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HOMOLOGY OF FREE ABELIANIZED EXTENSIONS OF GROUPS

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L. G. KOVACH [KOVÁCS], YU. V. KUZ'MIN, AND R. SHTER [RALPH STÖHR]

ABSTRACT. Let G be a group given by a free presentation $G = F/N$, and N' the commutator subgroup of N. The quotient F/N' is called a free abelianized extension of G. We study the homology of F/N' with trivial coefficients. In particular, for torsion-free G our main result yields a complete description of the odd torsion in the integral homology of F/N' in terms of the mod p homology of G.

§1. THE MAIN RESULT

Let G be an arbitrary group given as a quotient $G = F/N$ with F a free group and N a normal subgroup of F. The group $\Phi = F/N'$, where $N' = [N, N]$ is the commutator subgroup of N, is called a *free abelianized extension* of G. Set $M = N/N'$. By definition, there is an exact sequence

$$1 \to M \to \Phi \to G \to 1. \tag{1}$$

The homology of Φ with coefficients in $\mathbf{Z}[1/2]$ (the ring of rational numbers with 2-power denominators) has been studied in [5]–[7]. The present paper is a continuation of that work.

For an arbitrary abelian group A, we denote by tA the torsion subgroup of A, and by t_nA the subgroup of all those elements whose order divides some power of n. We write the n-fold exterior power of A as $\bigwedge^n A$. The abelian group M in (1) shall be regarded as a G-module with action coming from conjugation in Φ, and $\bigwedge^n M$ as a G-module with diagonal action.

The main result of [6], the outcome of the computation of the spectral sequence associated with the extension (1), can be stated as follows. *If $n \geq 2$ then*

$$tH_n(\Phi, \mathbf{Z}[1/2]) \cong t_n(\textstyle\bigwedge^n M \otimes_G \mathbf{Z}[1/2]) \tag{2}$$

and there is an exact sequence

$$\begin{aligned} 0 \to t_{n-1}H_2(G, \textstyle\bigwedge^{n-1} M \otimes \mathbf{Z}[1/2]) \to \textstyle\bigwedge^n M \otimes_G \mathbf{Z}[1/2] \\ \to H_n(\Phi, \mathbf{Z}[1/2]) \to t_{n-1}H_1(G, \textstyle\bigwedge^{n-1} M \otimes \mathbf{Z}[1/2]) \to 0. \end{aligned} \tag{3}$$

Further, it was shown that the quotient of $H_n(\Phi, \mathbf{Z}[1/2])$ over its torsion subgroup is a free $\mathbf{Z}[1/2]$-module. For finite G, the $\mathbf{Z}[1/2]$-free rank of this module has been computed by Zerck [9]. In the present paper we study the periodic groups involved in (2) and (3).

Let T_n denote the torsion subgroup of $H_n(\Phi, \mathbf{Z}[1/2])$, and let C_n and K_n stand for the cokernel and the kernel of the map $\bigwedge^n M \otimes_G \mathbf{Z}[1/2] \to H_n(\Phi, \mathbf{Z}[1/2])$ which is induced by the embedding $M \to \Phi$. It follows from (2) and (3) that the orders of the elements of T_n divide some power of n, and the orders of the elements of C_n and

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K_n divide some power of $n-1$. It was shown in [7] that multiplication by $n(n-1)$ annihilates the groups $H_k(G, \bigwedge^n M \otimes \mathbf{Z}[1/2])$ $(k > 0)$ and $t(\bigwedge^n M \otimes_G \mathbf{Z}[1/2])$; thus the orders of the elements in T_n and C_n, K_n divide n and $n-1$, respectively. Let $\pi(n)$ denote the product of the prime divisors of n. It was conjectured in [6] and [7] that the exponent of T_n divides $\pi(n)$ and the exponents of C_n and K_n divide $\pi(n-1)$. In this paper we confirm this conjecture for a large class of groups G including all torsion-free groups (see Corollary 1.2).

The second principal result of [6] was a description of the torsion part T_n in terms of the homology of G when $n = p$, an odd prime. Namely, *if G has no element of order p, then*

$$T_p \cong H_{p+2}(G, \mathbf{Z}_p), \tag{4}$$

where $\mathbf{Z}_p$ is $\mathbf{Z}/p\mathbf{Z}$ regarded as a trivial G-module. There is also a result for G with p-torsion, but the statement of that is more complicated. There are similar formulas for C_n and K_n when $n-1$ is a prime.

It was conjectured in [6] and [7] that for $n = p^t$

$$T_n \cong \bigoplus_{s=1}^{t} H_{n+2n/p^s}(G, \mathbf{Z}_p). \tag{5}$$

With $t = 1$ this gives (4). For $t = 2$, (5) has been proved by Hartley and the second author [3]. However, as we shall see below, for $t > 2$ the situation is more complicated. The main result of this paper is a complete description, for the case when G has no element of order p, of the p-torsion in T_n, C_n, and K_n in terms of the homology of G. To state it, we introduce the following notation.

For an abelian group A and a natural number m we put $mA = A \oplus \cdots \oplus A$ (with m summands), and $mA = 0$ when $m = 0$. For an arbitrary polynomial $f(x) = \sum_k m_k x^k$ with nonnegative integral coefficients and for any G-module A, we set

$$fH_n(G, A) = \bigoplus_k m_k H_{n+k}(G, A).$$

We shall show that the p-components of T_n, C_n, K_n have direct decompositions of this type, and we shall give recursive formulas for the computation of the corresponding polynomials. Namely, for $n > 1$ we define polynomials $f_n^{(p)}$ by

$$f_n^{(p)} = \begin{cases} 0 & \text{if } n \not\equiv 0,\ 1 \bmod p, \\ x^2 & \text{if } n = p, \\ x f_{n-1}^{(p)} & \text{if } n \equiv 1 \bmod p, \\ x^2 f_{n-p}^{(p)} + f_{n/p}^{(p)} & \text{if } n \equiv 0 \bmod p \text{ with } n > p. \end{cases}$$

Theorem 1. *Let p be an odd prime and G a group with no p-torsion. If $n \equiv 0 \bmod p$, then*

$$t_p T_n \cong f_n^{(p)} H_n(G, \mathbf{Z}_p),$$

while if $n \equiv 1 \bmod p$ but $n > 1$, then

$$t_p C_n \cong f_{n-1}^{(p)} H_n(G, \mathbf{Z}_p) \quad \textit{and} \quad t_p K_n \cong f_{n-1}^{(p)} H_{n+1}(G, \mathbf{Z}_p).$$

We remind the reader that (for any group G) $t_p T_n = 0$ if $n \not\equiv 0 \bmod p$ and $t_p C_n = t_p K_n = 0$ if $n \not\equiv 1 \bmod p$.

It follows from Theorem 1 that if G has no p-torsion, then the p-components of T_n, C_n, and K_n depend only on G, and not on the choice of the presentation $G = F/N$. Let us state two further consequences. We say that G *has no odd n-torsion* if it has no p-torsion for any odd prime p dividing n.

Corollary 1.1. *If G has no odd n-torsion, then*

$$T_n \cong \bigoplus_p f_n^{(p)} H_n(G, \mathbf{Z}_p),$$

where the direct sum is taken over all odd primes dividing n.

Corollary 1.2. *If G has no odd n-torsion, then the exponent of T_n divides $\pi(n)$.*

In particular, if G is torsion-free, these corollaries hold whenever $n > 1$. Both corollaries follow immediately from (2) and Theorem 1. Using (3), one can obtain similar corollaries for C_n and K_n. Thus our main result confirms the above conjecture concerning the exponent of T_n for G without odd n-torsion, and the conjecture concerning the exponents of C_n and K_n for G without odd $(n-1)$-torsion. On the other hand, our theorem shows that for $t > 2$ the right-hand side of (5) needs certain extra summands. For example, when $n = p^3$ the right-hand side of (5) corresponds to the polynomial $x^{2p^2} + x^{2p} + x^2$, whereas the relevant polynomial, $f_n^{(p)}$, is

$$x^{2p^2} + (x^2 + x) \sum_{r=1}^{p-1} x^{2(p^2 - r(p-1)-1)} + x^{2p} + x^2.$$

An interesting property of the formulas of Theorem 1 is that they are identical for all groups G, in the sense that the polynomials $f_n^{(p)}$ do not depend on G. In the last section of this paper we shall discuss some further properties of these polynomials, and give an application to the case when G is of finite homological dimension.

For a brief description of the behavior of cohomology, we need more notation: set $\bigwedge_n^* M = \operatorname{Hom}(\bigwedge^n M, \mathbf{Z}[1/2])$. The cohomology groups $H^n(\Phi, \mathbf{Z}[1/2])$ are members of exact sequences

$$\begin{aligned} 0 &\to t_{n-1} H^1(G, \textstyle\bigwedge_{n-1}^* M) \to H^n(\Phi, \mathbf{Z}[1/2]) \\ &\to \operatorname{Hom}_G(\mathbf{Z}[1/2], \textstyle\bigwedge_n^* M) \to t_{n-1} H^2(G, \textstyle\bigwedge_{n-1}^* M) \to 0 \end{aligned}$$

(see [6], §8) which are the cohomological analogues of (3). Since

$$\operatorname{Hom}_G(\mathbf{Z}[1/2], \textstyle\bigwedge_n^* M)$$

is torsion-free, the situation in cohomology is easier. Instead of three groups, T_n, C_n, and K_n, it suffices to consider two, namely the cokernel C^n and the kernel K^n of the restriction map $H^n(\Phi, \mathbf{Z}[1/2]) \to \operatorname{Hom}_G(\mathbf{Z}[1/2], \bigwedge_n^* M)$ (because $tH^n(\Phi, \mathbf{Z}[1/2]) = K^n$). The cohomological version of Theorem 1 states that for p-torsion-free G and $n \equiv 1 \bmod p$ there are isomorphisms

$$t_p C^n \cong f_{n-1}^{(p)} H^n(G, \mathbf{Z}_p) \quad \text{and} \quad t_p K^n \cong f_{n-1}^{(p)} H^{n-1}(G, \mathbf{Z}_p).$$

In particular, if G has no elements of order dividing $n-1$ then the exponents of C^n and K^n divide $\pi(n-1)$. Our results do not provide information about 2-torsion in the integral homology $H_*\Phi$, whereas $t_p H_n \Phi \cong t_p H_n(\Phi, \mathbf{Z}[1/2])$ for all odd primes p. It is known [4] that the exponent of $tH_2\Phi$ divides 4. The structure of this group was described in detail in [6] and [8]. Also, it is easily seen that the map $\bigwedge^2 M \otimes_G \mathbf{Z} \to H_2\Phi$ is an isomorphism. However, for $n > 2$ practically nothing is known about the 2-components of $tH_n\Phi$ or about the 2-components of the kernel and cokernel of the map $\bigwedge^n M \otimes_G \mathbf{Z} \to H_n\Phi$. In our proof of Theorem 1, the condition that $1/2$ belongs to the ring of coefficients will be used only to make (2) and (3) applicable. We have not excluded the possibility that (2) and (3) remain valid when $\mathbf{Z}[1/2]$ is replaced by $\mathbf{Z}$. This would mean that Theorem 1, with T_n, C_n,

and K_n replaced by their integral analogues, extends to the case $p = 2$. The same can be said about Corollaries 1.1 and 1.2. It was shown by Hannebauer and the third author [1] that, for an arbitrary group G, multiplication by $2n(n-1)$ annihilates the groups $H_k(G, \bigwedge^n M)$ $(k > 0)$ and $t(\bigwedge^n M \otimes_G \mathbf{Z})$. Consequently, if one could replace $\mathbf{Z}[1/2]$ by $\mathbf{Z}$ in (2) and (3), it would follow that the exponent of $tH_n\Phi$ divides n for odd n and $2n$ for even n, and that the exponents of the kernel and cokernel of the map $\bigwedge^n M \otimes_G \mathbf{Z} \to H_n\Phi$ divide $n-1$ for even n and $2(n-1)$ for odd n. It is tempting to conjecture that $\pi(n)$, $2\pi(n)$ and $\pi(n-1)$, $2\pi(n-1)$, respectively, are the optimal bounds on the corresponding exponents.

Most of this work was carried out, simultaneously but independently, by the second author in Moscow, and by the other two authors in Canberra during a visit of the third author to the Australian National University. He thanks the ANU for financial support and his hosts for their hospitality. The first complete proof, and the idea of expressing the results in terms of the polynomials $f_n^{(p)}$, came from the second author.

§2. Reduction to symmetric powers of the augmentation ideal

We shall keep the prime p fixed from now on, and assume that the group G has no p-torsion. It follows from (2) and (3) that Theorem 1 will be proved once we show that for $n > 1$ there exist isomorphisms

$$t_pH_k\left(G, \textstyle\bigwedge^n M\right) \cong f_n^{(p)} H_{k+n}(G, \mathbf{Z}_p) \qquad (k \geq 0). \tag{6}$$

For odd primes p, the formula for T_n then follows from (2) with $k = 0$, and the formulas for C_n and K_n follow from (3). The isomorphisms (6) will be established for arbitrary p (including $p = 2$).

It was proved in [3] that questions about torsion in the homology of G with coefficients in $\bigwedge^n M$ are equivalent to similar questions for symmetric powers of the argumentation ideal. For any abelian group A, we denote the nth symmetric power of A by A^n. If A is a G-module, A^n will be regarded a G-module with diagonal action. Let $\mathbf{Z}_{(p)}$ be the ring of integers localized at p; write R for the group ring $\mathbf{Z}_{(p)}G$ and Δ for the augmentation ideal of R. Then for all $n \geq 1$ and $k \geq 0$ there exist isomorphisms

$$t_pH_k\left(G, \textstyle\bigwedge^n M\right) \cong t_pH_{k+n}(G, \Delta^n)$$

(see [3], Proposition 1.1). On comparing this with (6), we see that our problem reduces to proving that for $n > 1$ there exist isomorphisms

$$t_pH_k(G, \Delta^n) \cong f_n^{(p)} H_k(G, \mathbf{Z}_p) \qquad (k \geq n). \tag{7}$$

As in [3], we study the symmetric powers Δ^n as submodules of the R^n. For $n > m \geq 0$, let K_m^n denote the submodule of R^n spanned by the elements

$$c_1 \circ c_2 \circ \cdots \circ c_i \circ \underbrace{1 \circ 1 \circ \cdots \circ 1}_{n-i \text{ terms}},$$

where $c_1, \ldots, c_i \in \Delta$ and $m < i \leq n$. These submodules form a chain

$$0 < \Delta^n = K_{n-1}^n < K_{n-2}^n < \cdots < K_0^n < R^n$$

in R^n with quotients $\Delta^n, \Delta^{n-1}, \ldots, \Delta, \mathbf{Z}_{(p)}$.

Now we are ready to state our main technical result. To simplify typography, in the rest of the paper we shall not indicate the group G in our notation for homology: thus whenever X is a G-module we shall write H_kX instead of $H_k(G, X)$.

Theorem 2. *Let G be a p-torsion-free group, $k \geq 1$, $m \geq 2$, and $n \geq 0$. Then*

$$H_k K^{mp^n}_{(m-1)p^n} \cong \begin{cases} 0 & \text{if } m \not\equiv 0,\ 1 \bmod p, \\ H_{k+2}\mathbf{Z}_p & \text{if } m = p, \\ H_{k+1} K^{(m-1)p^n}_{(m-2)p^n} & \text{if } m \equiv 1 \bmod p, \\ H_{k+2} K^{(m-p)p^n}_{(m-p-1)p^n} \oplus H_k K^{sp^{n+1}}_{(s-1)p^{n+1}} & \text{if } m = sp \text{ with } s \geq 2. \end{cases}$$

From Theorem 2 it is easy to see that $H_k K^{mp^n}_{(m-1)p^n}$ does not depend on n; that is, $H_k K^{mp^n}_{(m-1)p^n} \cong H_k K^m_{m-1}$ whenever $n \geq 1$. Hence the case $n = 0$ of Theorem 2 may also be put as follows.

Corollary 2.1. *Let G be a p-torsion-free group, $k \geq 1$, and $n \geq 2$. Then*

$$H_k \Delta^n \cong \begin{cases} 0 & \text{if } n \not\equiv 0,\ 1 \bmod p, \\ H_{k+2}\mathbf{Z}_p & \text{if } n = p, \\ H_{k+1}\Delta^{n-1} & \text{if } n \equiv 1 \bmod p, \\ H_{k+2}\Delta^{n-p} \oplus H_k \Delta^{n/p} & \text{if } p < n \equiv 0 \bmod p. \end{cases}$$

Finally, it is easy to see from the definition of $f_n^{(p)}$ that Corollary 2.1 implies the existence of the isomorphisms (7), and hence it also implies Theorem 1. The rest of the paper (except the last section) will therefore be devoted to the proof of Theorem 2.

§3. The homomorphisms π^n_m

Now we introduce the homomorphisms $\pi^n_m \colon R^n \to R^m$ which provide the key to our study of the homology of the symmetric powers Δ^n. First, let $n > m \geq 0$. As is well known, there exists a (unique) G-homomorphism $R^n \to R^m \otimes R^{n-m}$ such that, whenever $r_1, \dots, r_n \in R$,

$$r_1 \circ \cdots \circ r_n \mapsto \frac{1}{m!(n-m)!} \sum_{\omega} (r_{\omega(1)} \circ \cdots \circ r_{\omega(m)}) \otimes (r_{\omega(m+1)} \circ \cdots \circ r_{\omega(n)}),$$

where the sum is taken over all permutations ω of the indices $1, \dots, n$. (Our convention is that $R^0 = \mathbf{Z}_{(p)}$.) Further, on applying the augmentation map $R \to \mathbf{Z}_{(p)}$ to the last $n-m$ entries in $(r_1 \circ \cdots \circ r_m) \otimes (r_{m+1} \circ \cdots \circ r_n)$, we obtain a G-homomorphism

$$R^m \otimes R^{n-m} \to R^m \otimes \mathbf{Z}^{n-m}_{(p)} \cong R^m.$$

Let π^n_m be the composite of these two homomorphisms. For technical reasons it will be convenient to extend this definition to all n and m such that $n > m \geq -1$ by setting $R^{-1} = 0$, $\pi^n_{-1} = 0$, and $K^n_{-1} = R^n$. The convention $K^n_n = 0$ will also be useful.

If $n > m \geq l$, then π^n_m induces a homomorphism $K^n_{l-1}/K^n_l \to K^m_{l-1}/K^m_l$. The obvious isomorphisms of the quotients K^n_{l-1}/K^n_l and K^m_{l-1}/K^m_l with Δ^l combine to give an isomorphism $K^m_{l-1}/K^m_l \to K^n_{l-1}/K^n_l$. The composites of this with the homomorphism obtained from π^n_m are endomorphisms of the two quotients. As

$$(8) \qquad \pi^n_m : (c_1 \circ \cdots \circ c_l \circ \underbrace{1 \circ \cdots \circ 1}_{n-l \text{ terms}}) \mapsto \binom{n-l}{m-l} (c_1 \circ \cdots \circ c_l \circ \underbrace{1 \circ \cdots \circ 1}_{m-l \text{ terms}})$$

whenever $c_1, \dots, c_k \in \Delta$, both endomorphisms amount simply to multiplication by the binomial coefficient $\binom{n-l}{m-l}$. This is part (i) of our first technical result.

Lemma 1. (i) *If* $n > m \geq l$, *then the homomorphism* $K_{l-1}^n / K_l^n \to K_{l-1}^m / K_l^m$ *induced by* π_m^n *amounts to multiplication by* $\binom{n-l}{m-l}$.

(ii) *If* $n > m \geq -1$, *then* $K_m^n = \ker \pi_m^n$.

(iii) *Let* $n > m \geq k \geq -1$. *If all the numbers* $\binom{n-l}{m-l}$ *with* $m \geq l > k$ *are invertible in* $\mathbf{Z}_{(p)}$, *then the homomorphism* π_m^n *induces an isomorphism* $K_k^n / K_m^n \cong K_k^m$.

Part (ii) is an obvious consequence of the definitions and (i). Further, (i) yields (iii) because under the assumptions π_m^n induces isomorphisms on all canonical quotients Δ^l of the modules K_k^n/K_m^n and K_k^m.

For applications of this lemma, we have to be able to decide just when a binomial coefficient $\binom{b}{a}$ is divisible by p. As is well known, the answer lies in base p arithmetic. Write $a = \sum a_i p^i$ and $b = \sum b_i p^i$ (with $0 \leq a_i \leq p-1$ and $0 \leq b_i \leq p-1$, $i \geq 0$); then

$$\binom{b}{a} \equiv \prod \binom{b_i}{a_i} \quad \bmod p.$$

(For an easy proof, expand $(1+x)^b$ in $\mathbf{Z}_p[x]$ twice: once directly, and once using that pth powering is an endomorphism of every commutative ring of characteristic p.) We record here the relevant consequence.

Lemma 2. *The binomial coefficient* $\binom{b}{a}$ *is invertible in* $\mathbf{Z}_{(p)}$ *if (and only if) the base* p *digits of* a *are no larger than those of* b:

$$a_i \leq b_i \quad \textit{for all } i.$$

Part (iii) of Lemma 1 implies that, under the relevant conditions for n, m, and k, there is an exact sequence

$$0 \to K_m^n \to K_k^n \to K_k^m \to 0$$

of G-homomorphisms (the first being a natural inclusion and the second the relevant restriction of π_m^n). Using Lemma 2, we now obtain a number of such short exact sequences.

Lemma 3. (i) *If* $n \geq 0$ *and* $s > r \geq 1$, *then there exist short exact sequences*

$$0 \to K_{(rp-1)p^n}^{(sp-1)p^n} \to K_{(rp-p-1)p^n}^{(sp-1)p^n} \to K_{(rp-p-1)p^n}^{(rp-1)p^n} \to 0, \tag{9.a}$$

$$0 \to K_{(rp+1)p^n}^{(sp+1)p^n} \to K_{(rp-p+1)p^n}^{(sp+1)p^n} \to K_{(rp-p+1)p^n}^{(rp+1)p^n} \to 0, \tag{9.b}$$

$$0 \to K_{rp^n}^{sp^n} \to K_{(r-1)p^n}^{sp^n} \to K_{(r-1)p^n}^{rp^n} \to 0, \tag{9.c}$$

$$0 \to K_{rp^{n+1}}^{(rp+1)p^n} \to K_{(rp-1)p^n}^{(rp+1)p^n} \to K_{(rp-1)p^n}^{rp^{n+1}} \to 0, \tag{9.d}$$

$$0 \to K_{(sp-p)p^n}^{(sp-1)p^n} \to K_{(sp-p-1)p^n}^{(sp-1)p^n} \to K_{(sp-p-1)p^n}^{(sp-p)p^n} \to 0, \tag{9.e}$$

$$0 \to K_{(rp-1)p^n}^{(rp+1)p^n} \to K_{(rp-p+1)p^n}^{(rp+1)p^n} \to K_{(rp-p+1)p^n}^{(rp-1)p^n} \to 0, \tag{9.f}$$

$$0 \to K_{(sp-p+1)p^n}^{(sp-1)p^n} \to K_{(sp-p-1)p^n}^{(sp-1)p^n} \to K_{(sp-p-1)p^n}^{(sp-p+1)p^n} \to 0, \tag{9.g}$$

$$0 \to K_{(sp-1)p^n}^{sp^{n+1}} \to K_{(sp-p)p^n}^{sp^{n+1}} \to K_{(sp-p)p^n}^{(sp-1)p^n} \to 0. \tag{9.h}$$

(ii) *If* $n \geq 0$, $r \geq 1$, *and* $m > rp+1$ *with* $m \not\equiv 0 \bmod p$, *then there is a short exact sequence*

$$0 \to K_{(rp+1)p^n}^{mp^n} \to K_{(rp-1)p^n}^{mp^n} \to K_{(rp-1)p^n}^{(rp+1)p^n} \to 0. \tag{10}$$

(iii) *If* $n \geq 0$ *and* $s > 1$, *then there is a short exact sequence*

$$0 \to K^{(sp-1)p^n}_{(p-1)p^n} \to R^{(sp-1)p^n} \to R^{(p-1)p^n} \to 0. \tag{11}$$

(iv) *If* $n \geq 0$ *and* $m > 1$ *with* $m \not\equiv 0 \bmod p$, *then there is a short exact sequence*

$$0 \to K^{mp^n}_{p^n} \to R^{mp^n} \to R^{p^n} \to 0. \tag{12}$$

Proof. In all cases, the proof consists of an elementary verification that the corresponding binomial coefficients from part (iii) of Lemma 1 satisfy the criterion of Lemma 2. This verification becomes easier if one notes that the set of the $\binom{n-l}{m-l}$ with $m \geq l > k$ is the same as the set of the $\binom{n-m+j}{j}$ with $0 \leq j < m-k$. For example, in this way the proof of the existence of (9.a) reduces to checking that all the numbers $\binom{(s-r)p^{n+1}+j}{j}$ with $0 \leq j \leq p^{n+1}$ satisfy the conditions in Lemma 2, which is obvious. The verification of the other cases is left to the reader.

We shall need one more result of this kind.

Lemma 4. *If* $n \geq 0$, *then there is a* 4*-term exact sequence*

$$0 \to K^{p^{n+1}}_{(p-1)p^n} \to R^{p^{n+1}} \to R^{(p-1)p^n} \to \mathbf{Z}_p \to 0 \tag{13}$$

of G-homomorphisms.

Proof. We have to show that the cokernel of $\pi^{p^{n+1}}_{(p-1)p^n}$ is isomorphic to $\mathbf{Z}_p$. By Lemma 1(i), on the canonical factors Δ^l of its domain and codomain $\pi^{p^{n+1}}_{(p-1)p^n}$ induces multiplication by $\binom{p^{n+1}-l}{(p-1)p^n-l}$. It is easily seen that these binomial coefficients are invertible in $\mathbf{Z}_{(p)}$ whenever $0 < l \leq (p-1)p^n$. Indeed, these are just the numbers $\binom{p^n+j}{j}$ with $0 \leq j < (p-1)p^n$, which obviously satisfy the criterion of Lemma 2. On the other hand, when $l = 0$ we get for the top quotients

$$R^{p^{n+1}}/K^{p^{n+1}}_0 \cong R^{(p-1)p^n}/K^{(p-1)p^n}_0 \cong \mathbf{Z}_{(p)}$$

multiplication by $\binom{p^{n+1}}{(p-1)p^n}$, which is divisible by p but not by p^2. This proves that $\operatorname{coker} \pi^{p^{n+1}}_{(p-1)p^n} \cong \mathbf{Z}_p$ as required.

Finally, we mention that none of the results of this section depend on our standing assumption that G is p-torsion-free.

§4. Vanishing homology

In this section we show that the homology of some of the modules K^n_m vanishes in all positive dimensions. We shall use the following simple property of the homology functor: if $0 \to A \to B \to C \to 0$ is an exact sequence of G-homomorphisms and $H_kB = H_kC = 0$ whenever $k \geq 1$, then also $H_kA = 0$ for all positive k. When we refer to a short exact sequence from §3 in the present section, this amounts to a call for the application of this principle to that sequence. Applicability to the sequences (11) and (12) is justified by the first part of the following simple result, which is a special case of Lemma 2.3 of [3].

Lemma 5. *If* $k \geq 1$ *and* $n \geq 1$, *then* $H_kR^n = 0$ *and* $H_k(R^n \otimes \mathbf{Z}_p) = 0$.

We remark that this lemma does depend on our standing assumption that G has no p-torsion. For example, if G has an element of order p, then $H_kR^p \neq 0$ for all odd positive k (see [2], §4). For convenience, we introduce the convention that *in this section all statements involving* k *and* n *refer to all positive* k *and all nonnegative* n.

Lemma 6. *If $s > r \geq 1$, then*

$$H_k K_{(rp-p+1)p^n}^{(sp-p+1)p^n} = 0 \quad and \quad H_k K_{(rp-1)p^n}^{(sp-1)p^n} = 0.$$

Proof. The first statement will be proved by induction on r. For $r = 1$, the claim follows from (12) with $m = sp - p + 1$. For $r > 1$, the inductive hypothesis is that $H_k K_{(rp-2p+1)p^n}^{(sp-p+1)p^n} = 0$ whenever $s > r - 1$, so in particular also $H_k K_{(rp-2p+1)p^n}^{(rp-p+1)p^n} = 0$; hence an appeal to (9b) completes the inductive step. The second statement is proved similarly, by reference to (11) and to (9.a).

Corollary 6.1. *If $s > r \geq 1$, then*

$$H_k K_{(rp-p+1)p^n}^{(rp+1)p^n} = 0 \quad and \quad H_k K_{(sp-p-1)p^n}^{(sp-1)p^n} = 0.$$

Lemma 7. *If $r \geq 1$, then*

$$H_k K_{(rp-p+1)p^n}^{(rp-1)p^n} = 0 \quad and \quad H_k K_{(rp-1)p^n}^{(rp+1)p^n} = 0.$$

Proof. The two claims will be proved together by induction on r. When $r = 1$, (12) with $m = p - 1$ gives the first claim, and then that and the first half of Corollary 6.1 together form the basis of an appeal to (9.f) for a proof of the second.

Let $r > 1$. The inductive hypothesis concerning the second claim and the second half of Corollary 6.1 with $s = r$ together justify an application of (9.g), which proves the first claim. That and the first half of Corollary 6.1 then enable us to appeal again to (9.f), completing the inductive step.

Lemma 8. *If $s \geq 1$ with $s \not\equiv 0 \bmod p$, then*

$$H_k K_{sp^n}^{mp^n} = 0$$

whenever $m > s$ and $m \not\equiv 0 \bmod p$.

Proof. This will be done by induction on s. The case $s = 1$ follows from (12), so let $s > 1$. When $s \equiv 1 \bmod p$, say, $s = rp + 1$ with $r \geq 1$, the inductive hypothesis is that $H_{(rp-1)p^n}^{mp^n} = 0$; together with the second part of Lemma 7, this provides the basis for an appeal to (11). When $s \not\equiv 0, 1 \bmod p$, the inductive hypothesis is that $H_{(s-1)p^n}^{mp^n} = 0$ for all m with $m > s - 1$ and $m \not\equiv 0 \bmod p$, so in particular also $H_k K_{(s-1)p^n}^{sp^n} = 0$, and an appeal to (9.c) with $r = s - 1$ completes the inductive step.

Corollary 8.1. *If $m \geq 2$ and $m \not\equiv 0, 1 \bmod p$, then*

$$H_k K_{(m-1)p^n}^{mp^n} = 0.$$

This is just the case $m \not\equiv 0, 1 \bmod p$ in Theorem 2.

§5. Reduction modulo p

In this section we give the proof of Theorem 2, deferring one step to §6. We shall make use of the following simple property of homology. If

$$(14) \qquad 0 \to A \to X_1 \to X_2 \to \cdots \to X_m \to B \to 0$$

is an exact sequence of G-homomorphisms with $m \geq 1$ and $H_k X_2 = \cdots = H_k X_m = 0$ for all positive k, then there is a long exact sequence

$$(15) \qquad \cdots \to H_{k+1} X_1 \to H_{k+m} B \to H_k A \to H_k X_1 \to H_{k+m-1} B \to H_{k-1} A \to \cdots,$$

the maps being induced homomorphisms and connecting homomorphisms. This correspondence is functorial: to each morphism between two sequences like (14),

there is a morphism between the corresponding long exact sequences. If in (14) we have also $H_k X_1 = 0$ for all positive k, then of course all the maps $H_{k+m}B \to H_k A$ of (15) are isomorphisms ("dimension shifting"). This will often be used without specific reference.

A simple general observation will also be useful. For any G-module Y, consider the exact sequence

$$0 \to Y \xrightarrow{\pi} Y \xrightarrow{\rho} \overline{Y} \to 0,$$

in which $\overline{Y}$ stands for $Y \otimes \mathbf{Z}_p$, π is multiplication by p, and ρ is reduction modulo p. In the corresponding long exact homology sequence

$$\cdots \to H_k(Y) \xrightarrow{H_k(\pi)} H_k(Y) \xrightarrow{H_k(\rho)} H_k(\overline{Y}) \to \cdots,$$

if $H_k(Y)$ is annihilated by p, then $H_k(\rho)$ is *injective* (because then the preceding $H_k(\pi)$ is a zero map) and *split* (because its codomain is, in any case, elementary abelian).

Theorem 2 will be proved by induction on m (simultaneously for all n). The case $m \not\equiv 0, 1 \bmod p$ has already been established as Corollary 8.1. Dimension shifting gives the case $m \equiv 1 \bmod p$, as all relevant homology of the middle term of (9.d) vanishes by the second half of Lemma 7. The case $m = p$ also comes from dimension shifting, using Lemmas 4 and 5. Only the last case, $m = sp$ with $s \geq 2$, remains to be proved.

By combining the short exact sequences (9.h) and (9.e) we obtain a 4-term exact sequence

$$(16) \qquad 0 \to K^{sp^{n+1}}_{(sp-1)p^n} \to K^{sp^{n+1}}_{(sp-p)p^n} \to K^{(sp-1)p^n}_{(sp-p-1)p^n} \to K^{(sp-p)p^n}_{(sp-p-1)p^n} \to 0.$$

By the second half of Corollary 6.1, we have $H_k K^{(sp-1)p^n}_{(sp-p-1)p^n} = 0$ for all positive k. Hence (16) leads to a long exact sequence

$$\cdots \to H_{k+2}K^{(sp-p)p^n}_{(sp-p-1)p^n} \to H_k K^{sp^{n+1}}_{(sp-1)p^n} \to H_k K^{sp^{n+1}}_{(sp-p)p^n} \to H_{k+1}K^{(sp-p)p^n}_{(sp-p-1)p^n} \to \cdots.$$

What we have to show is that $H_k K^{sp^{n+1}}_{(sp-1)p^n}$ is the direct sum of its two neighbors in this sequence.

The terms of the exact sequences in Lemma 3 were all $\mathbf{Z}_{(p)}$-free, so all those sequences remain exact when tensored with $\mathbf{Z}_p$. The second half of Lemma 5 ensures that all the arguments of the previous section apply also after tensoring the relevant modules with $\mathbf{Z}_p$. In particular, tensoring (16) with $\mathbf{Z}_p$ yields another similar sequence, and reduction modulo p gives a morphism from (16) to that. The homology of the third term of the new sequence also vanishes in all positive dimensions, so the long exact sequences yield commutative diagrams

$$(17) \qquad \begin{array}{ccccccc} H_{k+2}K^{(sp-p)p^n}_{(sp-p-1)p^n} & \longrightarrow & H_k K^{sp^{n+1}}_{(sp-1)p^n} & \longrightarrow & H_k K^{sp^{n+1}}_{(sp-p)p^n} & \longrightarrow & H_{k+1}K^{(sp-p)p^n}_{(sp-p-1)p^n} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{k+2}\overline{K}^{(sp-p)p^n}_{(sp-p-1)p^n} & \longrightarrow & H_k \overline{K}^{sp^{n+1}}_{(sp-1)p^n} & \longrightarrow & H_k \overline{K}^{sp^{n+1}}_{(sp-p)p^n} & \longrightarrow & H_{k+1}\overline{K}^{(sp-p)p^n}_{(sp-p-1)p^n} \end{array}$$

in which the vertical maps come from reduction modulo p and where we have saved space by writing $K^b_a \otimes \mathbf{Z}_p$ as $\overline{K}^b_a$.

We shall show in the next section, as Corollary 10.1, that the three maps in the second row of this diagram are, in turn, injective, surjective, and zero. Taking that for granted here, the inductive step in the proof of Theorem 2 can now be completed as follows. As $K^{sp^{n+1}}_{(sp-p)p^n}$ and $K^{(sp-p)p^n}_{(sp-p-1)p^n}$ can be obtained from $K^{mp^n}_{(m-1)p^n}$ by decreasing

the value of m (namely, on replacing m and n by m/p and $n+1$ or by $m-1$ and n), the inductive hypothesis applies to the first, third, and fourth terms in the top row of our commutative diagram, enabling us to deduce that those three homology groups are annihilated by p. This yields that three of our vertical arrows are injective. It is then a matter of simple diagram chasing to see that the first map of the top row is injective; its last map is zero, so the middle map is surjective; and its second term is also annihilated by p. This proves that the second term of the first row is the direct sum of its two neighbors, as required.

§6. THE FROBENIUS MAPS

To complete the proof of Theorem 2, it remains to establish the claims made concerning the second row of the commutative diagram (17). All the work towards that will take place in characteristic p, so it will be convenient to extend the "bar convention" initiated in the previous section by writing not only $\overline{K}^n_m$ for $K^n_m \otimes \mathbf{Z}_p$ but also $\overline{\Delta}$ for $\Delta \otimes \mathbf{Z}_p$ and $\bar{\pi}^n_m \colon \overline{R}^n \to \overline{R}^m$ for the map obtained by tensoring $\pi^n_m \colon R^n \to R^m$ with $\mathbf{Z}_p$.

For the symmetric algebra $S(\overline{R}) = \bigoplus_{i=0}^{\infty} \overline{R}^i$, as for any commutative algebra over the field $\mathbf{Z}_p$, we have an algebra endomorphism

$$S(\overline{R}) \to S(\overline{R}), \qquad a \mapsto a^p,$$

called the Frobenius map. Obviously, this is an injective endomorphism of the graded G-module $S(\overline{R})$. Its restrictions to G-submodules, and the maps these induce on appropriate quotients, will also be called Frobenius maps: thus for example we may speak of a Frobenius map $\overline{K}^n_k/\overline{K}^n_m \to \overline{K}^{np}_{kp}/\overline{K}^{np}_{mp}$ whenever $n \geq m \geq k$.

The usefulness of these maps in our context stems from the fact that they "intertwine" with the homomorphisms $\bar{\pi}^n_m$ (and their restrictions to submodules, and the maps induced on appropriate quotients), in the following sense.

Lemma 9. *All diagrams like*

$$\begin{array}{ccc} \overline{R}^n & \longrightarrow & \overline{R}^m \\ \downarrow & & \downarrow \\ \overline{R}^{np} & \longrightarrow & \overline{R}^{mp} \end{array}$$

in which the horizontal arrows represent $\bar{\pi}^n_m$ and $\bar{\pi}^{np}_{mp}$ while the vertical arrows represent Frobenius maps, are commutative.

Proof. The module $\overline{R}^n = \overline{K}^n_{-1}$ is spanned by elements of the form

$$c_1 \circ \cdots \circ c_l \circ \underbrace{1 \circ \cdots \circ 1}_{n-l \text{ terms}},$$

where $c_1, \ldots, c_l \in \overline{\Delta}$ and $0 \leq l \leq n$. The first vertical map takes this element to

$$\underbrace{c_1 \circ \cdots \circ c_1}_{p \text{ terms}} \circ \cdots \circ \underbrace{c_l \circ \cdots \circ c_l}_{p \text{ terms}} \circ \underbrace{1 \circ \cdots \circ 1}_{(n-l)p \text{ terms}}.$$

The horizontal images of these two elements, as given by (8), involve binomial coefficients which are congruent mod p (as can be seen, for instance, from the base p arithmetic leading to Lemma 2). Therefore the second vertical map takes the first horizontal image to the second, and commutativity is verified.

As noted in the previous section, Lemma 3 and all of §4 remain valid when all modules are tensored with $\mathbf{Z}_p$ (that is, have a bar placed above them). The situation concerning Lemma 4 is not so simple. Instead of (13), we now have (for each nonnegative n) an exact sequence

$$\overline{R}^{p^{n+1}}/\overline{K}^{p^{n+1}}_{(p-1)p^n} \to \overline{R}^{(p-1)p^n} \to \overline{R}^0 \to 0$$

with the first map defined from $\bar{\pi}^{p^{n+1}}_{(p-1)p^n}$ and the second map $\bar{\pi}_0^{(p-1)p^n}$. It is straightforward to verify that the first map has image $\overline{K}_0^{(p-1)p^n}$, it is one-to-one on the submodule $\overline{K}_0^{p^{n+1}}/\overline{K}^{p^{n+1}}_{(p-1)p^n}$ of codimension 1, and it is zero on the 1-dimensional subspace, I_n say, spanned by the coset $(1\circ\cdots\circ 1)+\overline{K}^{p^{n+1}}_{(p-1)p^n}$. It follows that I_n is precisely the kernel of this map; indeed, that

$$\overline{R}^{p^{n+1}}/\overline{K}^{p^{n+1}}_{(p-1)p^n} \cong I_n \oplus (\overline{K}_0^{p^{n+1}}/\overline{K}^{p^{n+1}}_{(p-1)p^n}) \cong I_n \oplus \overline{K}_0^{(p-1)p^n},$$

and therefore also

$$H_k(\overline{R}^{p^{n+1}}/\overline{K}^{p^{n+1}}_{(p-1)p^n}) \cong H_k I_n \oplus H_k\overline{K}_0^{(p-1)p^n}.$$

Since this direct decomposition was derived entirely from maps which intertwine with the Frobenius maps, it follows that the map

$$H_k(\overline{R}^{p^{n+1}}/\overline{K}^{p^{n+1}}_{(p-1)p^n}) \to H_k(\overline{R}^{p^{n+2}}/\overline{K}^{p^{n+2}}_{(p-1)p^{n+1}}), \tag{18}$$

obtained by applying the functor H_k to the relevant Frobenius map, is the direct sum of the maps

$$H_k I_n \to H_k I_{n+1} \quad \text{and} \quad H_k\overline{K}_0^{(p-1)p^n} \to H_k\overline{K}_0^{(p-1)p^{n+1}} \tag{19}$$

similarly obtained. It is immediate to see that the Frobenius map takes I_n isomorphically onto I_{n+1}, so this claim makes sense; indeed, it follows that the first map in (19) is also an isomorphism. Consider next the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K}_0^{(p-1)p^n} & \longrightarrow & \overline{R}^{(p-1)p^n} & \longrightarrow & \overline{R}^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{K}_0^{(p-1)p^{n+1}} & \longrightarrow & \overline{R}^{(p-1)p^{n+1}} & \longrightarrow & \overline{R}^0 & \longrightarrow & 0
\end{array}$$

in which the rows are exact and the vertical arrows represent Frobenius maps. By the second part of Lemma 5 the middle terms of the rows have vanishing homology in positive dimensions, while the last vertical map is obviously the identity: so the corresponding commutative diagram

$$\begin{array}{ccc}
H_{k+1}\overline{R}^0 & \longrightarrow & H_k\overline{K}_0^{(p-1)p^n} \\
\downarrow & & \downarrow \\
H_{k+1}\overline{R}^0 & \longrightarrow & H_k\overline{K}_0^{(p-1)p^{n+1}}
\end{array}$$

involving the dimension-shifting isomorphisms proves that the second map in (19) is also an isomorphism. We may therefore conclude that the map (18) is an isomorphism.

By Lemma 5, in the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{K}^{p^{n+1}}_{(p-1)p^n} & \longrightarrow & \overline{R}^{p^{n+1}} & \longrightarrow & \overline{R}^{p^{n+1}}/\overline{K}^{p^{n+1}}_{(p-1)p^n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{K}^{p^{n+2}}_{(p-1)p^n} & \longrightarrow & \overline{R}^{p^{n+2}} & \longrightarrow & \overline{R}^{p^{n+2}}/\overline{K}^{p^{n+2}}_{(p-1)p^{n+1}} & \longrightarrow & 0
\end{array}$$

in which the horizontal maps are natural and the vertical arrows represent Frobenius maps, the middle terms have vanishing homology in all positive dimensions. Consequently, there is a commutative diagram

$$\begin{array}{ccc}
H_{k+1}(\overline{R}^{p^{n+1}}/\overline{K}^{p^{n+1}}_{(p-1)p^n}) & \longrightarrow & H_k\overline{K}^{p^{n+1}}_{(p-1)p^n} \\
\downarrow & & \downarrow \\
H_{k+1}(\overline{R}^{p^{n+2}}/\overline{K}^{p^{n+2}}_{(p-1)p^{n+1}}) & \longrightarrow & H_k\overline{K}^{p^{n+2}}_{(p-1)p^{n+1}}
\end{array}$$

with concerning isomorphisms as horizontal arrows and homomorphisms obtained from Frobenius maps as vertical arrows. The left vertical map, being (18), is an isomorphism; hence it follows that the right vertical map is also an isomorphism. We have proved the case $s = p$ of part (i) of the following result.

Lemma 10. *Let $n \geq 0$, $s \geq 2$, and $k \geq 1$. Then:*

(i) *The homomorphism $H_k\overline{K}^{sp^n}_{(s-1)p^n} \to H_k\overline{K}^{sp^{n+1}}_{(s-1)p^{n+1}}$ induced by the Frobenius map is an isomorphism.*

(ii) *The homomorphism $H_k\overline{K}^{sp^{n+1}}_{(sp-1)p^n} \to H_k\overline{K}^{sp^{n+1}}_{(sp-p)p^n}$ induced by the inclusion $\overline{K}^{sp^{n+1}}_{(sp-1)p^n} \to \overline{K}^{sp^{n+1}}_{(sp-p)p^n}$ is surjective.*

Proof. This will be done by induction on s (simultaneously for all n). The lemma holds when $s \not\equiv 0, 1 \bmod p$, for then the domain of the first map and the codomains of both maps vanish by Corollary 8.1. We have just seen that (i) holds when $s = p$. We shall proceed as follows. First, we show that if (i) holds for a given s then (ii) also holds for that s. Second, we complete the inductive step for (i).

Define a map $\overline{\Delta} \to \overline{K}^{sp^{n+1}}_{(sp-1)p^n}$ as the composite of $n+1$ Frobenius maps and an inclusion

$$\overline{\Delta}^s \to \overline{\Delta}^{sp} \to \overline{\Delta}^{sp^2} \to \cdots \to \overline{\Delta}^{sp^{n+1}} \to \overline{K}^{sp^{n+1}}_{(sp-1)p^n},$$

and a map $\overline{K}^s_{s-1} \to \overline{K}^{sp^{n+1}}_{(s-1)p^{n+1}}$ as the composite of $n+1$ Frobenius maps

$$\overline{K}^s_{s-1} \to \overline{K}^{sp}_{(s-1)p} \to \overline{K}^{sp^2}_{(s-1)p^2} \to \cdots \to \overline{K}^{sp^{n+1}}_{(s-1)p^{n+1}}. \tag{20}$$

Of course $\overline{K}^s_{s-1} = \overline{\Delta}^s$, so these maps and the inclusion $\overline{K}^{sp^{n+1}}_{(sp-1)p^n} \to \overline{K}^{sp^{n+1}}_{(sp-p)p^n}$ lead to a commutative diagram

$$\begin{array}{ccc}
H_k\overline{K}^s_{s-1} & \longrightarrow & H_k\overline{\Delta}^s \\
\downarrow & & \downarrow \\
H_k\overline{K}^{sp^{n+1}}_{(sp-1)p^n} & \longrightarrow & H_k\overline{K}^{sp^{n+1}}_{(sp-p)p^n}
\end{array}$$

in which the first horizontal map is the identity. If (i) holds for the given s and all n, then all the maps in (20) induce isomorphisms for the homology groups of positive dimension, so the second vertical map is an isomorphism. This proves that the second horizontal map is surjective, as claimed in (ii).

Now both assertions (i) and (ii) are established for $s \not\equiv 0, 1 \bmod p$ and for $s = p$, and we proceed to the inductive step for (i). Let $s > p$ and $s \equiv 0$ or $s \equiv 1 \bmod p$.

We take first the case $s \equiv 1 \bmod p$: say, $s = tp + 1$. Then there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{K}^{(tp+1)p^n}_{tp^{n+1}} \arrow[r] \arrow[d] & \overline{K}^{(tp+1)p^n}_{(tp-1)p^n} \arrow[r] \arrow[d] & \overline{K}^{tp^{n+1}}_{(tp-1)p^n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{K}^{(tp+1)p^{n+1}}_{tp^{n+2}} \arrow[r] & \overline{K}^{(tp+1)p^{n+1}}_{(tp-1)p^{n+1}} \arrow[r] & \overline{K}^{tp^{n+2}}_{(tp-1)p^{n+1}} \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact sequences of type (9.d) tensored with $\mathbf{Z}_p$ and the vertical maps are Frobenius maps. By Lemma 7, the homology of the middle terms vanishes in all positive dimensions. Hence there is a commutative square

$$\begin{tikzcd}
H_{k+1}\overline{K}^{tp^{n+1}}_{(tp-1)p^n} \arrow[r] \arrow[d] & H_k\overline{K}^{(tp+1)p^n}_{tp^{n+1}} \arrow[d] \\
H_{k+1}\overline{K}^{tp^{n+2}}_{(tp-1)p^{n+1}} \arrow[r] & H_k\overline{K}^{(tp+1)p^{n+1}}_{tp^{n+2}}
\end{tikzcd}$$

in which the horizontal maps are dimension-shifting isomorphisms. The inductive hypothesis concerning (i), with $s - 1$ in place of s, gives that the first vertical map is an isomorphism, and it follows that the second vertical map must also be an isomorphism.

Now we turn to the case $s \equiv 0 \bmod p$: say, $s = tp$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \overline{K}^{tp^{n+1}}_{(tp-1)p^n} \arrow[r] \arrow[d] & \overline{K}^{tp^{n+1}}_{(t-1)p^{n+1}} \arrow[r] \arrow[d] & \overline{K}^{(tp-1)p^n}_{(tp-p-1)p^n} \arrow[r] \arrow[d] & \overline{K}^{(tp-p)p^n}_{(tp-p-1)p^n} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overline{K}^{tp^{n+2}}_{(tp-1)p^{n+1}} \arrow[r] & \overline{K}^{tp^{n+2}}_{(t-1)p^{n+2}} \arrow[r] & \overline{K}^{(tp-1)p^{n+1}}_{(tp-p-1)p^{n+1}} \arrow[r] & \overline{K}^{(tp-p)p^{n+1}}_{(tp-p-1)p^{n+1}} \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact sequences like (16) tensored with $\mathbf{Z}_p$ and the vertical arrows represent Frobenius maps. By the second part of Corollary 6.1, the homology of the third terms vanishes in all positive dimensions, so there is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_{k+2}\overline{K}^{(tp-p)p^n}_{(tp-p-1)p^n} \arrow[r] \arrow[d] & H_k\overline{K}^{tp^{n+1}}_{(tp-1)p^n} \arrow[r, "*"] \arrow[d] & H_k\overline{K}^{tp^{n+1}}_{(t-1)p^{n+1}} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{k+2}\overline{K}^{(tp-p)p^{n+1}}_{(tp-p-1)p^{n+1}} \arrow[r] & H_k\overline{K}^{tp^{n+2}}_{(tp-1)p^{n+1}} \arrow[r, "*"] & H_k\overline{K}^{tp^{n+2}}_{(t-1)p^{n+2}} \arrow[r] & \cdots
\end{tikzcd}$$

involving the corresponding long exact sequences. The inductive hypothesis concerning (ii), with t, n, and t, $n+1$ in place of s, n, gives that the homomorphisms marked with asterisks $*$ are surjective: hence in this commutative diagram each of the four sets of dots can be replaced by 0. On the other hand, the inductive hypothesis concerning (i), with $s-p$, n and t, $n+1$ in place of s, n, gives that the left and the right vertical maps are isomorphisms. Thus our long commutative diagram yields commutative diagrams with short exact sequences as rows and isomorphisms as first and last vertical maps. This implies that the middle vertical maps are also isomorphisms, as required.

The proof of Lemma 10 is now complete.

Part (ii) of Lemma 10 means that in the long exact sequence

$$\cdots \to H_{k+2}\overline{K}^{(sp-p)p^n}_{(sp-p-1)p^n} \to H_k\overline{K}^{sp^{n+1}}_{(sp-1)p^n} \to H_k\overline{K}^{sp^{n+1}}_{(sp-p)p^n} \to H_{k+1}\overline{K}^{(sp-p)p^n}_{(sp-p-1)p^n} \to \cdots,$$

from which the second row of (17) was taken, every third map is surjective. The right-hand neighbors of these are then zero maps, and of course the right-hand neighbors of those are injective.

Corollary 10.1. *The maps in the second row of the commutative diagram* (17) *are, in turn, injective, surjective, and zero.*

This completes the proof of Theorem 2.

§7. The coefficients of $f_n^{(p)}$

In this section we discuss some properties of the polynomials $f_n^{(p)}$, and give an application in the context of groups of finite homological dimension.

Let $n \equiv 0$ or $n \equiv 1 \bmod p$. It is easy to see by induction on n that in this case the coefficient of the highest power of x in $f_n^{(p)}$ is 1, and that this highest power is x^{2m} when $n = mp$ and x^{2m+1} when $n = mp+1$ (with $m \geq 1$). Now we determine the lowest power of x which occurs in $f_n^{(p)}$ (with nonzero coefficient). Let $n = \sum n_i p^i$ (with $0 \leq n_i \leq p-1$ and $i \geq 0$) be the expression of n to base p. Write $\sigma_p(n)$ for $\sum n_i$, the sum of the base p digits of n; denote the number of nonzero digits by $\varepsilon_p(n)$; and put $\delta_p(n) = 2\sigma_p(n) - \varepsilon_p(n) + 1$. We claim that *the term of lowest degree in the polynomial $f_n^{(p)}$ is just $x^{\delta_p(n)}$*. This is obviously the case for $n = p$ and for $n = p+1$, so we have the initial step for an induction on n. For the inductive step, let $n \geq 2p$ and consider first the case $n \equiv 1 \bmod p$. Then $f_n^{(p)} = x f_{n-1}^{(p)}$, so the terms of lowest degree in $f_n^{(p)}$ and $f_{n-1}^{(p)}$ have equal coefficients, and their degrees differ by 1. This confirms our claim, because $\sigma_p(n) = \sigma_p(n-1)+1$ and $\varepsilon_p(n) = \varepsilon_p(n-1)+1$, and so $\delta_p(n) = \delta_p(n-1)+1$. Next, let $n \equiv 0 \bmod p$. Then $f_n^{(p)} = x^2 f_{n-p}^{(p)} + f_{n/p}^{(p)}$. It is immediate to see that $\delta_p(n/p) = \delta_p(n)$ so (unless $n/p \not\equiv 0, 1 \bmod p$ and therefore $f_{n/p}^{(p)} = 0$) by the inductive hypothesis the lowest degree term in $f_{n/p}^{(p)}$ is $x^{\delta_p(n)}$. The inductive hypothesis is always applicable to $f_{n-p}^{(p)}$, so our claim will follow if we prove that $2 + \delta_p(n-p) \geq \delta_p(n)$, with equality if and only if $n/p \not\equiv 0, 1 \bmod p$. Let p^t be the highest power of p to divide n, so that in this case $n = \sum_{i \geq t} n_i p^i$, with $t \geq 1$ and $n_t > 0$. Then

$$n - p = \sum_{i=1}^{t-1}(p-1)p^i + (n_t - 1)p^t + \sum_{i>t} n_i p^i ,$$

so $\sigma_p(n-p) = (t-1)(p-1)+\sigma_p(n)-1$ and $\varepsilon_p(n-1) = (t-1)+\varepsilon_p(n)-\max\{0, 2-n_t\}$. This yields that

$$\delta_p(n-p) + 2 = \delta_p(n) + (t-1)(2p-3) + \max\{0, 2-n_t\} \geq \delta_p(n).$$

Clearly, equality holds here if and only if $t = 1$ and $n_t \geq 2$, and this condition is equivalent to $n/p \not\equiv 0, 1 \bmod p$. We have proved the following result.

Theorem 3. *Let p be an odd prime, $n \equiv 0 \bmod p$, and G a group without p-torsion. Then in the decomposition*

$$t_p T_n = f_n^{(p)} H_n(G, \mathbf{Z}_p) = \bigoplus_k m_k H_{n+k}(G, \mathbf{Z}_p)$$

the homology groups of maximal and minimal dimension occur with multiplicity 1. *The maximal dimension is $n + 2n/p$, and the minimal dimension is $n + \delta_p(n)$.*

Corollary 3.1. *Let p be an odd prime and G a group of finite homological dimension d.*

(i) *If $d < n + \delta_p(n)$, then $t_p T_n = 0$.*
(ii) *If $d = n + \delta_p(n)$, then $t_p T_n \cong H_{n+\delta_p(n)}(G, \mathbf{Z}_p)$.*

There are similar results for the groups C_n and K_n.

This corollary improves a result of [6], where it was proved that $d < n+2$ implies $T_n = C_n = K_n = 0$. The point is that $\delta_p(n) \geq 2$ for all positive n, with equality only when n is a power of p. Indeed, $\delta_p(n)$ is not even bounded above: when $t \geq 2$, the largest among the $\delta_p(n)$ with $n \equiv 0 \bmod p$ and $n \leq p^t$ is $\delta_p(p^t - p) = (t-1)(2p-3)+1$. Therefore the exact condition $d < n + \delta_p(n)$ given in Corollary 3.1 is much better than the condition $d < n+2$ in [6].

Finally, to avoid the impression that all nonzero coefficients must equal 1, we mention that $x^{2(p+1)}$ occurs in $f^{(p)}_{p^3+p^2}$ with coefficient greater than 1.

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Mathematics Research Section
School of Mathematical Sciences
Australian National University
Canberra, A.C.T. 2601, Australia
E-mail address: kovacs@pell.anu.edu.au

Department of Mathematics
Moscow Institute of Railway Transportation Engineers
101475 Moscow, USSR

Department of Mathematics
UMIST
Manchester M60 1QD, England
E-mail address: r.stohr@uk.ac.umist

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