Mathematics Research Report No. MRR 061–94

A projective geometry problem related to tensor factorizations of group algebras

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7 October 1994

ABSTRACT. Let $\mathbb{F}$ be a field, $\mathbb{E}$ a subfield of $\mathbb{F}$, and U a finite dimensional vector space over $\mathbb{E}$. Denote by $U^{\mathbb{F}}$ the $\mathbb{F}$-space $U \otimes_{\mathbb{E}} \mathbb{F}$ obtained from U 'by extending the scalars', and regard U a subset of $U^{\mathbb{F}}$ (via the embedding $u \mapsto u \otimes 1$). Then each $\mathbb{E}$-basis of U is an $\mathbb{F}$-basis of $U^{\mathbb{F}}$.

PROBLEM. *If $\{x_1, \ldots, x_m\}$ is any $\mathbb{F}$-basis of $U^{\mathbb{F}}$, must U have an $\mathbb{E}$-basis $\{v_1, \ldots, v_m\}$ such that, for every subset I of $\{1, \ldots, m\}$,*

$$\{\, x_i \mid i \in I \,\} \cup \{\, v_j \mid j \notin I \,\}$$

is also an $\mathbb{F}$-basis of $U^{\mathbb{F}}$?

A positive answer would enable one to strengthen the conclusion and simplify the proof of a theorem in a forthcoming paper, 'Tensor factorizations of group algebras and modules', by Jon F. Carlson and the speaker.

1991 *Mathematics Subject Classification*: primary 20C05; secondary 16S34.

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This paper was presented at the Twelfth Annual Victorian Algebra Conference, Swinburne University of Technology, 26-27 September 1994.

A projective geometry problem related to tensor factorizations of group algebras

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Let $\mathbb{F}$ be a field, $\mathbb{E}$ a subfield of $\mathbb{F}$, and U a finite dimensional vector space over $\mathbb{E}$. Denote by $U^{\mathbb{F}}$ the $\mathbb{F}$-space $U \otimes_{\mathbb{E}} \mathbb{F}$ obtained from U by 'extending the scalars', and regard U a subset of $U^{\mathbb{F}}$ (via the embedding $u \mapsto u \otimes 1$). Then each $\mathbb{E}$-basis of U is an $\mathbb{F}$-basis of $U^{\mathbb{F}}$.

PROBLEM. *If $\{x_1, \ldots, x_n\}$ is any $\mathbb{F}$-basis of $U^{\mathbb{F}}$, must U have an $\mathbb{E}$-basis $\{v_1, \ldots, v_n\}$ such that, for every subset I of $\{1, \ldots, n\}$,*

$$\{\, x_i \mid i \in I \,\} \cup \{\, v_j \mid j \notin I \,\} \tag{1}$$

is also an $\mathbb{F}$-basis of $U^{\mathbb{F}}$?

A positive answer would enable one to strengthen the conclusion and simplify the proof of a theorem in a forthcoming paper [1] by Jon F. Carlson and the author.

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1. Why is this a projective geometry problem? Because it is insensitive to (basis) vectors being replaced by (nonzero) scalar multiples.

2. Haven't we seen this before? Isn't there a theorem that guarantees a positive answer? Yes, we have all seen something like it before. It is known as the Exchange Theorem: it asserts that if $\{x_1, \ldots, x_n\}$ and $\{v_1, \ldots, v_n\}$ are two bases of one vector space, then it is possible to re-number one of the bases so that each $\{x_i\} \cup \{\, v_j \mid j \neq i \,\}$ is also a basis. The symmetry of the hypothesis dictates that the re-numbering can also be done so that each $\{\, x_i \mid i \neq j \,\} \cup \{v_j\}$ is a basis.

However, as the example below will show, it is not always possible to find *one* re-numbering which will achieve *both* conclusions. The first conclusion corresponds to (1) being a basis whenever I is a singleton, and the second asserts the same for the I whose complements are singletons: if these two cases cannot be handled simultaneously, what hope is there for dealing with all I at once? Of course, in our Problem the second (ordered) basis can be chosen much more freely, not just by re-ordering some given basis, and the hope must lie in that extra freedom.

3. EXAMPLE. Instead of bases, consider direct decompositions with summands of dimension 1. Such decompositions of 3-space correspond to nondegenerate triangles in the projective plane. Let ABC be such a triangle, and let P be a

point which is not on any of the lines AB, AC, BC. Choose Q as the intersection of AP and BC; note that $Q \neq P$, and that A lies on PQ. Similarly, let R be the intersection of PB and AC; note that $R \neq P$, and that B lies on PR. The triangle PQR is nondegenerate, else the lines PQ and PR would coincide with each other and therefore also with AB, but P was chosen so that it does not lie on AB. We need to investigate the possibility of a bijection between the vertex sets of ABC and PQR such that swapping matched vertices keeps the triangles nondegenerate. Since QBC and PQA are degenerate, A cannot be so swapped with Q or with R. Similarly, PBR and ARC being degenerate shows that B cannot be so swapped with Q or R. The conclusion is that there can be no bijection of the desired kind, for it would have to match both A and B to P.

4. Our Problem becomes much easier if instead of direct decompositions with many summands (each of dimension 1) we consider direct decompositions with just two summands (of arbitrary dimension). For that case, a positive answer can easily be had.

LEMMA 2.3(b) of [1]. *If $U^{\mathbb{F}} = X \oplus Y$, then U has a direct decomposition $U = V \oplus W$ such that $U^{\mathbb{F}} = X \oplus W^{\mathbb{F}} = V^{\mathbb{F}} \oplus Y$.*

We reproduce the short and elementary proof in full. It is based on the following.

LEMMA 2.3(a) of [1]. *If R and S are $\mathbb{F}$-subspaces in $U^{\mathbb{F}}$ with $\dim R = \dim S$, then there is an $\mathbb{E}$-subspace T in U such that $T^{\mathbb{F}}$ is a common complement to R and S in $U^{\mathbb{F}}$.*

Proof. (a) Consider U an $\mathbb{E}$-subspace in $U^{\mathbb{F}}$. If the common codimension of R and S in $U^{\mathbb{F}}$ is 0, then $T = 0$ will do. This provides the initial step for a proof by induction on that common codimension. For the inductive step, suppose the common codimension is positive. Then the intersections $R \cap U$ and $S \cap U$ are proper subspaces in U, and no vector space can be the set-union of just two proper subspaces. Thus there is an element, u say, which is in U but neither in R nor in S. The inductive hypothesis applies with $\mathbb{F}u \oplus R$ and $\mathbb{F}u \oplus S$ in place of R and S, and so there is a subspace, T_0 say, in U such that $T_0^{\mathbb{F}}$ is a common complement to $\mathbb{F}u \oplus R$ and $\mathbb{F}u \oplus S$. It is easy to see that $T = \mathbb{E}u \oplus T_0$ will do.

(b) Apply (a) twice. First, with $R = S = Y$. The T so obtained will be our V. Second, with $R = V^{\mathbb{F}}$ and $S = X$, and the T now obtained will serve as W. $\square$

5. Where can one see pairs of bases with the strong exchange property asked for in our Problem? Let p be a prime and G a finite direct product of several cyclic groups of equal, p-power, order. If two bases of G are congruent modulo

the Frattini subgroup $\Phi(G)$ of G, then as bases of G they clearly have this strong exchange property. (There can be no hesitation about the choice of the bijection between the two bases.) Of course these bases are not vector space bases until we pass to the quotient $G/\Phi(G)$, and then they are no longer different, so this is cheating. No, not cheating, just heuristic.

6. Let p and G be as above, write $\mathbb{F}_p$ for the field of p elements, and $\mathbb{F}_pG$ for the group algebra of G over $\mathbb{F}_p$. If this algebra is written as a tensor product of subalgebras,

$$\mathbb{F}_pG = \textstyle\bigotimes_{i=1}^{m} A_i, \tag{2}$$

then there is a corresponding direct decomposition

$$(\operatorname{rad}\mathbb{F}_pG)/(\operatorname{rad}\mathbb{F}_pG)^2 = \textstyle\bigoplus_{i=1}^{m}(\operatorname{rad}A_i + (\operatorname{rad}\mathbb{F}_pG)^2)/(\operatorname{rad}\mathbb{F}_pG)^2. \tag{3}$$

It is easy to see that if another tensor factorization of the algebra gives rise to the same direct decomposition of $(\operatorname{rad}\mathbb{F}_pG)/(\operatorname{rad}\mathbb{F}_pG)^2$, then the two tensor factorizations have the strong exchange property. (Once more, the choice of the bijection is not an issue.)

Some tensor factorizations of $\mathbb{F}_pG$ come from direct decompositions of G. In [1] we were considering just how close the connection must be between arbitrary tensor factorizations (2) of the group algebra and tensor factorizations that come from direct decompositions of the group. There is a natural isomorphism between $(\operatorname{rad}\mathbb{F}_pG)/(\operatorname{rad}\mathbb{F}_pG)^2$ and $G/\Phi(G)$, and the direct decomposition of $G/\Phi(G)$ obtained from (3) along that lifts (usually in several ways) to a direct decomposition of G itself. Let

$$G = \textstyle\prod_{i=1}^{m} G_i \tag{4}$$

be any one of these direct decompositions of G. Then

$$\mathbb{F}_pG = \textstyle\bigotimes_{i=1}^{m} \mathbb{F}_pG_i \tag{5}$$

is also a tensor factorization of $\mathbb{F}_pG$, and because (2) and (5) correspond to the same direct decomposition (3) they have the strong exchange property:

$$\mathbb{F}_pG = (\textstyle\bigotimes_{i\in I} A_i) \otimes (\bigotimes_{j\notin I} \mathbb{F}_pG_j) \qquad \text{whenever } I \subseteq \{1,\ldots,m\}. \tag{6}$$

Of course one may well feel that the strong exchange property is a relatively weak consequence of the fact that (2) and (5) correspond to the same (3), but it is a consequence that is easier to 'carry forward'.

7. A graded version of this argument ensures that the same conclusion can be reached without assuming that G is a direct sum of pairwise isomorphic cyclic subgroups.

THEOREM. *If G is any finite abelian p-group, then to each tensor factorization (2) of the group algebra $\mathbb{F}_p G$ there is a direct decomposition (4) of G such that (6) is true.*

8. In many considerations which involve elements of algebraic geometry, infinite fields take precedence over finite fields. For this reason, in [1] we did not even pause to state this Theorem but hastened to try to carry it forward through an 'extension of scalars'. Using (a graded version of) the lemma reproduced above, we succeeded with the case of two-factor decompositions, but not with the general, many-factor case: the Problem under discussion stood in our way. For the many-factor case we had to be content with the following result.

THEOREM 2.2 of [1]. *Let $\mathbb{F}$ a field of characteristic p, and G a finite abelian p-group. To each tensor factorization $\mathbb{F}G = \bigotimes_{i=1}^m A_i$ of $\mathbb{F}G$, there is a direct decomposition $G = \prod_{i=1}^m G_i$ such that $A_i \cong \mathbb{F}G_i$ and*

$$\mathbb{F}G = A_i \otimes \left(\bigotimes_{j\neq i} \mathbb{F}G_j\right) \qquad \textit{for } \ i = 1, \dots, m.$$

Of course even this amounts to a very strong conection between arbitrary tensor factorizations of $\mathbb{F}G$ and tensor factorizations that come from direct decompositions of G; in particular, it implies that each of the former can be had by 'shifting' one of the latter by an algebra automorphism of $\mathbb{F}G$ (use the automorphism obtained by combining the isomorphisms $\mathbb{F}G_i \to A_i$).

9. Appendix. We conclude by setting out formally how a positive answer to the Problem could be used for proving the Theorem above for arbitrary $\mathbb{F}$ (of characteristic p) in place of $\mathbb{F}_p$. In doing so we use the notation of [1], so this Appendix is intended to be read only in conjunction with that paper.

To minimize inessential formal complications, let us keep to elementary abelian G, with $\mathbb{F}_p G = \bigotimes_{i=1}^m A_i$ and $U = \overline{G}$, $\alpha\colon U^{\mathbb{F}} \to \overline{\operatorname{rad}\mathbb{F}G}$. Choose $b_1, \dots, b_n$ in the set-union of the A_i so that $x_i\alpha = \overline{b_i}$ defines an $\mathbb{F}$-basis $\{x_1, \dots, x_n\}$ for $U^{\mathbb{F}}$, and write the elements of a matching $\mathbb{F}_p$-basis $\{v_1, \dots, v_n\}$ of $\overline{G}$ as $v_i = \overline{h_i}$, with suitable elements h_i of G. Write B_i and H_i for the subalgebra generated by b_i and the subgroup generated by h_i, respectively, noting that none of the B_i and $\mathbb{F}H_i$ can have dimension greater than p. Given any subset I of $\{1, \dots, n\}$, we know that $\{\overline{b_i} \mid i \in I\} \cup \{\overline{h_j} \mid j \notin I\}$ spans $\overline{\operatorname{rad}\mathbb{F}G}$ and so $\{b_i \mid i \in I\} \cup \{h_j \mid j \notin I\}$ generates $\mathbb{F}G$. It follows that $\mathbb{F}G$ is a homomorphic image of the external tensor product

$$(\bigotimes_{i\in I} B_i) \otimes (\bigotimes_{j\notin I} \mathbb{F}H_j),$$

and dimension comparison shows that $\mathbb{F}G$ must therefore be the internal tensor product of these subalgebras. In particular, with $I = \varnothing$ this gives that $\mathbb{F}G = \bigotimes \mathbb{F}H_j$, and so G is the direct product of the H_j. Similarly, with $I = \{1, \ldots, n\}$ we get that $\mathbb{F}G = \bigotimes_{i=1}^{m} A_i = \bigotimes_{j=1}^{n} B_j$. Since each B_j is a subalgebra of some A_i, it follows that each A_i is the tensor product of the B_j that it contains. Define G_i as the product of the H_j which are such that $B_j \subseteq A_i$; then G is the direct product of the G_i, and (6) holds. (Yes, in (6) I do mean for every subset I of $\{1, \ldots, m\}$, even if earlier in this paragraph I ranged through the set of subsets of $\{1, \ldots, n\}$.)

Reference

1. Jon F. Carlson and L. G. Kovács, *Tensor factorizations of group algebras and modules*, J. Algebra, to appear. Also, Australian National University Mathematics Research Report No. MRR 017-94.