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Unitary units in modular group algebras

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Let p be a prime, K a field of characteristic p, G a locally finite p-group, KG the group algebra, and V the group of the units of KG with augmentation 1. The anti-automorphism $g \mapsto g^{-1}$ of G extends linearly to KG; this extension leaves V setwise invariant, and its restriction to V followed by $v \mapsto v^{-1}$ gives an automorphism of V. The elements of V fixed by this automorphism are called *unitary*; they form a subgroup. Our first theorem describes the K and G for which this subgroup is normal in V.

For each element g in G, let $\overline{g}$ denote the sum (in KG) of the distinct powers of g. The elements $1+(g-1)h\overline{g}$ with g, $h \in G$ are the *bicyclic* units of KG. Our second theorem describes the K and G for which all bicyclic units are unitary.

1. Introduction

Let KG be the group algebra of a group G over a commutative ring K (with 1) and $V(KG)$ the group of normalized units (that is, of the units with augmentation 1) in KG. The anti-automorphism $g \mapsto g^{-1}$ extends linearly to an anti-automorphism $a \mapsto a^*$ of KG; this extension leaves $V(KG)$ setwise invariant, and its restriction to $V(KG)$ followed by $v \mapsto v^{-1}$ gives an automorphism of $V(KG)$. The elements of $V(KG)$ fixed by this automorphism are the *unitary normalized units* of KG; they form a subgroup which we denote by $V_*(KG)$. (Interest in unitary units arose in algebraic topology, and a more general definition, involving an 'orientation homomorphism', is also current; the special case we use here arises when the orientation homomorphism is trivial.)

The first question considered here is to find the pairs K, G for which $V_*(KG)$ is normal in $V(KG)$. (Since each unit of a group algebra is a scalar multiple of a normalized unit, if $V_*(KG)$ is normal in $V(KG)$ then it is normal also in the group of all units of KG.) For $K = \mathbb{Z}$, this question was discussed

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(without any restriction on the orientation homomorphism) by A. A. Bovdi and S. K. Sehgal in [4]. Here we deal with the 'modular' case, that is, with the case of K a field of prime characteristic p and G a locally finite p-group.

Theorem 1.1. *Let K be a field of prime characteristic p and let G be a nonabelian locally finite p-group. The subgroup $V_*(KG)$ is normal in $V(KG)$ if and only if $p = 2$ and G is the direct product of an elementary abelian group with a group H for which one of the following holds:*

(i) *H has no direct factor of order 2, but it is a semidirect product of a group $\langle h \rangle$ of order 2 and an abelian 2-group A, with $h^{-1}ah = a^{-1}$ for all a in A;*

(ii) *H is an extraspecial 2-group, or the central product of such a group with a cyclic group of order 4.*

We work with the definition that a p-group is extraspecial if its centre, commutator subgroup and Frattini subgroup are equal and have order p: we do not require the group itself to be finite.

The proof of this theorem will be given in Section 2. The reason we take the p-group G locally finite is that, as is well known, this ensures that each non-unit of KG lies in the augmentation ideal.

Every group G may be written (see Lemma 2.3) as a direct product of an elementary abelian 2-group E and a group H which has no direct factor of order 2 (we do not exclude $E = 1$ or $H = 1$). The isomorphism type of G determines the isomorphism types of E and H, and vice versa.

It is easy to verify that if H satisfies (i) then $A = \langle a \in H \mid a^2 \neq 1 \rangle$ and A has no direct factor of order 2. Conversely, if A is a nontrivial abelian 2-group without a direct factor of order 2 and H is formed as the semidirect product indicated, then (i) holds. The classification of the groups H of this kind is thus reduced to the classification of abelian 2-groups, a problem whose solution in terms of Ulm invariants is well known in the finite or countably infinite case but is beyond reach in general.

As to case (ii), the classification of finite extraspecial groups is well known. Equally conclusive results were obtained for extraspecial groups of countably infinite order by M. F. Newman in [9]; he also showed there that no such results can be expected for extraspecial groups of arbitrary order.

The only group H which satisfies both conditions (i) and (ii) is the dihedral group of order 8.

The second part of the paper concerns the bicyclic units introduced in Ritter and Sehgal [11]. For K a commutative ring and g an element of finite order $|g|$ in a group G, let $\overline{g}$ denote the sum (in KG) of the distinct powers of g:

$$\overline{g} = \sum_{i=0}^{|g|-1} g^i.$$

If also $h \in G$, put

$$u_{g,h} = 1 + (g-1)h\overline{g}.$$

Note that $1-(g-1)h\overline{g}$ is a two-sided inverse for $u_{g,h}$ and the augmentation of $u_{g,h}$ is 1, so $u_{g,h}$ is a normalized unit. The elements of this form are called *bicyclic units.*

The problem considered here is to find the K and G for which each bicyclic unit of KG is unitary. It is easy to see that $u_{g,h}=1$ if and only if the cyclic group $\langle g\rangle$ is normalized by h, and that if $K=\mathbb{Z}$ then 1 is the only bicyclic unit which is unitary. Thus the G which can partner $K=\mathbb{Z}$ are precisely the groups in which every subgroup of finite order is normal. (The situation is not so simple for $K=\mathbb{Z}$ when unitarity is defined with reference to a nontrivial orientation homomorphism: see A. A. Bovdi and S. K. Sehgal [3].) In private communication, Professor Sehgal has directed attention to the modular case: what can one say when K is of characteristic p and G is a p-group?

Theorem 1.2. *Let p be a prime, K a commutative ring of characteristic p, and G a nonabelian p-group. All bicyclic units of KG are unitary if and only if $p=2$ and G is the direct product of an elementary abelian group and a group H for which one of the following holds*:

(i) *H has an abelian subgroup A of index 2 such that conjugation by an element of H outside A inverts each element of A*;

(ii) *H is an extraspecial 2-group, or the central product of such a group with a cyclic group of order 4*;

(iii) *H is the direct product of a quaternion group of order 8 and a cyclic group of order 4, or the direct product of two quaternion groups of order 8*;

(iv) *H is the central product of the group $\langle x,y \mid x^4=y^4=1,\ x^2=[y,x]\rangle$ with a quaternion group of order 8, the nontrivial element common to the two central factors being x^2y^2*;

(v) *H is isomorphic to one of the groups H_{32} and H_{245} defined below.*

The relevant definitions are:

$$\begin{aligned} H_{32} = \langle\, x,y,u \mid{}& x^4=y^4=1,\\ & x^2=[y,x],\\ & y^2=u^2=[u,x],\\ & x^2y^2=[u,y]\,\rangle,\\ H_{245} = \langle\, x,y,u,v \mid{}& x^4=y^4=[v,u]=1,\\ & x^2=v^2=[y,x]=[v,y],\\ & y^2=u^2=[u,x],\\ & x^2y^2=[u,y]=[v,x]\,\rangle. \end{aligned}$$

The subscripts are the serial numbers of these groups in the CAYLEY library of groups of order dividing 128 described by Newman and O'Brien in [10]. It is a mere coincidence that H_{32} has order 32. The other group, H_{245}, is one of the two Suzuki 2-groups (see § VIII.7 in [7]) of order 64. That CAYLEY library

provided a key step in the first version of our proof of Theorem 1.2: we are indebted to Dr E. A. O'Brien for extracting from it the list (Lemma 4.1) of the groups H of order dividing 128 whose Frattini subgroup is central, noncyclic, of order 4, and contains all elements of order 2 in H. His list showed that all groups of this kind have order dividing 64. Professor P. P. Pálfy then drew our attention to the fact that this much follows already from a result of Blackburn (Theorem VIII.5.4 in [7]), even without assuming that the 2-group in question has order dividing 128. Once this is available, we do not need CAYLEY: Lemma 4.1 can be read off the tables of Hall and Senior [6], where H_{32} and H_{245} are labelled $32\,\Gamma_4 c_3$ and $64\,\Gamma_{13} a_5$.

Several of the comments we made after Theorem 1.1 apply here as well. In (i) we can assume that H has no direct factor of order 2: the isomorphism type of H is then determined by G. We may also assume there that $|H| > 8$, for the two nonabelian groups of order 8 occur also under (ii). It is easy to see that then H has only one abelian subgroup of index 2, so the isomorphism type of A is in turn determined by H. Moreover, the squares of all the elements of H outside A are equal to each other, and this element, a_0 say, has order at most 2. Of course the height of a_0 in A is also an isomorphism invariant of H. Mackey's proof of Ulm's Theorem (given in Kaplansky [8]) shows that if two countable p-groups have the same Ulm invariants and we are given a height-preserving isomorphism from a finite subgroup of one to a subgroup of the other, then this will extend to an isomorphism of the two groups. It follows that in the finite or countably infinite case the Ulm invariants of A together with the height in A of the common square of the elements outside A form a *complete* set of invariants for H. Conversely, if a_0 is any element of order at most 2 in an abelian 2-group A with $|A| > 4$, then the group H defined by

$$H = \big\langle A, h \;\big|\; h^2 = a_0,\ h^{-1}ah = a^{-1} \text{ for all } a \text{ in } A \big\rangle$$

satisfies (i) and $|H| > 8$. (The reader may like to work out how the relevant invariants must be restricted to ensure that H is nonabelian and has no direct factor of order 2.)

The proof of Theorem 1.2 splits naturally into a ring-theoretic part and a group-theoretic part, which are presented in Section 3 and Section 4. The ring-theoretic step also confirms the comment above that in $\mathbb{Z}G$ the only unitary bicyclic unit is 1. (In fact, $V_*(\mathbb{Z}G)$ is always G itself, as was observed in Lemma 1 of A. A. Bovdi [1].)

Lemma 1.3. *Let K be a commutative ring with 1 and G a group. Suppose that g is an element of finite order in G and h is an element of G which does not normalize $\langle g\rangle$. The bicyclic unit $u_{g,h}$ is unitary if and only if the characteristic of K is 2 while $\langle g^2\rangle$ is normalized by h and contains either h^2 or $(hg)^2$.*

All Hamiltonian p-groups are 2-groups, so this implies that p cannot be odd in Theorem 1.2, and the proof of that theorem is reduced to the following.

Lemma 1.4. *Let G be a nonabelian 2-group such that if $g, h \in G$ then $\langle g^2 \rangle$ is normal in G and $\langle g, h\rangle / \langle g^2 \rangle$ is either abelian or dihedral. Then G is the direct product of an elementary abelian group with a nonabelian group H for which one of the conditions* (i)–(v) *of Theorem* 1.2 *holds. Conversely, every 2-group G of this kind satisfies our hypotheses.*

2. Normal unitary subgroup

The aim of this section is to prove Theorem 1.1. It will be convenient to use $y^* = y^{-1}$ as the test of whether a unit y is unitary. At first, K can be any commutative ring with 1 and G any group.

Lemma 2.1. *For $x \in V(KG)$ and $y \in V_*(KG)$, we have $x^{-1}yx \in V_*(KG)$ if and only if xx^* commutes with y.*

Proof. Clearly, $(x^{-1}yx)^* = (x^{-1}yx)^{-1}$ means that $x^*y^*(x^*)^{-1} = x^{-1}y^{-1}x$ which in turn is equivalent to $xx^*y^* = y^{-1}xx^*$. As we are given that $y^* = y^{-1}$, this proves the lemma. □

As $G \leq V_*(KG)$, an element which commutes with every element of $V_*(KG)$ is central in KG. Thus Lemma 2.1 gives the following.

Corollary 2.2. *The subgroup $V_*(KG)$ is normal in $V(KG)$ if and only if all elements of the form xx^* with $x \in V(KG)$ are central in KG.* □

(For the case of $K = \mathbb{Z}$, this is a special case of Lemma 2 of [4]; the proof we have given comes from that paper.)

Proof of Theorem 1.1. From the simple fact that over a field of characteristic p a finite p-group has only one irreducible representation, it follows readily that under the hypotheses of the theorem the augmentation ideal of KG is locally nilpotent and so each element outside that ideal is a unit. Differently put, if x is a non-unit in KG then $1 + x \in V(KG)$.

Suppose first that $V_*(KG)$ is normal in $V(KG)$. By Corollary 2.2, if x is a normalized unit then xx^* is central. As each unit is a scalar multiple of a normalized unit, the same conclusion is available whenever x is a unit. It follows that if x is any unit then $xx^* = x^{-1}(xx^*)x = x^*x$. If x is a non-unit in KG, then $1 + x$ is a unit and so $(1 + x)(1 + x)^* = (1 + x)^*(1 + x)$, whence again $xx^* = x^*x$.

A group algebra in which $xx^* = x^*x$ holds for every element x is called *normal.* A.A. Bovdi, P.M. Gudivok and M.S. Semirot proved in [2] that the group algebra of a nonabelian group G over a commutative ring K is normal if and only if either G is hamiltonian or the characteristic of K is 2 and G is a direct product of an elementary abelian 2-group with a group H such that (i) or (ii) holds. Thus the proof of our 'only if' claim is complete.

Suppose next that $p = 2$ and $G = E \times H$ with E elementary abelian and H satisfying (i) or (ii). In view Corollary 2.2, what we have to show is that xx^* is central whenever $x \in V(KG)$.

Consider first the case (i). Then each element x of KG can be written as $x = x_1 + x_2 h$ with $x_1, x_2 \in K(E \times A)$, and of course $hx_ih = x_i^*$. Using again that $h^2 = 1$, that $K(E \times A)$ is commutative, and that the characteristic is 2, we see that $xx^* = (x_1 + x_2 h)(x_1^* + hx_2^*) = x_1x_1^* + 2x_1x_2h + x_2x_2^* = x_1x_1^* + x_2x_2^*$. Thus xx^* lies in the commutative algebra $K(E \times H)$ and is easily seen to commute with h, so it is central in KG.

Consider next the case (ii). Then the commutator subgroup of G has only one nontrivial element; call that c, and write I for the ideal of KG generated by $1 + c$. This element c is central in G, while if $g, h \in G$ then either $hg = gh$ or $hg = ghc$: so either $hg(1 + c) = gh(1 + c) = g(1 + c)h$ or $hg(1 + c) = ghc(1 + c) = gh(1 + c) = g(1 + c)h$ proves that $g(1 + c)$ commutes with h. It follows that every element of I, and therefore also every element of $1 + I$, is central in KG.

Let γ be the natural homomorphism of KG onto $K(G/\langle c\rangle)$ defined by $g\gamma = g\langle c\rangle$ for all g in G. Of course γ intertwines the augmentation maps of the two group algebras, so if x is a normalized unit in KG then $x\gamma$ is a normalized unit in $K(G/\langle c\rangle)$. Further, γ intertwines the anti-automorphism $*$ of KG with the similarly defined anti-automorphism of $K(G/\langle c\rangle)$; we shall use $*$ also for the latter anti-automorphism. Note that $K(G/\langle c\rangle)$ is elementary abelian. It is an easy exercise to see that in a characteristic 2 group algebra of an elementary abelian 2-group each normalized unit is unitary. In particular, if $x \in V(KG)$ then $x\gamma$ is unitary, so $(xx^*)\gamma = (x\gamma)(x^*\gamma) = (x\gamma)(x\gamma)^* = 1$, that is, $xx^* \in 1 + \ker\gamma$. Since I is minimal among the ideals for which $c \equiv 1 \mod I$, it is precisely $\ker\gamma$. We have proved that $xx^* \in 1 + I$. By the conclusion of the previous paragraph, xx^* is therefore central in KG. The proof of the theorem is now complete. □

Remarks. On any group algebra of an elementary abelian 2-group, the anti-automorphism $*$ is in fact just the identity map.

For a generalization of the result of [2], see [5].

We conclude this section with a purely group-theoretic lemma which was mentioned in the introduction's comments on Theorem 1.1.

Lemma 2.3. *Every group G is a direct product $E \times H$ of an elementary abelian 2-group E and a group H which has no direct factor of order 2. If $G = E_1 \times H_1$ is another such decomposition of G, then $E_1 \cong E$ and $H_1 \cong H$.*

Proof. Let $\mathsf{Z}(G)$ denote the centre of G; set $A = \mathsf{A}(G) = \langle a \in \mathsf{Z}(G) \mid a^2 = 1\rangle$ and $B = \mathsf{B}(G) = \langle g^2 \mid g \in G\rangle$. Let E be a direct complement to $A \cap B$ in A, and H/B a direct complement to AB/B in G/B: then

$$G = ABH = EBH = EH \quad \text{while} \quad E \cap H \leq E \cap AB \cap H = E \cap B = 1,$$

so $G = E \times H$. Here E is elementary abelian because A is. If $H = C \times K$ with $|C| \leq 2$, then $G = E \times C \times K$ and $B = \mathsf{B}(K)$, so $C \leq A = E \times (A \cap B) \leq E \times K$

yields that $C = 1$. This proves that H has no direct factor of order 2, that is, $\mathsf{A}(H) \le \mathsf{B}(H)$. If $G = E_1 \times H_1$ is another decomposition with E_1 elementary abelian and $\mathsf{A}(H_1) \le \mathsf{B}(H_1)$, then $E_1 \cong A/(A \cap B) \cong E$, and H_1/B is another direct complement to AB/B in G/B: so we also have $G = E \times H_1$ and therefore $H_1 \cong G/E \cong H$. □

3. Unitary bicyclic units

The aim of this section is to prove Lemma 1.3. Accordingly, K is once again an arbitrary commutative ring with 1 and G is an arbitrary group. Recall the definition $u_{g,h} = 1 + (g-1)h\overline{g}$, and note that $u_{g,h} = u_{g,hg} \in V(KG)$, with

$$u_{g,h}^{-1} = 1 - (g-1)h\overline{g} \qquad \text{and} \qquad u_{g,h}^{*} = 1 + \overline{g}h^{-1}(g^{-1} - 1).$$

The *support* of an element a of KG is the set of those elements of G which occur with nonzero coefficient in the expression of a as K-linear combination of elements of G:

$$\operatorname{supp} \textstyle\sum_{g \in G} \alpha_g g = \{\, g \in G \mid \alpha_g \neq 0 \,\}.$$

Two simple observations about bicyclic units will be used without reference. First, if h normalizes $\langle g \rangle$ then $h\overline{g} = \overline{g}h$ and so $u_{g,h} = 1$. Second, if h does not normalize $\langle g \rangle$ then $u_{g,h} \neq 1$; indeed, in this case no element of G can occur more than once in the expansion of $1 + (g-1)h\overline{g}$, so the support of $u_{g,h}$ has cardinality $1 + |g| + |g|$. Explicitly, if $h \notin \mathsf{N}(\langle g \rangle)$ then

$$\operatorname{supp} u_{g,h} = \{1\} \cup \{\, ghg^i \mid 0 \le i < |g| \,\} \cup \{\, hg^i \mid 0 \le i < |g| \,\}.$$

Proof of Lemma 1.3. Suppose that $u_{g,h}^{*} = u_{g,h}^{-1}$. If the characteristic of K were not 2, we could argue that in the expression of $u_{g,h}^{-1}$ as $1 - (g-1)h\overline{g}$ both h and hg have coefficient 1 while in that of $u_{g,h}^{*}$ the only nontrivial elements of G with coefficient 1 are the $g^i h^{-1} g^{-1}$, hence there exist i, j such that $h = g^i h^{-1} g^{-1}$ and $hg = g^j h^{-1} g^{-1}$, and then

$$hgh^{-1} = \left(g^j h^{-1} g^{-1}\right)\left(g^i h^{-1} g^{-1}\right)^{-1} = g^{j-i} \in \langle g \rangle$$

contradicts the assumption that $h \notin \mathsf{N}(\langle g \rangle)$. Thus the characteristic of K is 2.

Note that $|g^2|$ can only be $|g|$ or $|g|/2$. We exploit this repeatedly, for it yields that once we show $h^{-1}g^2h \in \langle g \rangle$, it follows that h normalizes $\langle g^2 \rangle$. Namely, if $h^{-1}g^2h \in \langle g \rangle$ then $h \notin \mathsf{N}(\langle g \rangle)$ implies that $\langle h^{-1}g^2h \rangle < \langle g \rangle$, whence $|h^{-1}g^2h| = |g^2| = |g|/2$, and then $\langle h^{-1}g^2h \rangle$ must be the unique subgroup, $\langle g^2 \rangle$, of index 2 in $\langle g \rangle$.

Since $u_{g,h} \neq 1$, the support of $u_{g,h}^{-1}$ is given by

$$\operatorname{supp} u_{g,h}^{-1} = \{1\} \cup \{\, ghg^i \mid 0 \le i < |g| \,\} \cup \{\, hg^i \mid 0 \le i < |g| \,\},$$

while

$$\text{supp } u^*_{g,h} = \{1\} \cup \{\, g^i h^{-1} g^{-1} \mid 0 \le i < |g| \,\} \cup \{\, g^i h^{-1} \mid 0 \le i < |g| \,\}.$$

Given our assumption that $u^*_{g,h} = u^{-1}_{g,h}$, these two supports are equal. We now distinguish a number of cases according to the form in which various elements of supp $u^{-1}_{g,h}$ appear in supp $u^*_{g,h}$.

Suppose first that $h = g^i h^{-1}$, so $ghg = g^{i+1}h^{-1}g$. If $ghg = g^j h^{-1}$, then $h^{-1}gh = g^{j-i-1} \in \langle g\rangle$, contrary to $h \notin \mathsf{N}(\langle g\rangle)$. Thus $ghg = g^j h^{-1} g^{-1}$, and then $h^{-1}g^2h = g^{j-i-1} \in \langle g\rangle$, so h normalizes $\langle g^2\rangle$. Of course now $h^2 = g^i \in \langle g\rangle$, so $h^2 \notin \langle g^2\rangle$ would imply that $\langle g\rangle = \langle\langle g^2\rangle, h^2\rangle$, which is impossible because $h \in \mathsf{N}(\langle g^2\rangle)$ but $h \notin \mathsf{N}(\langle g\rangle)$. This proves that in this case $h^2 \in \langle g^2\rangle$.

Suppose next that $h = g^i h^{-1} g^{-1}$, and note that $h \notin \mathsf{N}(\langle g\rangle)$ implies that $i \neq 0$. If $|g| = 2$ then this forces $i = 1$, so conjugation by g inverts h and therefore $(hg)^2 = 1$. Suppose that $|g| > 2$; then $g^i h^{-1} g = hg^2 \in$ supp $u^{-1}_{g,h}$. If $hg^2 = g^j h^{-1}$, then $h^{-1}gh = g^{j-i} \in \langle g\rangle$, contrary to $h \notin \mathsf{N}(\langle g\rangle)$. Thus $hg^2 = g^j h^{-1} g^{-1}$, and then $h^{-1}g^2h = g^{j-i} \in \langle g\rangle$, so h normalizes $\langle g^2\rangle$. If i is even, then in the factor group $\langle g,h\rangle/\langle g^2\rangle$ the image of g is the square of the image of h, but this is impossible because $h \notin \mathsf{N}(\langle g\rangle)$. So i is odd, and then in $\langle g,h\rangle/\langle g^2\rangle$ conjugation by the image of g inverts the image of h and therefore the image of hg has order 2: thus again $(hg)^2 \in \langle g^2\rangle$.

This completes the proof of the 'only if' claim. For the proof of the 'if' claim, assume first that the characteristic of K is 2, and note that then each bicyclic unit of KG is its own inverse. Next, assume that $\langle g^2\rangle$ is normalized by h and contains either h^2 or $(hg)^2$. Since in any case $u_{g,h} = u_{g,hg}$, we may assume without loss of generality that in fact $h^2 \in \langle g^2\rangle$. Since $\langle g^2\rangle$ is normalized by h but $\langle g\rangle$ is not, $|g|$ must be even, whence

$$\overline{g} = (g+1)\overline{g^2} = \overline{g^2}(g^{-1}+1).$$

Further, as both g and h normalize $\langle g^2\rangle$, both commute with $\overline{g^2}$, while $h^2 \in \langle g^2\rangle$ implies that $h^2\overline{g^2} = \overline{g^2}$ and so

$$h\overline{g^2} = \overline{g^2}h^{-1}.$$

Using again that the characteristic of K is 2, we can therefore argue that

$$\begin{aligned} u^{-1}_{g,h} = u_{g,h} &= 1 + (g+1)h\overline{g} \\ &= 1 + (g+1)h\overline{g^2}(g^{-1}+1) \\ &= 1 + (g+1)\overline{g^2}h^{-1}(g^{-1}+1) \\ &= 1 + \overline{g}h^{-1}(g^{-1}+1) \\ &= u^*_{g,h}\,, \end{aligned}$$

as required. $\square$

4. A certain class of groups

The rest of the paper will be taken up by the proof of Lemma 1.4.

As usual, the Frattini subgroup of a group H will be written $\Phi(H)$. Recall that if H is a finite 2-group then $\Phi(H) = \langle h^2 \mid h \in H \rangle$. If H is any 2-group, we write $\Omega(H) = \langle h \in H \mid h^2 = 1 \rangle$. As we mentioned in the introduction, Theorem VIII.5.4 of [7] and the tables of [6] together yield the following.

Lemma 4.1 (O'Brien). *The finite 2-groups H in which $\Phi(H)$ and $\Omega(H)$ are equal, central, and of order 4, are precisely the following: $C_4 \times C_4$, $C_4 \rtimes C_4$, $C_4 \rtimes Q_8$, and the groups named in parts* (iii)–(v) *of Theorem* 1.2. □

Here C_4 and Q_8 stand for a cyclic group of order 4 and a quaternion group of order 8, while $C_4 \rtimes C_4$ and $C_4 \rtimes Q_8$ indicate semidirect products which are not direct products (the last-named semidirect factor not being normal): in each of these two cases, there is only one isomorphism type of groups of this kind. Both groups satisfy condition (i) of Theorem 1.2.

It will be convenient to have a short temporary name for the 2-groups G such that if $g, h \in G$ then $\langle g^2 \rangle$ is normal in G and $\langle g, h\rangle / \langle g^2 \rangle$ is either abelian or dihedral: let us call these groups G *good*. Obviously, a group is good if and only if each of its two-generator subgroups is good, and so all subgroups of good groups are good. A little more thought shows that all factor groups of good groups are also good, and that the direct product of an elementary abelian 2-group with a good group is always good.

Of course, all abelian or dihedral 2-groups are good. The next exploratory step is to look (for example, by using [6]) at each of the groups of order dividing 16, and check that all but three of them are good. The three that fail do so because they are of the form $\langle g, h\rangle$ with $g^2 = 1$ but are neither abelian nor dihedral; they are $(C_2 \times C_2) \rtimes C_4$ and the two semidirect products $C_8 \rtimes C_2$ in which the action of C_2 on C_8 is neither trivial nor inverting. (The nonabelian $(C_2 \times C_2) \rtimes C_4$ form a single isomorphism class.) We note for future use that the reasons which make the generalized quaternion group of order 16 good but the semidihedral group of order 16 bad, yield the same conclusions for generalized quaternion groups and semidihedral groups of larger orders as well.

Proof of the last sentence of Lemma 1.4. In case (i), all two-generator subgroups of H are abelian or dihedral so G is good. In cases (ii)-(v) we have $|\Phi(H)| \leq 4$, so the two-generator subgroups K of H are of order dividing 16. No K can be a $C_8 \rtimes C_2$, because that would mean $\Phi(H) \geq \Phi(K) \cong C_4$ but $\Phi(H)$ contains no C_4. No K can be a $(C_2 \times C_2) \rtimes C_4$, because then $\Phi(H) \geq \Phi(K) \cong C_2 \times C_2$ would exclude case (ii) and so (see Lemma 4.1) ensure $|\Omega(H)| = 4$, contrary to $\Omega(K) \leq \Omega(H)$ and $|\Omega(K)| = 8$. Thus in every case we may conclude that G is good. This proves the converse part of Lemma 1.4. □

The proof of the direct part needs more preparation. To avoid cumbersome circumlocution, *we count Q_8 among the generalized quaternion groups.* As usual, $x^y = y^{-1}xy$, and an *involution* is a group element of order 2.

The first step is rather trivial, and the second is not much harder.

Lemma 4.2. *If H is a nonabelian good group with $|\Phi(H)| = 2$ and no direct factor of order 2, then it satisfies* (ii) *of Theorem* 1.2.

Proof. Since H has no direct factor of order 2, it has no central involution outside $\Phi(H)$; thus $|\Phi(H)| = 2$ implies that $\mathsf{Z}(H)$ is cyclic of order at most 4. If $|\mathsf{Z}(H)| = 2$ then H is extraspecial; otherwise there is a maximal subgroup M which does not contain $\mathsf{Z}(H)$ and is easily seen to be extraspecial. □

Lemma 4.3. *An involution in a good group normalizes every cyclic subgroup of order greater than 2 and centralizes every subgroup isomorphic to $C_4 \rtimes C_4$.*

Proof. Let g be an involution in a good group G. If h is any element of G, then by the definition of 'good' $\langle g,h\rangle$ is abelian or dihedral: in either case, if $\langle h\rangle$ is of order greater than 2 then it is normalized by g. Suppose now that x, y are elements of G such that $\langle x,y\rangle \cong C_4 \rtimes C_4$ with $x^y = x^{-1}$. If $x^g = x^{-1}$ and $y^g = y^{\pm 1}$, then g fails to normalize $\langle xy\rangle$, while if $x^g = x$ and $y^g = y^{-1}$, then yg is an involution which does not normalize $\langle xy\rangle$: so the only option is that g centralizes $\langle x,y\rangle$. □

We shall make repeated use of the fact that if $\langle x,y\rangle$ is a nonabelian dihedral 2-group then $\langle x\rangle$, $\langle y\rangle$, $\langle xy\rangle$ are pairwise distinct and precisely two of them are non-normal subgroups of order 2, while the third is normal and has order divisible by 4.

If $\langle g,h\rangle$ is good and $\langle g\rangle$ is not normal in it, then $\langle g,h\rangle/\langle g^2\rangle$ is a nonabelian dihedral 2-group, so it follows that $\langle g^2\rangle$ contains precisely one of $\langle h^2\rangle$ and $\langle (gh)^2\rangle$. Suppose further that neither $\langle h\rangle$ nor $\langle gh\rangle$ is normal in $\langle g,h\rangle$: then, by this argument, each of $\langle g^2\rangle$, $\langle h^2\rangle$ and $\langle (gh)^2\rangle$ must contain one and only one of the other two. As it is impossible to order a three-element set in this manner, we have a contradiction, which proves that at least one of $\langle g\rangle$, $\langle h\rangle$ and $\langle gh\rangle$ must be normal in $\langle g,h\rangle$. We have proved that *in a good group, every two-generator subgroup is metacyclic* (in the sense of having a cyclic normal subgroup with cyclic quotient).

If a nonabelian 2-group has a cyclic normal subgroup of order 4 with cyclic quotient, then it is isomorphic to one of

$$P_k = \left\langle\, w,y \;\middle|\; w^4 = 1,\ w^y = w^{-1},\ y^{2^k} = w^2 \,\right\rangle,$$
$$R_k = \left\langle\, w,y \;\middle|\; w^4 = 1,\ w^y = w^{-1},\ y^{2^{k+1}} = 1 \,\right\rangle.$$

(This notation is not intended for use beyond the proof of the next lemma.) We claim that *such a group is good if and only if $k \le 1$*. If $k \le 1$ then both groups are of order dividing 16 and we have said nothing new. If $k > 1$ then $P_k = \langle wy^{2^{k-1}}, y\rangle$ and $\left(wy^{2^{k-1}}\right)^2 = 1$ but P_k is neither abelian nor dihedral and so cannot be good. As P_k is a homomorphic image of R_k, in this case R_k cannot be good either.

Lemma 4.4. *If $\langle x,y\rangle$ is good and $\langle x\rangle$ is normal in it but $\langle y\rangle$ is not, then $|x| \ge 4 \ge |y|$ and $x^y = x^{-1}$.*

Proof. Since in $\langle x, y\rangle/\langle y^2\rangle$ the image of $\langle x\rangle$ is normal but the image of $\langle y\rangle$ is not, $\langle x, y\rangle/\langle y^2\rangle$ is a nonabelian dihedral group and the image of $\langle x\rangle$ has order divisible by 4: thus

$$x^2 \notin \langle y^2\rangle \tag{1}$$

and

$$x^y \equiv x^{-1} \mod \langle y^2\rangle. \tag{2}$$

Of course (2) and the normality of $\langle x\rangle$ give that

$$x^y \equiv x^{-1} \mod \langle x\rangle \cap \langle y^2\rangle. \tag{3}$$

From (1) we know that $\langle x\rangle \cap \langle y^2\rangle \leq \langle x^4\rangle$, so by (3)

$$x^y \equiv x^{-1} \mod \langle x^4\rangle. \tag{4}$$

It also follows from (1) that there is in $\langle x\rangle$ and element, w say, of order 4, and (4) implies that $w^y = w^{-1}$. Thus $\langle w, y\rangle$ is a P_k or an R_k, and the argument leading up to the lemma may be invoked for the conclusion that $y^4 = 1$. If $y^2 \notin \langle x\rangle$, then $\langle x\rangle \cap \langle y^2\rangle = 1$ and so (3) gives that $x^y = x^{-1}$. If $y^2 \in \langle x\rangle$, then $\langle x, y\rangle$ has a cyclic maximal subgroup and a nonabelian dihedral quotient, so it can only be dihedral or semidihedral or generalized quaternion. We have already seen that semidihedral groups are not good, so $x^y = x^{-1}$ holds in this case as well. Finally, $|x| \geq 4$ because $\langle x, y\rangle$ is nonabelian. This completes the proof of the lemma. □

It follows that under the hypotheses of Lemma 4.4 the group $\langle x, y\rangle$ is either dihedral or generalized quaternion or a semidirect product $\langle x\rangle \rtimes \langle y\rangle$ with $|x| \geq |y| = 4$ and $x^y = x^{-1}$.

Lemma 4.5. *If $\langle x, u\rangle$ is good and both $\langle x\rangle$ and $\langle u\rangle$ are normal in it, then $\langle x, u\rangle$ is either abelian or isomorphic to Q_8.*

Proof. We argue by contradiction. If the conclusion is false then $\langle x, u\rangle$ cannot be hamiltonian: so there is a cyclic subgroup, $\langle y\rangle$ say, which at least one of x and u fails to normalize. We may assume without loss of generality that x does not normalize $\langle y\rangle$, and then Lemma 4.4 is conveniently applicable. At first we only use the conclusion that $x^y = x^{-1}$. By conjugation, $\langle x, u\rangle$ induces a cyclic group of automorphisms on $\langle x\rangle$, and now we know that this includes the inverting automorphism. In the automorphism group of a cyclic 2-group, the subgroup generated by the inverting automorphism is a maximal cyclic subgroup (-1 is not a square mod 2^n when $n > 1$): so the group induced by $\langle x, u\rangle$ is of order 2. This proves that the centralizer $C_{\langle x,u\rangle}(x)$, which contains x but not y, is of index 2. Thus $y \notin \langle x, \Phi(\langle x, u\rangle)\rangle$, and therefore $\langle x, y\rangle = \langle x, u\rangle$. However, of the groups of Lemma 4.4, only Q_8 can be generated by two normal cyclic subgroups, and we have assumed that $\langle x, u\rangle$ is not that group. This contradiction completes the proof of Lemma 4.5. □

Since a good two-generator group is metacyclic, it has a generating set which satisfies the hypotheses of one of these two lemmas.

Corollary 4.6. *A two-generator 2-group is good if and only if it is either abelian or dihedral or generalized quaternion or a semidirect product $\langle x\rangle \rtimes \langle y\rangle$ with $|x| \geq |y| = 4$ and $x^y = x^{-1}$.* □

Lemma 4.7. *If H is a nonabelian good group of exponent greater than 4, then H satisfies* (i) *of Theorem* 1.2.

Proof. In the semidirect products of Corollary 4.6, all the elements outside the abelian group $\langle x, y^2\rangle$ have order 4, and every cyclic subgroup of $\langle x, y^2\rangle$ is normal in $\langle x, y\rangle$. In a dihedral or generalized quaternion group which does not have exponent 4, all cyclic subgroups of order greater than 4 are normal and lie in the unique cyclic maximal subgroup. It follows that in a good group every cyclic subgroup of order greater than 4 is normal and any two elements of order greater than 4 commute.

Let $A = \langle a \in H \mid a^4 \neq 1\rangle$: this is now an abelian subgroup of H. Let $a \in H$ with $a^4 \neq 1$, and $h \in H$ but $h \notin A$, so $h^4 = 1$. Then a and h cannot commute (else $(ah)^4 \neq 1$ and hence $ah \in A$, $h \in A$ would follow). Lemma 4.5 cannot apply with $x = a$, $u = h$, because $\langle a, h\rangle$ is neither abelian nor of exponent 4. Hence Lemma 4.4 must apply with $x = a$, $y = h$. It follows that every element of H outside A must invert every element of A. If H had more than one nontrivial coset modulo A, the quotient of two elements chosen from distinct nontrivial cosets would still lie outside A: it would have to fix as well as invert every element of A. This being impossible, the index of A in H must be 2. □

Lemma 4.8. *If H is a good group of exponent 4, then its Frattini subgroup is elementary abelian and central.*

Proof. If $g, h \in H$ then $[g^2, h] = 1$ because, by Corollary 4.6, $\langle g, h\rangle$ is either abelian or D_8 or Q_8 or $C_4 \rtimes C_4$, and $[g^2, h] = 1$ holds for every pair of elements g, h in each of these groups. This shows that the Frattini subgroup is generated by central involutions. □

In the proof of our next lemma, we shall make use of two properties of $C_4 \rtimes C_4$. First, it has only two nontrivial elements that are squares. Second, as it can be generated by two non-commuting elements of order 4 whose product is also of order 4, no automorphism of it can invert all elements of order 4.

Lemma 4.9. *If a, x, y are elements of a good group H of exponent 4 such that $x^y = x^{-1}$ and $a^2 \notin \langle x, y\rangle \cong C_4 \rtimes C_4$, then x must centralize and y must invert a. If also $b \in H$ and $a^2 \neq b^2 \notin \langle x, y\rangle$, then a and b commute.*

Proof. The subgroup $\langle a^2\rangle$ is normal and the image of $\langle x, y\rangle$ in the quotient $H/\langle a^2\rangle$ is still a $C_4 \rtimes C_4$. By Lemma 4.3, this image must centralize the image of a. It follows that $\langle x, y\rangle$ normalizes $\langle a\rangle$. It cannot centralize a, for then we would have $\langle a, x, y\rangle = \langle a\rangle \times \langle x, y\rangle$, and we know that $C_4 \times (C_4 \rtimes C_4)$ has a

quotient $C_4 \times D_8$ which is not good. Since $\langle xy, y\rangle = \langle x, y\rangle$, it follows that at least one of xy and y must invert a.

Suppose only one of them does: say, $a^{xy} = a^{-1}$ but $a^y = a$. Then $(ay)^2 \notin \langle x, y\rangle$, and the above argument may be repeated with ay in place of a, giving the conclusion that at least one of xy and y must invert ay. However, now $(ay)^{xy} \neq (ay)^{-1}$ because $a^{-1}x^2y \neq a^{-1}y^{-1}$, and $(ay)^y = ay \neq (ay)^{-1}$: we have reached a contradiction. A similar argument gives a contradiction if we assume that $a^{xy} = a$ and $a^y = a^{-1}$. This proves that both xy and y must invert a, that is, x must centralize and y must invert a.

By Corollary 4.6, the only nonabelian good groups of exponent 4 generated by two elements of order 4 are Q_8 and $C_4 \rtimes C_4$. Since $a^2 \neq b^2$, we cannot have $\langle a, b\rangle \cong Q_8$. If $\langle a, b\rangle \cong C_4 \rtimes C_4$, then a^2 and b^2 are the only nontrivial squares in $\langle a, b\rangle$, and by assumption neither of these lies in $\langle x, y\rangle$: thus all cyclic subgroups of order 4 in $\langle a, b\rangle$ avoid $\langle x, y\rangle$ and are therefore inverted by y. Since no automorphism of $C_4 \rtimes C_4$ can act like that, a and b must commute. □

One of the two nontrivial squares in $C_4 \rtimes C_4$ generates the commutator subgroup; hence if two cyclic subgroups of order 4 in $C_4 \rtimes C_4$ intersect trivially, one of them must be normal. This will also be used in the proof of the next lemma.

In view of Lemma 4.8, if H is a good group of exponent 4 then $\Phi(H)$ is an elementary abelian group spanned by squares, so it has a basis consisting of squares: that is, H has a subset X such that $\Phi(H) = \prod_{x\in X}\langle x^2\rangle$ and each x^2 is nontrivial.

Lemma 4.10. *Let H be a good group of exponent 4 and X any subset such that $\Phi(H) = \prod_{x\in X}\langle x^2\rangle$ and each x^2 is nontrivial. Then either $\langle X\rangle$ is abelian or all but one of the elements of X commute with each other and are inverted by the remaining one.*

Proof. Suppose that $\langle X\rangle$ is nonabelian, and that x, y is a noncommuting pair of elements of X. Since $\langle x\rangle \cap \langle y\rangle = 1$, one of $\langle x\rangle$ and $\langle y\rangle$ normalizes the other: say, $x^y = x^{-1}$. By Lemma 4.9, then each element of $X \setminus \{x, y\}$ is centralized by x and inverted by y, and any two elements of $X \setminus \{x, y\}$ commute with each other. □

Lemma 4.11. *If H is a nonabelian good group of exponent 4, then the Frattini subgroup of its centre has order at most 2.*

Proof. Suppose not: then $Z(H) = \langle a\rangle \times \langle b\rangle \times \cdots$ with $|a| = |b| = 4$. Since H is nonabelian, it has a nonabelian two-generator subgroup K. By Corollary 4.6, K is either D_8 or Q_8 or $C_4 \rtimes C_4$. We propose to show that $\langle a, b, K\rangle$ must contain a subgroup with a quotient isomorphic to $C_4 \times D_8$. Since $C_4 \times D_8$ has a subgroup $(C_2 \times C_2) \rtimes C_4$ which we know is not good, this will contradict the assumption that H is good, and so prove the lemma. If $\langle a^2, b^2\rangle \leq K$, we must have $K \cong C_4 \rtimes C_4$, and then no generality is lost by assuming that $K = \langle x, y \mid x^4 = y^4 = 1,\ x^y = x^{-1}\rangle$ and $a^2 = x^2$, $b^2 = y^2$, so

$\langle a, ax, by\rangle = \langle a\rangle \times \langle ax, by\rangle = C_4 \times D_8$. This argument tacitly involved changing a and b if necessary (without changing $\langle a, b\rangle$). The same flexibility allows us to assume that if $\langle a^2, b^2\rangle \not\leq K$ then $\langle a, b, K\rangle = \langle a\rangle \times \langle b, K\rangle$, and then what we need is that D_8 is a quotient of a subgroup of $\langle b, K\rangle$. Since D_8 is a quotient of $C_4 \rtimes C_4$, this is only an issue if $K \cong Q_8$, but then the central product $C_4 \mathsf{Y} Q_8$ is a quotient of $\langle b, K\rangle$ and of course $C_4 \mathsf{Y} Q_8 \cong C_4 \mathsf{Y} D_8 > D_8$. □

Lemma 4.12. *If H is a nonabelian good group of exponent 4 with $|\Phi(H)| > 4$, then H satisfies* (i) *of Theorem* 1.2.

Proof. Let X be a subset of H of the kind discussed in Lemma 4.10.

First, suppose that $\langle X\rangle$ is abelian; then $\langle X\rangle = \prod_{x\in X}\langle x\rangle$. Set $A = \mathrm{C}(\langle X\rangle)$: by Lemma 4.11, this centralizer is abelian. As A is of exponent precisely 4, it is generated by its elements of order 4. Thus if an element h normalizes every cyclic subgroup of order 4 in A then it either centralizes or inverts A. If h is an involution, then by Lemma 4.3 this comment is applicable. If we can show that each h of order 4 acts on A in this way, the claim of the lemma will follow.

Suppose then that $h \in H$, $h \notin A$, and $|h| = 4$. If necessary, one can change X without changing $\langle X\rangle$ (and therefore without changing A) so as to achieve that X has an element, x_1 say, with $x_1^2 = h^2$. Let x_2 be another element of X, and set $X' = X \setminus \{x_1, x_2\}$. Then $\{h, x_2\} \cup X'$ and $\{h, x_1x_2\} \cup X'$ can also play the role of X in Lemma 4.10. Since X' is nonempty (this is where we use the assumption that $|\Phi(H)| > 4$) and commutes with x_2 and with x_1x_2, in each case h is the only element which could invert all the others. Since X' is a nonempty common part of 'all the others', h behaves the same way in both cases. If it centralizes in both cases, then it centralizes all of X, contrary to $h \notin A$. Thus h inverts all elements of $\langle X\rangle$.

This proves that the centralizer A of X has at most one nontrivial coset in H, so $|H : A|$ is at most 2. It cannot be 1, because A is abelian but H is not. If a is any element of order 4 in A, then there is an x in X such that for $Y = \{a\} \cup (X \setminus \{x\})$ we have $\Phi(H) = \prod_{y\in Y}\langle y^2\rangle$, so Y can play the role of X in all this. The centralizer of Y contains and therefore equals A, so an element h of order 4 outside A inverts every element of $\langle Y\rangle$ as well. This proves that such an h inverts every element of order 4 in A, and so it inverts every element of A, as required.

Second, suppose that $\langle X\rangle$ is not abelian: say, y is the element of X which inverts all the others, and all the others commute with each other. Set $X' = X \setminus \{y\}$ and $A = \mathrm{C}(X')$. By Lemma 4.11, A is abelian. If $\Phi(A) = \Phi(H)$, we can replace X by a subset of A and appeal to the half of the lemma which we have already proved. It remains to deal with the case $\Phi(A) < \Phi(H)$. Of course then $\Phi(A) = \Phi(\langle X'\rangle)$, so A is the direct product of $\langle X'\rangle$ with an elementary abelian group. We shall show that every element h outside A inverts A.

Now $\langle x, y\rangle \cong C_4 \rtimes C_4$ whenever $x \in X'$, so Lemma 4.3 ensures that every involution lies in A: we need only consider the h of order 4. If $h^2 \notin \langle x^2 \mid x \in X'\rangle$ then $\{h\} \cup X'$ can play the role of X in Lemma 4.11; given that X' has at least two elements and they commute, this means that h must either centralize

or invert every element of X', that is, $h \in \langle y \rangle A$. If $h^2 \in \langle x^2 \mid x \in X' \rangle$, one can change X' without changing $\langle X' \rangle$ (and therefore without changing A) so as to achieve that X' has an element, x_1 say, with $x_1^2 = h^2$. Let x_2 be another element of X' (here we use again the assumption that $|\Phi(H)| > 4$), and set $X'' = X' \setminus \{x_1, x_2\}$. Both $\{y, h, x_2\} \cup X''$ and $\{y, h, x_1x_2\} \cup X''$ can play the role of X in Lemma 4.11. Because y inverts both x_2 and x_1x_2, we can conclude that h commutes with x_2, with x_1x_2, and with every element of X'' (and is inverted by y). What matters is that in this case $h \in A$.

We have proved that $H = \langle y \rangle A$, and noted that y centralizes, that is, inverts, every involution. We have also seen that y inverts $\langle X' \rangle$. Since A is the direct product of $\langle X' \rangle$ with an elementary abelian group, it follows that y inverts A, and then so does every element of H outside A. This completes the proof of the lemma. □

Lemma 4.13. *If h is a noncentral involution in a good group H of exponent 4 with $|\Phi(H)| > 2$, then H has an abelian subgroup A of index 2 such that every element of A is inverted by h.*

Proof. Since h is noncentral it is noncentral already in some nonabelian two-generator subgroup which by Corollary 4.6 can only be dihedral: thus there is in H an element a of order 4 such that $a^h = a^{-1}$. Since $|\Phi(H)| > 2$, there is also in H an element b such that $a^2 \neq b^2 \neq 1$. For any such b, by Corollary 4.6, $\langle a, b \rangle$ is either abelian of a $C_4 \rtimes C_4$. It cannot be the latter, because h does not centralize it and we have Lemma 4.3. It follows that every such b commutes with a. Further by Lemma 4.3, h must normalize both $\langle b \rangle$ and $\langle ab \rangle$, which is now only possible if h inverts b. If $c \in H$ and $c^2 \neq 1$, then either $c^2 \neq a^2$ or $c^2 \neq b^2$, and the above argument with a, c or b, c in place of a, b yields that h inverts c. We have proved that h inverts every element of order 4 in H. Further, any two elements of order 4 commute: else by Corollary 4.6 the subgroup they generate would be a Q_8 or a $C_4 \rtimes C_4$, and we have observed just before stating this lemma that neither of these groups has an automorphism that inverts all elements of order 4. Set $A = \langle a \in H \mid a^2 \neq 1 \rangle$; this is an abelian subgroup, and every element of it is inverted by h. If g is an element of H outside A, then g is an involution (by the definition of A). If $\langle a, g \rangle$ were abelian, it would be generated by elements of order 4 and so would lie in A, contrary to $g \notin A$: thus g is a noncentral involution and, like h, inverts every element of A. Hence gh centralizes A and therefore cannot lie outside it. This means that every element g of H outside A lies in the coset Ah, that is, that $|H : A| = 2$. □

Proof of the direct part of Lemma 1.4. Let G be a nonabelian good group. By Lemma 2.3, we may write G as $E \times H$ with E elementary abelian and H having no direct factor of order 2, and of course H is also nonabelian and good. If the exponent of H is greater than 4, Lemma 4.7 shows that H satisfies (i) of Theorem 1.2. Suppose that the exponent of H is 4. By Lemma 4.8, then $\Phi(H) \leq Z(H)$. If $|\Phi(H)| = 2$ then Lemma 4.2 shows that H satisfies (ii), while if $|\Phi(H)| > 4$ then (i) holds by Lemma 4.12. In the remaining case,

$|\Phi(H)| = 4$. If H has a noncentral involution, (i) holds by Lemma 4.13. If all involutions are central, then $\Omega(H) \leq \Phi(H)$ because H has no direct factor of order 2. We cannot have $|\Omega(H)| = 2$, for then H would be cyclic or generalized quaternion and (as H has exponent 4) this is excluded by $|\Phi(H)| = 4$. Thus $\Omega(H) = \Phi(H)$, and Lemma 4.1 shows that in this case H satisfies (i) or (iii) or (iv) or (v).

This completes the proof of Lemma 1.4, and so also the proof of Theorem 1.2.

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