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Generating finite soluble groups

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ABSTRACT

We prove that if a finite soluble group G can be generated by s subgroups with pairwise coprime orders, each of these subgroups being generatable by r elements, then G itself can be generated by $r+s-1$ elements. No stronger conclusion can hold in general: we construct such groups, even with $s-1$ of the relevant subgroups actually cyclic, which cannot be generated by fewer than $r+s-1$ elements. We also show that if a finite soluble group G has a family of d-generator subgroups whose indices have no common divisor, then G can be generated by $d+1$ elements.

Let d be a positive integer, G a finite soluble group, U a simple G-module, $\mathbb{C}_G(U)$ the kernel of the action of G on U, and $\mathbb{D}_G(U)$ the intersection of the normal subgroups N of G such that $\mathbb{C}_G(U)/N$ is G-isomorphic to U. The quotient $\mathbb{C}_G(U)/\mathbb{D}_G(U)$ is known as the *crown* of G corresponding to (the isomorphism type of) U; as G-module, this crown is a direct sum of isomorphic copies of U, with the number of summands called the *U-rank* of G. Of course U is finite and its ring of G-endomorphisms is a finite field: call the dimension of U over that field the *absolute dimension* of U. It is an immediate consequence of two theorems of Gaschütz (Satz 4.1 of [2] and Satz 4 of [1]) that G can be generated by $d+1$ elements if and only if the U-rank of G is at most $d+1$ whenever G acts trivially on U and at most d times the absolute dimension of U when the action on U is nontrivial. Since the absolute dimension of U is also the multiplicity of U as direct summand in the largest semisimple quotient of the regular G-module over the relevant prime field, this conclusion may also be phrased as follows.

LEMMA. *A finite soluble group G can be generated by $d+1$ elements if and only if G/G' can be so generated and each crown of G on which G acts nontrivially can be generated (as G-module) by d elements.*

Our aim here is to draw attention to another consequence of these ideas of Gaschütz. It answers a question which had been put to us by Professor Luis Ribes, and shows that the Grushko-Neumann Theorem has no simple analogue in the context of pro-(finite soluble) groups; for a discussion of the relevant issues, see Ribes and Wong [6].

THEOREM 1. *If a finite soluble group G is generated by s subgroups of pairwise coprime orders, and if each of these subgroups can be generated by r elements, then G can be generated by $r+s-1$ elements.*

PROOF. If G satisfies the hypotheses, so does every homomorphic image of G. The claim is obvious when $r=1$ or $s=1$, and also when G is abelian. In view of the Lemma, it suffices therefore to show that the crown of G corresponding to an arbitrarily chosen nontrivial U can be generated (as module) by $r+s-2$ elements. In doing so, we may replace G by its quotient modulo $\mathbb{D}_G(U)$ or, more conveniently, assume that $\mathbb{D}_G(U)=1$. Write C for $\mathbb{C}_G(U)$. By Satz 5.1 of Gaschütz [2], C is then complemented in G; let K denote one of its complements, and p the unique prime divisor of the order of C. Let $H_1, \ldots, H_s$ be the subgroups provided by the hypothesis; arrange the listing of these subgroups so that $H_2, \ldots, H_s$ are all p'-groups. If H is a Hall p'-subgroup of K, it is also a Hall subgroup of G, so each of $H_2, \ldots, H_s$ has G-conjugates contained in H: therefore they have C-conjugates contained in K. On replacing K by a conjugate if necessary, one can arrange that one of these subgroups, H_2 say, is actually contained in K. That done, for $i=3, \ldots, s$ choose elements c_i in C so that $H_i^{c_i} \leq K$. Finally, choose a family of r elements to generate H_1, and write these elements as $x_1y_1, \ldots, x_ry_r$ with $x_j \in K$, $y_j \in C$. Then G is clearly generated by H_1, H_2, $H_3^{c_3}, \ldots, H_s^{c_s}$, $c_3, \ldots, c_s$, and hence also by K and $y_1, \ldots, y_r$, $c_3, \ldots, c_s$. It follows that G is the product of K with the G-submodule of C generated by the $r+s-2$ elements last listed. As K and C intersect trivially, that submodule must therefore be C itself, and so we see that C can be generated by the required number of elements.

Given positive integers r and s, it is not hard to construct examples, even with $H_1, \ldots, H_{s-1}$ cyclic, such that G cannot be generated by fewer than $r+s-1$ elements. When $s=1$ there is nothing to do, and the case of $r=1$ will be left to the reader. In the remaining case, let $p_1, \ldots, p_s$ be distinct primes, and q a prime which is congruent to 1 modulo $p_1 \cdots p_{s-1}p_s^2$. Further, let E be an extraspecial group of order p_s^{r+1} when r is even, or the central product of a cyclic group of order p_s^2 with an extraspecial group of order p_s^r when r is odd. For $i=1, \ldots, s-1$, let H_i be a group of order p_i, generated, say, by h_i. The direct product $H_1 \times \cdots \times H_{s-1} \times E$ then has a faithful simple module U of order

$q^{[r/2]}$. Any such U is absolutely simple even as E-module; the h_i and the central elements of E act on it as scalars or, in multiplicative terminology, as powering automorphisms. Let W be the direct product of $(r+s-2)[r/2]$ copies of such a U, and $\{w_1, \ldots, w_{r+s-2}\}$ an E-module generating set for it. Form G as the semidirect product of W with the direct product acting on it, and let H_s be the subgroup generated by E and $w_s, \ldots, w_{r+s-2}$. Using the Lemma, it is easy to see that H_s can be generated by r elements, and that G cannot be generated by fewer than $r+s-1$ elements: so we shall be done if we show that $H_1^{w_1}, \ldots, H_{s-1}^{w_{s-1}}, \mathrm{H}_s$ together generate G. Let h_s be a nontrivial element in the centre of E; for $i = 1, \ldots, s$, let k_i be an integer such that the action of h_i on U is given by $h_i : u \mapsto u^{k_i}$; note that $k_i \not\equiv 1 \pmod q$. The outstanding claim now follows from

$$[h_i^{w_i}, h_s] = [h_i[h_i, w_i], h_s] = [[h_i, w_i], h_s] = [w_i^{1-k_i}, h_s] = w_i^{(1-k_i)(k_s-1)}.$$

The hypothesis of Theorem 1 may be seen as a weakening of the hypothesis of Theorem 2 in [4], which asserts that if each Sylow subgroup of a finite soluble group can be generated by d elements then the group itself can be generated by $d+1$ elements. The examples just constructed show that Theorem 1 cannot be strengthened to provide a generalization of that result. Instead, one can readily establish the following.

THEOREM 2. *If a finite soluble group G has a family of d-generator subgroups whose indices have no common divisor, then G can be generated by $d+1$ elements.*

The PROOF begins like that of Theorem 1, with a reduction to the consideration of $C = \mathbb{C}_G(U)$ for some U such that $\mathbb{D}_G(U) = 1$, and of a complement K of C in G. The coprimality assumption then ensures that one of the d-generator subgroups, H say, contains C. In view of the Lemma, it will suffice to prove that C can be generated by d elements as H-module. By Nakayama's Lemma, this will follow if we show it for the largest semisimple quotient of C instead. Given a simple H-module V, let M denote the intersection of those normal subgroups N of H which are contained in C and such that C/N is H-isomorphic to V. The proof will be completed by showing that, for every choice of V, the H-module C/M can be generated by d elements. To this end, note first that $H \cap K$ complements C in H, and that $\mathbb{C}_{H\cap K}(V)$ is normal in $H \cap K$: thus $\mathbb{C}_{H\cap K}(V)N/N$ is normalized by $(H\cap K)N/N$. Of course it is also centralized by C/N, so it is normal in H/N. Thus $\mathbb{C}_{H\cap K}(V)N$ is normal in H, and by an isomorphism theorem $\mathbb{C}_H(V)/\mathbb{C}_{H\cap K}(V)N$ is H-isomorphic to C/N and hence to V. It follows first that $\mathbb{C}_{H\cap K}(V)N \geq \mathbb{D}_H(V)$, and then that $M \geq C \cap \mathbb{D}_H(V)$. Another appeal to an isomorphism theorem now yields that C/M is H-isomorphic to a section of the crown of H corresponding to V. By our Lemma, that crown can be generated by d elements; hence so can C/M, and we are done.

In conclusion, we note that recently the solubility hypothesis has been re-

moved from Theorem 2 of [4] by Lucchini [5] and Guralnick [3]. One is therefore encouraged to ask: can it also be removed from our theorems? Further, using that a nontrivial simple module can never have order 2, it is easy to see that Theorem 2 of [4] remains valid even if the Sylow 2-subgroup is allowed $d+1$ generators; does that slight generalization also survive the omission of the solubility hypothesis?

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