\mathbb{F}H$ such that $u_0 y_j = u_0 a_j$. Then $u y_j = u a_j$ for all $u \in U$, because $u \in u_0 \mathbb{F}H$ and the group is abelian. It follows in particular that $a_j^p = 0$ and so, by Lemma 4.3, $a_j = \sum_{i=1}^m b_{ji} y_i$ for suitable $b_{ji}$ in $\mathbb{F}H$. Further, one can write $b_{ji} = f_{ji} + c_{ji}$ with $f_{ji} \in \mathbb{F}$ and $c_{ji} \in \operatorname{rad} \mathbb{F}H$.

If $a \in J$ then $a \equiv \sum_{k=1}^n f_k y_k \bmod J^2$ for a unique $(f_1, \ldots, f_n)$ in $\mathbb{F}^n$. We know that $\langle 1 + a \rangle$ acts on $U$ freely if and only if $\langle 1 + \sum_k f_k y_k \rangle$ does. On the other hand, $1 + \sum_k f_k y_k$ acts on $U$ as $1 + \sum_{i=1}^m b_i y_i$ acts on $\mathbb{F}H$, where

$$b_i = f_i + \sum_{j=m+1}^{n} f_j (f_{ji} + c_{ji}) \in \mathbb{F}H.$$

By Lemmas 4.4 and 4.5, the action of $\langle 1 + \sum_{i=1}^m b_i y_i \rangle$ on $\mathbb{F}H$ is not free if and only if all the $b_i$ lie in $\operatorname{rad} \mathbb{F}H$. Since the $c_{ji}$ are in $\operatorname{rad} \mathbb{F}H$ while the $f_i + \sum_{j=m+1}^n f_j f_{ji}$ are scalars, the condition amounts to

$$f_i + \sum_{j=m+1}^{n} f_j f_{ji} = 0 \qquad \text{for } i = 1, \ldots, m. \tag{9}$$

This proves that $U_{\langle 1+a \rangle}$ is not free if and only if $(f_1, \ldots, f_n)$ is a nonzero solution of the set of $m$ simultaneous linear equations given in (9). The solution set of (9) is, of course, a linear subspace of $\mathbb{F}^n$, and the set of equations is obviously independent, so the codimension of the solution sets is $m$. ∎

In the next theorem, we again use the convention that the 1-dimensional trivial $\mathbb{F}G$-module is written simply as $\mathbb{F}$. For a subgroup $H$ of $G$, the restriction $\mathbb{F}_H$ of that module to $H$ is, of course, just the 1-dimensional trivial $\mathbb{F}H$-module. The $\mathbb{F}G$-module induced from $\mathbb{F}_H$ will be written as $\mathbb{F}_H \uparrow^G$. We also exploit the fact that $\mathsf{V}_G(\mathbb{F})$ is another name for what in the previous theorem we called $J/J^2$.

4.7. THEOREM. *Let $p$ be a prime, $\mathbb{F}$ an algebraically closed field of characteristic $p$, and $G$ a finite abelian $p$-group. Suppose that $U$ and $V$ are $\mathbb{F}G$-modules such that $G$ acts regularly on $U \otimes V$. Then $\mathsf{V}_G(U)$ and $\mathsf{V}_G(V)$*

*are linear subspaces of* $\mathsf{V}_G(\mathbb{F})$, *with* $\mathsf{V}_G(\mathbb{F}) = \mathsf{V}_G(U) \oplus \mathsf{V}_G(V)$. *Further, there exist subgroups* $H$ *and* $K$ *of* $G$ *such that* $G = H \times K$, *the varieties of* $\mathbb{F}_H \uparrow^G$ *and* $\mathbb{F}_K \uparrow^G$ *are also linear subspaces of* $\mathsf{V}_G(\mathbb{F})$, *and*

$$\mathsf{V}_G(U) \oplus \mathsf{V}_G(\mathbb{F}_H \uparrow^G) = \mathsf{V}_G(\mathbb{F}) = \mathsf{V}_G(\mathbb{F}_K \uparrow^G) \oplus \mathsf{V}_G(V).$$

*Proof.* By Theorem 4.1, there exist subgroups $H$ and $K$ such that $G = H \times K$, $H$ acts regularly on $U$, and $K$ acts regularly on $V$. By Mackey's Subgroup Theorem, $H$ acts regularly on $\mathbb{F}_K \uparrow^G$ and $K$ acts regularly on $\mathbb{F}_H \uparrow^G$. By a familiar rule,

$$U \otimes (\mathbb{F}_H \uparrow^G) \cong (U_H \otimes \mathbb{F}_H) \uparrow^G \cong U_H \uparrow^G \cong \mathbb{F}H \uparrow^G \cong \mathbb{F}G,$$

and similarly $(\mathbb{F}_K \uparrow^G) \otimes V \cong \mathbb{F}G$, all these being isomorphisms of $\mathbb{F}G$-modules. The variety of the tensor product of two modules is the intersection of the varieties of the modules [3] (or see [1] or [5]). Moreover, the variety of any free module such as $\mathbb{F}G$ is 0. Consequently we have that

$$\mathsf{V}_G(U) \cap \mathsf{V}_G(\mathbb{F}_H \uparrow^G) = 0 = \mathsf{V}_G(\mathbb{F}_K \uparrow^G) \cap \mathsf{V}_G(V)$$

as well as $\mathsf{V}_G(U) \cap \mathsf{V}_G(V) = 0$. All that remains is to apply the previous theorem several times, use that the rank of $G$ is the sum of the ranks of $H$ and $K$, and count codimensions in $\mathsf{V}_G(\mathbb{F})$. ∎

Of course, one also has $(\mathbb{F}_H \uparrow^G) \otimes (\mathbb{F}_K \uparrow^G) \cong \mathbb{F}G$ and

$$\mathsf{V}_G(\mathbb{F}_H \uparrow^G) \oplus \mathsf{V}_G(\mathbb{F}_K \uparrow^G) = \mathsf{V}_G(\mathbb{F}).$$

## 5. ON THE CASE OF NONABELIAN GROUPS

We gave a number of examples to indicate that various hypotheses cannot be omitted. None of those examples involved group algebras of finite $p$-groups over fields of characteristic $p$, and this raises many questions. To mention just the simplest: does the group algebra of a directly indecomposable $p$-group over a field of characteristic $p$ ever admit a nontrivial tensor factorization (as algebra)? Theorem 2.1 says "no" if the group is abelian. At this stage, we can add only that the answer is "no" if the group is nonabelian of order 8 (and $p = 2$): the case of the smallest relevant nonabelian groups. To show this, we use the following.

5.1. LEMMA. *The centre of a tensor product of algebras is the tensor product of the centres of the tensor factors.*

A nonabelian group of order 8 has five conjugacy classes of elements, so the centre of its group algebra has dimension 5. In a proper tensor factorization of the group algebra, one tensor factor would have dimension 2. All 2-dimensional algebras are commutative, so by the lemma the 5-dimensional centre would have a 2-dimensional tensor factor: this is clearly impossible.

Lemma 5.1 must be well known but we have no reference for it and so give a proof. Let $X$ and $Y$ be the regular modules for two $\mathbb{F}$-algebras, $B$ and $C$ say. Considering $B$ as a subalgebra of $\mathrm{End}_{\mathbb{F}} X$, we see that the centre of $B$ is $B \cap \mathrm{End}_B X$. With this point of view, our claim is that

$$(B \otimes C) \cap \mathrm{End}_{B\otimes C}\left(X \sharp Y\right) = (B \cap \mathrm{End}_B X) \otimes (C \cap \mathrm{End}_C Y).$$

First, note that $B \otimes C = [B \otimes (\mathrm{End}_{\mathbb{F}} Y)] \cap [(\mathrm{End}_{\mathbb{F}} X) \otimes C]$. Second, as the subalgebras $B$ and $C$ generate $B \otimes C$,

$$\mathrm{End}_{B\otimes C}\left(X\sharp Y\right) = \mathrm{End}_B\left(X\sharp Y\right) \cap \mathrm{End}_C\left(X\sharp Y\right).$$

It is obvious that $\mathrm{End}_B(X\sharp Y)$ contains $\mathrm{End}_B X \otimes \mathrm{End}_{\mathbb{F}} Y$. On the other hand, $B$-module $X\sharp Y$ is the direct sum of $\dim Y$ copies of $X$ and therefore $\mathrm{End}_B(X\sharp Y)$ is isomorphic to the algebra of all $(\dim Y)$-by-$(\dim Y)$ matrices over $\mathrm{End}_B X$. Counting dimensions now yields that

$$\mathrm{End}_B\left(X\sharp Y\right) = (\mathrm{End}_B X) \otimes (\mathrm{End}_{\mathbb{F}} Y).$$

Similarly,

$$\mathrm{End}_C\left(X\sharp Y\right) = (\mathrm{End}_{\mathbb{F}} X) \otimes (\mathrm{End}_C Y).$$

Consequently

$$\begin{aligned}
&(B \otimes C) \cap \mathrm{End}_{B\otimes C}\left(X\sharp Y\right)\\
&\quad = \left[B \otimes (\mathrm{End}_{\mathbb{F}} Y)\right] \cap \left[(\mathrm{End}_{\mathbb{F}} X) \otimes C\right]\\
&\qquad \cap \left[(\mathrm{End}_B X) \otimes (\mathrm{End}_{\mathbb{F}} Y)\right] \cap \left[(\mathrm{End}_{\mathbb{F}} X) \otimes (\mathrm{End}_C Y)\right]\\
&\quad = \left[B \otimes (\mathrm{End}_{\mathbb{F}} Y)\right] \cap \left[(\mathrm{End}_B X) \otimes (\mathrm{End}_{\mathbb{F}} Y)\right]\\
&\qquad \cap \left[(\mathrm{End}_{\mathbb{F}} X) \otimes C\right] \cap \left[(\mathrm{End}_{\mathbb{F}} X) \otimes (\mathrm{End}_C Y)\right]\\
&\quad = \left[(B \cap (\mathrm{End}_{\mathbb{F}} X)) \otimes (\mathrm{End}_{\mathbb{F}} Y)\right]\\
&\qquad \cap \left[(\mathrm{End}_{\mathbb{F}} X) \otimes (C \cap (\mathrm{End}_C Y))\right]\\
&\quad = (B \cap \mathrm{End}_B X) \otimes (C \cap \mathrm{End}_C Y).
\end{aligned}$$

This completes the proof of the lemma. ∎

If $G$ is a dihedral group of order 8 then $G$ has a cyclic subgroup $H$ of order 4 and a nonnormal subgroup $K$ of order 2, such that $G = HK$, a semidirect product. The coset spaces $\mathbb{F}(G/H)$ and $\mathbb{F}(G/K)$ are $\mathbb{F}G$-modules and $\mathbb{F}G = \mathbb{F}(G/H) \otimes \mathbb{F}(G/K)$. Of course, there is no such decomposition of $\mathbb{F}G$ as a tensor product of algebras. However, this is one of many decompositions of $\mathbb{F}G$ as a tensor product of modules. Still, some analogue of Theorem 4.6 should hold in the case of a nonabelian $p$-group, although it is not clear what the formulation of such a result should be.

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