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KARIN ERDMANN AND L. G. KOVÁCS

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ABSTRACT. In a recent paper, Jon F. Carlson and the second author showed that all tensor factorizations of the group algebra of a finite abelian p-group over a field of characteristic p are related to direct decompositions of the group. They asked whether over such a field the group algebra of a nonabelian directly indecomposable p-group can ever be written as a nontrivial tensor product, and gave a negative answer for groups of order 8. We extend this here by proving that the answer is negative if the group can be generated by two elements or if the group has cyclic centre and nilpotency class 2.

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Postal addresses:

The Mathematical Institute, 24–29 St Giles, Oxford OX1 3BL, England
(*E-mail*: erdmann@vax.ox.ac.uk)

School of Mathematical Sciences, Australian National University, Canberra, ACT 0200, Australia
(*E-mail*: kovacs@maths.anu.edu.au)

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Some tensor-indecomposable modular group algebras

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In a recent paper [1], Jon F. Carlson and the second author showed that all tensor factorizations of the group algebra of a finite abelian p-group over a field of characteristic p are related to direct decompositions of the group. As a first step towards exploring whether their results may be extended to group algebras of nonabelian groups, they asked whether over such a field the group algebra of a nonabelian directly indecomposable p-group can ever be written as a nontrivial tensor product. They gave a negative answer for groups of order 8. We extend this here by proving that the answer is negative if the group can be generated by two elements or if the group has cyclic centre and nilpotency class 2.

All algebras considered will be finite dimensional associative algebras with 1, over (commutative) fields. Algebra homomorphisms must respect 1, subalgebras must have the same 1 as the whole algebra. An algebra generated by elementwise commuting subalgebras is a homomorphic image of the 'external' tensor product of those subalgebras. The corresponding natural homomorphism is an isomorphism, and the algebra is the 'internal' tensor product of these subalgebras, if and only if the product of the dimensions of the subalgebras is no larger than the dimension of the algebra. To give a 'tensor factorization' amounts to naming such subalgebras. The tensor product (or tensor factorization) is 'nontrivial' if no tensor factor has dimension 1. An algebra is said to be tensor-indecomposable if it has no nontrivial tensor factorization.

Let us start with some general observations. If the radical of an algebra has codimension 1, then the same is true for each subalgebra (because each subalgebra has a nilpotent ideal of codimension 1), and so in particular it is true for each tensor factor of the algebra. In any algebra whose radical is nonzero and of codimension 1, the annihilator of the radical is central and nonzero, so the dimension of the center is greater than 1. By Lemma 5.1 of [1], the centre of a tensor product is the tensor product of the centres of the tensor factors. The preceding comments show that if an algebra whose radical has codimension 1 has a nontrivial tensor factorization, then the factorization of the centre as tensor product of the centres of the tensor factors is also nontrivial. Hence in proving such an algebra tensor-indecomposable it is sufficient to show that its centre is tensor-indecomposable.

In Lemma 2.6 of [1] and in the discussion leading to it, commutativity was assumed but not used seriously until the claim that $a \mapsto a^p$ is a ring endomorphism. The following weaker variant is therefore available even in the noncommutative case.

LEMMA 1. *Let $\mathbb{F}$ be any field and A any $\mathbb{F}$-algebra whose radical has codimension 1. If X is a subalgebra such that $X + (\operatorname{rad} A)^2 = A$, then $X = A$. If $A = B \otimes C$, then $(\operatorname{rad} A)/(\operatorname{rad} A)^2$ is the direct sum of $(\operatorname{rad} B + (\operatorname{rad} A)^2)/(\operatorname{rad} A)^2$ and $(\operatorname{rad} C + (\operatorname{rad} A)^2)/(\operatorname{rad} A)^2$. If the tensor factorization of A is nontrivial, then so is this direct decomposition of $(\operatorname{rad} A)/(\operatorname{rad} A)^2$.* □

To complete preparations for our first result, it remains to recall that if p is a prime, $\mathbb{F}$ is a field of characteristic p, and G is a group of p-power order, and if the cosets containing $g_1, \dots, g_r$ form a basis for the Frattini factor group of G, then the cosets containing $g_1 - 1, \dots, g_r - 1$ form a basis for $(\operatorname{rad} \mathbb{F}G)/(\operatorname{rad} \mathbb{F}G)^2$.

THEOREM 1. *If p is a prime, $\mathbb{F}$ is a field of characteristic p, and G is a group of p-power order generated by two elements which do not commute with each other, then the group algebra $\mathbb{F}G$ is tensor-indecomposable.*

Proof. By the preceding comment, in this case $(\operatorname{rad} \mathbb{F}G)/(\operatorname{rad} \mathbb{F}G)^2$ is 2-dimensional. Suppose that $\mathbb{F}G = B \otimes C$ is a nontrivial tensor factorization. By the second part of Lemma 1, one can then choose an element b from $\operatorname{rad} B$ and an element c from $\operatorname{rad} C$ such that their cosets together span $(\operatorname{rad} \mathbb{F}G)/(\operatorname{rad} \mathbb{F}G)^2$. By the first part of that lemma, the subalgebra generated by b and c is $\mathbb{F}G$ itself. On the other hand, this subalgebra is commutative (because B and C commute elementwise). This contradicts the assumption that G is nonabelian. □

Further preparations are needed for the second result. For an algebra A and a nonnegative integer i, set $\alpha_i = \dim(\operatorname{rad} A)^i/(\operatorname{rad} A)^{i+1}$, with the convention that $(\operatorname{rad} A)^0 = A$. Note that $\alpha_0, \alpha_1, \dots$ is a sequence nonnegative integers almost all of which are 0, and that once one term is 0 then so are all subsequent terms.

LEMMA 2. *If A admits a nontrivial tensor factorization and if $\alpha_i = 1$ with $i \neq 0$, then $\alpha_{i+1} = 0$.*

Proof. Suppose that $A = B \otimes C$ is a nontrivial tensor factorization, but let i be any nonnegative integer at first (that is, do not assume $\alpha_i = 1$ for the time being). Choose a basis X for B so that each $(\operatorname{rad} B)^j$ is spanned by the elements of X that it contains, and choose a basis Y for C similarly. Of course then $\{ x \otimes y \mid x \in X,\ y \in Y \}$ is a basis for A. It is not hard to see that $(\operatorname{rad} A)^i$ is spanned by the $x \otimes y$ such that for some j we have $x \in X \cap (\operatorname{rad} B)^j$ and $y \in Y \cap (\operatorname{rad} C)^{i-j}$. This yields that

$$\alpha_i = \sum_{j=0}^{i} \beta_j \gamma_{i-j}. \tag{1}$$

Let r, s be chosen so that $\beta_r > \beta_{r+1} = 0$ and $\gamma_s > \gamma_{s+1} = 0$. Consider the summands on the right hand side of (1): $\beta_j \gamma_{i-j}$ is nonzero if and only if

$0 \leqslant j \leqslant r$ and $0 \leqslant i-j \leqslant s$. This shows that if $i > r+s$ then $\alpha_i = 0$ because all summands are 0. Similarly, if $0 < i < r+s$ then $\alpha_i > 1$ because more than one summand is positive, except when $rs = 0$. Only in this exceptional case do we make use of the assumption that the tensor factorization $A = B \otimes C$ is nontrivial: it guarantees that if $r = 0$ then $\beta_0 > 1$ while if $s = 0$ then $\gamma_0 > 1$. In either sub-case, $\alpha_i > 1$ because the unique nonzero summand ($\beta_0\gamma_i$ or $\beta_i\gamma_0$, respectively) is greater than 1. It follows that if $i > 0$ and $\alpha_i = 1$ then $i = r+s$ and $\alpha_{i+1} = 0$, as required. □

We can now proceed to the second result of this note.

THEOREM 2. *If p is a prime, $\mathbb{F}$ is a field of characteristic p, and G is a nonabelian group of p-power order with nilpotency class 2 and cyclic centre, then the group algebra $\mathbb{F}G$ is tensor-indecomposable.*

(Of course if the word 'nonabelian' is omitted the conclusion remains valid, for then G itself is cyclic and what we have is a very easy special case of Theorem 2.1 of [1].)

Proof. Let $Z(G) = \langle z \rangle$ and $|G| = p^m$, $|z| = p^n$. Since G is nonabelian, $m \geqslant 3$. Write $Z(\mathbb{F}G) = A$: as we noted in the introduction, it will be sufficient to prove that A is tensor-indecomposable.

We shall deal first with the case $n = 1$. Denote by e the sum of the elements of $Z(G)$. If g is a noncentral element of G, then the sum of the conjugates of g is just ge. There are $(p^m - p)/p$ such class sums; together with e itself, they span a subspace of dimension p^{m-1}: call that U. (Of course, $\dim A = p^{m-1} + p - 1$.) Using that $e^2 = 0$, one readily sees that U annihilates $\operatorname{rad} A$. Suppose that $A = B \otimes C$. Since products of basis elements of B with basis elements of C are basis elements of $B \otimes C$ and so cannot be 0 while U annihilates $\operatorname{rad} B$, it is clear that $U \cap C = 0$. The codimension of U in A being $p-1$, this implies that $\dim C \leqslant p-1$. Similarly, $\dim B \leqslant p-1$, and so $p^{n-1} + p - 1 = \dim A = (\dim B)(\dim C) \leqslant (p-1)^2$. As $m \geqslant 3$, this is impossible, and so A must be tensor-indecomposable.

To deal with another exceptional case, let $p^n = 4$. Then the sum of the elements of $\langle z \rangle$ is $(z-1)^3$ and the sum of the elements of $\langle z^2 \rangle$ is $(z-1)^2$, so $\operatorname{rad} A$ has a basis consisting of $z-1$ and of elements of the form $g(z-1)^2$ or $g(z-1)^3$ with $g \in G$. Call V the subspace spanned by the basis elements other than $z-1$, and note that the product of any two elements of V is 0. It is now easy to see that if $d \in (\operatorname{rad} A) \setminus V$ then $d^3 = (z-1)^3 \neq 0$, whence it follows that

(2) no subalgebra D of A can have $(\operatorname{rad} D)^2 > (\operatorname{rad} D)^3 = 0$.

Of course, we have $(\operatorname{rad} A)^3 > (\operatorname{rad} A)^4 = 0$, and therefore $A = B \otimes C$ would

imply that $(\operatorname{rad} A)^3 = (\operatorname{rad} B)^k \otimes (\operatorname{rad} C)^{3-k}$ for some k such that

$$\begin{gathered} 0 < k < 3, \\ (\operatorname{rad} B)^k > (\operatorname{rad} B)^{k+1} = 0, \text{ and} \\ (\operatorname{rad} C)^{3-k} > (\operatorname{rad} C)^{4-k} = 0. \end{gathered}$$

If $k = 2$ the second line contradicts (2) with $D = B$, while if $k = 1$ the third line contradicts (2) with $D = C$. The first line allows no other value for k, so we conclude that A is tensor-indecomposable in this case as well.

For the sequel we may therefore assume that $n > 1$ and $p^n > 4$, so that

$$p^{n-1} + 1 < p^n - 1. \tag{3}$$

Given that G has class 2, for any given g in G the map $G \to Z(G)$, $x \mapsto [g,x]$ is a homomorphism, and therefore the sum of the conjugates of g is ge where e is the sum of the elements of some subgroup of $Z(G)$. Each such subgroup is of the form $\langle z^{p^k} \rangle$ with $0 \leqslant k \leqslant n$. As is well known and easy to see,

$$\binom{p^{n-k} - 1}{i} \equiv (-1)^i \mod p,$$

whence

$$(z-1)^{p^{n-k}-1} = \sum_{i=0}^{p^{n-k}-1} z^i. \tag{4}$$

Using that $x \mapsto x^{p^k}$ is an endomorphism of every commutative ring of characteristic p, we see from (4) that the sum of the elements of $\langle z^{p^k} \rangle$ is $(z-1)^{p^n - p^k}$. The conclusion we have been aiming for is that A has a basis consisting of powers of z and of elements of the form $g(z-1)^{p^n-p^k}$ with $g \in G \setminus Z(G)$ and $k < n$. As $(z-1)^{p^n} = 0$, this yields that $\operatorname{rad} A$ has a basis consisting of powers of $(z-1)$ and of elements of the form $g(z-1)^t$ with $g \in G$ and $t \geqslant p^n - p^{n-1}$.

If $r > p^{n-1}$ and if a product of r factors, each factor an element of this basis, has a factor of the form $g(z-1)^t$, then the product is a multiple of $(z-1)^{p^n}$ and is therefore 0. Thus $(\operatorname{rad} A)^r$ is spanned by the $(z-1)^s$ with $s \geqslant r$. This is where we use (3): it enables us to conclude that (in the notation of Lemma 2) $\alpha_i = \alpha_{i+1} = 1$ for $i = p^{n-1} + 1$. In view of Lemma 2, this proves that A is tensor-indecomposable. □

Reference

1. Jon F. Carlson and L. G. Kovács, *Tensor factorizations of group algebras and modules*, J. Algebra, to appear. Also, Australian National University Mathematics Research Report No. MRR 017-94.