A PROBLEM OF WIELANDT ON FINITE PERMUTATION GROUPS

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1. *Introduction*

Problem 6.6 in the Kourovka Notebook [**9**], posed by H. Wielandt, reads as follows.

'Let P, Q be permutation representations of a finite group G with the same character. Suppose $P(G)$ is a primitive permutation group. Is $Q(G)$ necessarily primitive? Equivalently: Let A, B be subgroups of a finite group G such that for each class C of conjugate elements of G their intersection with C has the same cardinality: $|A \cap C| = |B \cap C|$. Suppose A is a maximal subgroup of G. Is B necessarily maximal? The answer is known to be affirmative if G is soluble.'

Note that two permutation representations with the same character have the same kernel, and that one may as well consider the question as one concerning the factor group over that common kernel; we may therefore restrict attention to faithful representations. Our aim is to reduce the restricted question to the case of almost simple G (we call a group *almost simple* if its socle is non-abelian and simple). The solution of the problem may then be approached by examining permutation representations of the almost simple groups. While much progress has been made lately towards understanding maximal subgroups, and therefore primitive permutation representations, of almost simple groups, here one would need to know also many non-maximal subgroups; thus we are far from having reduced the issue to scanning existing tabulations. Still it seems plausible that the almost simple case can eventually be settled and that for that case the answer will be affirmative—at worst, with an explicit list of exceptions. Our reduction is reversible in the sense that it will convert such a result into a general theorem by showing how to generate the full list of exceptions from that of the almost simple ones. (If there are no exceptions to list, restriction to the faithful case will have been immaterial; but if there are any exceptions at all, the non-faithful ones are clearly beyond accounting, as the hypothesis places no restriction whatsoever on the common kernel.)

2. *Statement of results*

To make these statements explicit, we need to develop some terminology. If G, A, B are as in the second half of Wielandt's problem, but B is not maximal, we shall say that G, A, B is an *exception*. If A and B are corefree in G (that is, if neither of them

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contains any non-trivial normal subgroup of G), the exception will be called *faithful*. If G is almost simple, the exception will be called *almost simple*. (All almost simple exceptions, if any, are faithful: else the quotient modulo the common normal core of A and B would be a soluble exception because of the Schreier 'conjecture', while as Wielandt had already noted soluble exceptions do not exist.)

If G, A, B is an exception and $x, y \in G$ then G, A^x, B^y is also an exception, and it would not make sense for us to consider two such exceptions different. We shall say that G, A, B is *isomorphic* to another exception G_1, A_1, B_1 if there is an isomorphism $G \cong G_1$ which matches the conjugacy class of A (in G) to the conjugacy class of A_1 (in G_1), and the conjugacy class of B to that of B_1.

THEOREM 1. *Let G, A, B be any faithful exception. Let K be any maximal normal subgroup of the socle M of G, and put $N = \mathbf{N}_G(K)$ and $Z = \mathbf{C}_N(M/K)$.*

Then M is a non-abelian minimal normal subgroup of G, and

$$N/Z,\ (A \cap N)Z/Z,\ (B \cap N)Z/Z$$

is an almost simple exception. Moreover, the socle of N/Z is MZ/Z.

Note that as M is a non-abelian minimal normal subgroup, the maximal normal subgroups of M form a single conjugacy class of subgroups in G; hence the almost simple exceptions so obtained from different choices of K are all isomorphic. Also, Theorem 1 implies that if there are no almost simple exceptions then there are no exceptions at all.

By a wreath product $U \operatorname{Wr} S_n$ we mean the usual semidirect product W of the symmetric group S_n and the n-fold direct power U^n of the (abstract) group U. The projection of W onto S_n corresponding to this semidirect decomposition will be denoted by π. Consider π a permutation representation of W and take a point stabilizer W_o. This has an obvious direct factorization

$$W_o = S_{n-1} U^n = U \times (S_{n-1} U^{n-1}) = U \times (U \operatorname{Wr} S_{n-1});$$

let π_o denote the corresponding projection of W_o onto the first direct factor U. A subgroup V of W will be called *large* if $V\pi$ is transitive (as subgroup of S_n) and $(V \cap W_o)\pi_o = U$. Note that once $V\pi$ is transitive the validity of $(V \cap W_o)\pi = U$ is independent of the point o whose stabilizers in S_n and in W were taken as S_{n-1} and W_o, respectively. Consequently, conjugates of large subgroups are large.

THEOREM 2. *Let G, A, B be an almost simple exception, M the socle of G, and V a large subgroup of any $(G/M) \operatorname{Wr} S_n$ with $n > 1$. Further, let G^* denote the complete inverse image of V under the obvious homomorphism of $G \operatorname{Wr} S_n$ onto $(G/M) \operatorname{Wr} S_n$, and set*

$$A^* = G^* \cap (S_n A^n), \qquad B^* = G^* \cap (S_n B^n).$$

Then G^*, A^*, B^* is a faithful exception.

THEOREM 3. *Let G_1^*, A_1^*, B_1^* be defined in the manner indicated in Theorem 2 from another almost simple exception G_1, A_1, B_1 and from a large subgroup V_1 of $(G_1/M_1)\,\mathrm{Wr}\,S_{n_1}$, where M_1 stands for the socle of G_1.*

Then the exceptions G^*, A^*, B^* and G_1^*, A_1^*, B_1^* are isomorphic if and only if $n = n_1$ and there is an isomorphism $G \cong G_1$ between G, A, B and G_1, A_1, B_1 such that the corresponding isomorphism $(G/M)\,\mathrm{Wr}\,S_n \cong (G_1/M_1)\,\mathrm{Wr}\,S_n$ matches the conjugacy class of V to that of V_1.

The point is, of course, that every faithful exception which is not almost simple is so constructed from an almost simple exception. Given a faithful but not almost simple exception G, A, B, Theorem 1 shows how to choose the relevant almost simple exception $N/Z, (A \cap N)Z/Z, (B \cap N)Z/Z$. The socle of N/Z being MZ/Z, the quotient of N/Z modulo its socle may be thought of as N/MZ, so the relevant V will be a large subgroup of some $(N/MZ)\,\mathrm{Wr}\,S_n$, namely that with $n = |G:N|$. Recall that the embedding theorem gives an embedding of G in $N\,\mathrm{Wr}\,S_n$ which is unique up to composition with inner automorphisms of this wreath product (see **[12]**), so its image is unique up to conjugacy. It is straightforward to check (or see **[12]**) that the image is a large subgroup. Compose this embedding with the obvious homomorphism of $N\,\mathrm{Wr}\,S_n$ onto $(N/MZ)\,\mathrm{Wr}\,S_n$, and choose the image of the composite as V. Clearly, V is a large subgroup and is well defined up to conjugacy.

THEOREM 4. *If G, A, B and M, N, Z are as in Theorem 1 and G is not almost simple, then $(N/Z)^*, [(A \cap N)Z/Z]^*, [(B \cap N)Z/Z]^*$, defined as in Theorem 2 with respect to the V chosen above, is isomorphic to G, A, B.*

The combined effect of these theorems may be put as follows.

COROLLARY. *Given a list $\mathscr{L}$ containing precisely one representative G, A, B of each isomorphism class of almost simple exceptions, a list $\mathscr{L}^*$ containing precisely one representative of each isomorphism class of faithful but not almost simple exceptions could be made up as follows.*

For each G, A, B on $\mathscr{L}$, calculate first the group Λ of those automorphisms of G which stabilize setwise both the conjugacy class of A and the conjugacy class of B. For each integer $n > 1$, let Λ act on S_n trivially and on G^n diagonally, and form the semidirect product $\Lambda(G\,\mathrm{Wr}\,S_n)$ accordingly. Sort the large subgroups of $G\,\mathrm{Wr}\,S_n$ containing M into $\Lambda(G\,\mathrm{Wr}\,S_n)$-conjugacy classes, take one representative G^* from each such class, set $A^* = G^* \cap S_n A^n$, $B^* = G^* \cap S_n B^n$, and let the G^*, A^*, B^* so obtained make up $\mathscr{L}^*$.

Of course, if $\mathscr{L}$ is non-empty then $\mathscr{L}^*$ is infinite; thus even if $\mathscr{L}$ is finite and so can be written out in full, $\mathscr{L}^*$ can not. To illustrate that the corollary need not be pointless in such a case, consider the possibility that $\mathscr{L}$ turns out to consist of a single G, A, B, with G simple and $\mathbf{Out}\,G = 1$; then the corollary would give that the G^* on $\mathscr{L}^*$ are precisely the $G\,\mathrm{Wr}\,T$ with T ranging through one representative of each permutational isomorphism class of transitive groups.

The proofs of the theorems will take up the rest of the paper. Each proof is set out as a separate section: §§3, 4, 5, 6 prove Theorems 1, 4, 2, 3, respectively.

3. *Proof of Theorem* 1

The proof of Theorem 1 will take several steps. Throughout, G, A, B denotes a faithful exception, M is the socle of G and K is a maximal normal subgroup of M, with $N = \mathbf{N}_G(K)$ and $Z = \mathbf{C}_N(M/K)$. Of course, A is a corefree maximal subgroup of G, while B is corefree but not maximal.

We shall not make much direct use of Wielandt's condition that $|A \cap C| = |B \cap C|$ for each conjugacy class C of elements in G. Mostly, we shall use instead that the character $1_A\uparrow^G$ of G induced from the trivial character 1_A of A coincides with the character $1_B\uparrow^G$ similarly defined with B in place of A. (To see that the two conditions are equivalent, first note that summing each side of $|A \cap C| = |B \cap C|$ over all C yields that $|A| = |B|$, as does the agreement of the degrees of the two characters. Next, take a transversal T for A in G and count the cardinality of the set $\{(c, t) \in C \times T \mid t^{-1}ct \in A\}$ in two ways to obtain that the value of $1_A\uparrow^G$ at an element c of C is just $|G:A|\,|A \cap C|\,|C|^{-1}$.)

One immediate consequence of $1_A\uparrow^G = 1_B\uparrow^G$ is that *a subgroup H of G complements A in G* (in the sense that $G = AH$ and $A \cap H = 1$) *if and only if it complements B*. (This holds because, by Mackey's subgroup theorem, H complements A if and only if the restriction $1_A\uparrow^G\downarrow_H$ is the regular character of H.) We shall apply this fact several times.

The first application yields that

(3.1) *M is non-abelian.*

For, if M were abelian it would have to complement every corefree maximal subgroup and every complement of M would be maximal, so M would complement A but could not complement B.

Note that (3.1) confirms Wielandt's comment that there are no soluble exceptions.

The second application leads to the conclusion that

(3.2) *M is a minimal normal subgroup of G.*

As is well known (see [2, §2]), a finite group which has a corefree maximal subgroup can have at most two minimal normal subgroups, and if it does have two, then the corefree maximal subgroups are precisely their common complements. Thus if G had two minimal normal subgroups both would complement A but at least one would not complement B.

Conditions (3.1) and (3.2) prove Theorem 1 whenever $K = 1$; so *from now on it will be assumed that $K > 1$.*

There are also the following immediate consequences of (3.1) and (3.2).

(3.3) *M is a direct product of non-abelian simple groups, K is the product of all but one of the simple direct factors of M, with $\mathbf{C}_M(K)$ being the other simple direct factor. The simple direct factors of M are precisely the G-conjugates of $\mathbf{C}_M(K)$, and the maximal normal subgroups of M are the G-conjugates of K. Moreover, $\mathbf{N}_G(\mathbf{C}_M(K)) = N$ and $\mathbf{C}_G(\mathbf{C}_M(K)) = Z$.*

Our next aim is to show that

(3.4) $G = AM, A \cap M > 1, G = BM, B \cap M > 1$ *and* $|A \cap M| = |B \cap M|$.

The maximality of A guarantees that $AM = G$. It is an immediate consequence of Wielandt's condition that

$$|A| = |B| \quad \text{and} \quad |A \cap M| = |B \cap M|.$$

Thus $G = BM$ (as $|G:BM| = |B \cap M|/|A \cap M| = 1$), and the only alternative to (3.4) is that M complements both A and B. Suppose that this alternative holds; we shall derive a contradiction. The first point to establish is that if $a \in A$, $b \in B$, and $aM = bM$, then $a^x = b$ for some x in M. As $|A \cap C| = |B \cap C|$ holds for the conjugacy class C of a in G, there is g in G such that $a^g \in B$. As $G = MB$, we have $g = xy$ with some x in M and y in B, and then

$$a^x = a[a,x] = y(a^g)y^{-1} \in aM \cap B = bM \cap B.$$

Here $bM \cap B = \{b\}$ follows from $B \cap M = 1$, and hence $a^x = b$. The second point is that $(A \cap Z)M = (B \cap Z)M$: indeed, if $a \in A \cap Z$, then there is b in B such that $aM = bM$; as this b is of the form a^x with x in M, we have that $b = a^x \in Z^x = Z$, and hence $a \in bM \subseteq (B \cap Z)M$. Since the argument is symmetric in A and B, the second point is established. The critical point is to note that A and B are 'top groups' in two twisted wreath decompositions of G, both decompositiosn having 'first coordinate subgroup' $\mathbf{C}_M(K)$. Recall (from [**15**]) that a twisted wreath decomposition of a group G consists of two subgroups: the 'first coordinate subgroup', which is such that its normal closure (called the 'base group') is the direct product of its distinct conjugates, and the 'top group', which is a complement to the base group.

As we have seen, now these conditions hold (with M being the base group in both decompositions). It follows that G is isomorphic to two ('external') twisted wreath products of $\mathbf{C}_M(K)$ by G/M, the two isomorphisms matching A and B to the respective top groups. In both constructions, the 'twisting subgroup' is N/M. In one, the 'twisting action' of N/M is obtained by composing the obvious isomorphism of N/M onto $A \cap N$ with the conjugation action of the latter on $\mathbf{C}_M(K)$; so the kernel of the twisting action is $(A \cap Z)M/M$. In the other, the twisting action is defined using B in place of A. The second point $(A \cap Z)M = (B \cap Z)M$ established above ensures that the two twisting actions of N/M have a common kernel. There is a criterion for deciding whether in a twisted wreath product of a non-abelian simple group (such as $\mathbf{C}_M(K)$) by any finite group (such as G/M) the base group is the socle and the top group is a maximal subgroup. In both cases considered here, the base group is the socle, so the criterion tests only the maximality of the top group; therefore it must hold in the case of A and fail in the case of B. However, the criterion can be stated in a form (see [**6, 7**]) which involves only the twisting subgroup and the kernel of the twisting action, not the twisting action itself; so it cannot distinguish between A and B. This contradiction establishes (3.4).

Since A is a corefree maximal subgroup of G, one may view G as a primitive permutation group (in its obvious action on the set $G:A$ of its cosets modulo A). In this action, A is a point stabilizer, and (3.4) shows that the socle M is not regular. Thus [**14**, (4.1)] becomes applicable, with A playing the role of what was there called H. As in [**14**, (4.1)], let P denote the intersection of those maximal normal subgroups $K_1, \ldots, K_k$ of M for which $A \cap K_i = A \cap K$; let $P_1, \ldots, P_l$ be the distinct conjugates of P in G, and set $R_j = \bigcap\{P_{j'} \mid j' \neq j\}$. As M is minimal normal, [**14**, (4.1)] gives the following.

(3.5) *If* $k > 1$ *then* $AK = G$. *If* $l > 1$ *then* $\{R_1, \ldots, R_l\}$ *is a (single, complete) conjugacy class of subgroups in* G *such that* $M = R_1 \times \ldots \times R_l$ *and* $A \cap M = \prod (A \cap R_j)$.

We may as well agree to number the K_i and the P_j so that

$$K = K_1 \quad \text{and} \quad P = P_1 = R_2 \times \ldots \times R_l.$$

In addition, set

$$Q = \mathbf{N}_G(P) \quad \text{and} \quad S = \mathrm{core}_Q((A \cap Q)P).$$

(Beware: in [**14**] this normal core was called Z.) As was noted at the end of the proof of [**14**, (4.1)], if $l > 1$ then the sublattice generated in the subgroup lattice of G by A, K, M, N, P, Q, S is as pictured on Figure 1, the case in which $k = 1$ being shown on the left and $k > 1$ on the right.

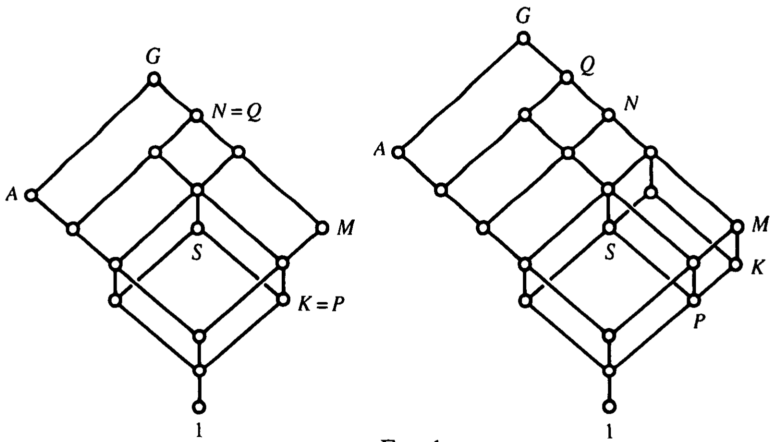

FIG. 1

The non-trivial part of this claim is that $S \cap M = P$; we shall also use the point that if $k > 1$ then $(A \cap Q)P/S$ intersects KS/S trivially. The last quotation we shall need from [**14**] in this section is [**14**, (3.2)], which we restate here as follows.

(3.6) $S = \mathbf{C}_Q(M/P)$.

This shows that when $k = 1$ (so $P = K$), we have $S = Z$ (with Z as defined in §2, and trivially also as defined in [**14**]).

To make the second half of (3.5) applicable, we show that

(3.7) $A \cap K > 1$ *and hence* $l > 1$.

If $l = 1$, then $P = 1$ and $A \cap K_i = A \cap K$ for all maximal normal subgroups K_i of M, whence $A \cap K = 1$. In this case k, being the number of all maximal normal subgroups of M, is not 1, for $K > 1$ by assumption. We shall prove (3.7) by showing that $A \cap K = 1$ leads to a contradiction. (Beware: as $l = 1$, we cannot appeal to Figure 1!) By (3.5) in this case K complements A, so it must also complement B. The fact that K complements A means that K is regular as permutation group on $G{:}A$ and so G is 'of simple diagonal type'. In turn, this implies that N is a maximal subgroup of G (see [**3**, Remark 2 on p. 6] or [**13**, Corollary to Theorem 1]). Since B is not maximal in G, there is a subgroup D with $B < D < G$. Set $L = \mathbf{N}_D(D \cap K)M$. Since $DM = G$ (for K complements B and $DM \geqslant BK$), we have that $N = (D \cap N)M$; thus

$\mathbf{N}_D(D \cap K) \geqslant D \cap N$ yields that $L \geqslant N$. Further, $D > B$ and $BK = G$ together imply that $D \cap K > 1$. If $L = G$, then all G-conjugates of K are of the form K^x with $x \in \mathbf{N}_D(D \cap K)$. In this case

$$1 < D \cap K \leqslant \bigcap (D \cap K)^x \leqslant \bigcap K^x = 1$$

and we have reached a contradiction as required. If $L = N$, then $\mathbf{N}_D(D \cap K) = D \cap N$ and $|D : \mathbf{N}_D(D \cap K)| = |G:N|$, so $D \cap M$ contains $|G:N|$ distinct D-conjugates $(D \cap K)^d$ of $D \cap K$. As

$$(D \cap M)/(D \cap K)^d \cong (D \cap M)/(D \cap K) \cong (D \cap M)K/K = M/K,$$

each quotient $(D \cap M)/(D \cap K)^d$ is a non-abelian simple group of order $|M/K|$. Hence $|D \cap M| \geqslant |M/K|^{|G:N|}$. On the other hand, $|M/K|^{|G:N|} = |M|$ by (3.3), so in this case $D \geqslant M$, contradicting $B < D < G = BM$. Since N is maximal, there are no other options for L, and the proof of (3.7) is complete.

Our next aim is to prove that

(3.8) $B \cap M = \prod (B \cap R_j)$.

To this end we return to the character equality $1_A\!\uparrow^G = 1_B\!\uparrow^G$. By 'the lemma that is not Burnside's' (see [**16**]), the number of the orbits of a permutation representation can be read off the character of that representation. Recall from (3.4) that $G = AM = BM$. Since R_j is normal in the transitive M, all orbits of R_j on $G:A$ have the same length. The same holds for the orbits of R_j on $G:B$, though at this stage we cannot yet say that the orbits on $G:A$ are equal in length to those on $G:B$. That equality follows only because the restriction of the character condition to R_j implies that the number of orbits on $G:A$ is the same as the number of orbits on $G:B$ and also that $|G:A| = |G:B|$. In particular, this proves that

$$|R_j : (A \cap R_j)| = |R_j : (B \cap R_j)|;$$

equivalently,

$$|A \cap R_j| = |B \cap R_j|.$$

By (3.4), $|B \cap M| = |A \cap M|$. Since $l > 1$ by (3.7), we know from (3.5) that $|A \cap M| = \prod |A \cap R_j|$. The conclusions of the last three sentences combined yield that $|B \cap M| = \prod |B \cap R_j| = |\prod (B \cap R_j)|$, whence $B \cap M = \prod (B \cap R_j)$.

(3.9) *$(A \cap Q)P$ is maximal in Q, but $(B \cap M)P$ is not.*

In preparation for the proof of this, let $B < D < G$; we show that then $DP \neq G$. Suppose that $DP = G$; then $(D \cap M)P = M$. As $D \cap R_1$ is normalized by $D \cap M$ (because $R_1 \trianglelefteq M$) and centralized by P (because R_1 is), it follows that $D \cap R_1 \trianglelefteq M$; so the non-trivial group $D \cap R_1$ is the product of some of the simple direct factors of M. As $DM \geqslant DP = G$, it follows that D permutes the simple direct factors of M transitively, and therefore $M \leqslant D$ and $G = DM = D$. This contradiction proves that $DP \neq G$.

Set $E = \bigcap DP_j$. By [**8**, Lemma 2.2] this E is a subgroup (even if perhaps none of the products DP_j is), and $E \cap M = \prod (E \cap R_j)$. Obviously, $B < D \leqslant E < G$. Now (3.9) follows from the results of Gross and Kovács [**8**], though [**11**] provides a more convenient reference for seeing this. Namely, by (3.7) and (3.5), the hypothesis called

(*) in [**8**, **11**] is satisfied by the present G, M, P, Q (in place of what were there called G, M, K, N); by (3.8) and what we have just proved, the subgroups A, B, E are, in the terminology of [**11**], high with respect to P; so [**11**, (3.01)] directly gives that $(A \cap Q)P/P$ is maximal in Q/P, but $(B \cap Q)P/P$ is not.

The last step towards the proof of Theorem 1 is to show that

(3.10) $1_{(A \cap Q)P}\uparrow^Q = 1_{(B \cap Q)P}\uparrow^Q$.

Since $AM = BM = G$ and $Q \geqslant M$ show that $AQ = BQ = G$, and as $1_A\uparrow^G = 1_B\uparrow^G$, Mackey's subgroup theorem yields that

$$1_{A \cap Q}\uparrow^Q = 1_{B \cap Q}\uparrow^Q.$$

Consider an arbitrary irreducible character χ of Q. If the normal subgroup P of Q is not contained in $\ker\chi$, then χ cannot be involved in either side of (3.10); so we may restrict attention to the case of $P \leqslant \ker\chi$. Let X be a Q-module which affords χ. By the Frobenius reciprocity theorem, the multiplicity of χ in $1_{(A \cap Q)P}\uparrow^Q$ is the dimension of $\mathbf{C}_X((A \cap Q)P)$. As P acts trivially on X, in fact $\mathbf{C}_X((A \cap Q)P) = \mathbf{C}_X(A \cap Q)$. Consequently, the multiplicities of χ in $1_{(A \cap Q)P}\uparrow^Q$ and in $1_{A \cap Q}\uparrow^Q$ are the same. This holds also with B in place of A, so the last displayed equation implies (3.10).

Now (3.9) and (3.10) show that $Q, (A \cap Q)P, (B \cap Q)P$ is an exception. Consequently, $\operatorname{core}_Q((B \cap Q)P)$ is the same as the normal core S of $(A \cap Q)P$, and $Q/S, (A \cap Q)P/S, (B \cap Q)P/S$ is a faithful exception. In view of (3.7), we may appeal to Figure 1. In particular, $MS/S \cong M/P$, so MS/S is a non-trivial normal subgroup of Q/S and a direct product of non-abelian simple groups. This implies that MS/S is contained in the socle of Q/S. We know from (3.2) that in a faithful exception the socle is minimal normal, so MS/S must be precisely the socle of Q/S. The right-hand picture in Figure 1 shows that if $k > 1$ then KS/S is a maximal normal subgroup of MS/S and, as we have noted commenting on Figure 1, the intersection of $(A \cap Q)P/S$ with KS/S is trivial. By (3.7), this cannot happen in a faithful exception, so $k = 1$. Of course, then $P = K$, $Q = N$, and $S = Z$ (by (3.6)). We noted as a consequence of (3.10) that $(B \cap Q)P \geqslant S$, so now $(B \cap N)K \geqslant Z$ and hence (see Figure 1) $Q/S, (A \cap Q)P/S, (B \cap Q)P/S$ may also be written as $N/Z, (A \cap N)Z/Z, (B \cap N)Z/Z$. Since M is a non-abelian minimal normal subgroup, the definition of Z directly implies that N/Z is almost simple, with socle MZ/Z. This completes the proof of Theorem 1.

4. *Proof of Theorem* 4

It will be convenient to continue this argument without a break, proceeding to the proof of Theorem 4. In view of (3.4) and (3.5) we are in a position to apply [**14**, Theorem (3.4)]. Since $k = 1$, $P = K$, $Q = N$, $S = Z$, and $l = |G:Q| = n$, the conclusion may be put as follows.

Choose a transversal for N in G, and use the embedding theorem to obtain a corresponding embedding ϕ of G in $N \operatorname{Wr} S_n$. Let

$$\psi: N \operatorname{Wr} S_n \longrightarrow (N/Z) \operatorname{Wr} S_n$$

and

$$\chi: (N/Z) \operatorname{Wr} S_n \longrightarrow (N/MZ) \operatorname{Wr} S_n$$

be the homomorphisms corresponding to the natural projections $N \to N/Z$ *and* $N/Z \to N/MZ$, *respectively. Set* $V = G\phi\psi\chi$.

Then V *is large as subgroup of* $(N/MZ)\,\mathrm{Wr}\,S_n$, *and its complete inverse image under* χ *is* $G\phi\psi$. *Moreover,* $\phi\psi$ *is one-to-one, and* $A\phi\psi$ *is conjugate in* $G\phi\psi$ *to* $G\phi\psi \cap S_n[(A \cap N)Z/Z]^n$.

This means that $\phi\psi$ maps G isomorphically onto $(N/Z)^*$ and it maps the conjugacy class of A to that of $[(A \cap N)Z/Z]^*$. Theorem 4 will therefore follow if we can prove that $\phi\psi$ maps the conjugacy class of B to that of $[(B \cap N)Z/Z]^*$; equivalently, that $B\phi\psi$ is conjugate in $G\phi\psi$ to $G\phi\psi \cap S_n[(B \cap N)Z/Z]^n$.

To this end we first establish that for some element x of K we have

$$(B^x)\phi = G\phi \cap S_n[(B \cap N)K]^n.$$

Apply [**8**, Theorem 4.2], with $L = (B \cap N)K$ and $H_1 = B$. The fifth sentence of the proof of that theorem indicates that, for the H defined there, $H\phi = G\phi \cap S_n L^n$; because of (3.8), the last sentence of [**8**, Theorem 4.2] gives that $H_1^x = H$. (There our x was called k.)

We have already noted, at the end of the proof of Theorem 1, that

$$(B \cap N)K = (B \cap N)Z;$$

hence $S_n[(B \cap N)K]^n$ contains the kernel Z^n of ψ. It is a general rule that if X and Y are subsets of the domain of a mapping ψ and Y contains $\ker\psi$, then $(X \cap Y)\psi = X\psi \cap Y\psi$. This allows us to deduce from $(B^x)\phi = G\phi \cap S_n[(B \cap N)K]^n$ that

$$(B^x)\phi\psi = G\phi\psi \cap S_n[(B \cap N)Z/Z]^n,$$

as required. This completes the proof of Theorem 4.

5. *Proof of Theorem* 2

In preparation for the proof of Theorem 2, we have to consider the connection between product action and tensor induction. On the latter, our principal reference is [**5**, §12A], though we shall use a slightly different notation; in particular, we shall continue to write our maps on the right and their composites accordingly.

Let ρ be a transitive permutation representation of a group U on a set Ω, and T a point stabilizer in U relative to ρ. Set $W = U\,\mathrm{Wr}\,S_n = S_n U^n$; then $S_n T^n$ is a point stabilizer relative to the corresponding product action of W on the cartesian power Ω^n. As before, let S_{n-1} be a point stabilizer in S_n, set $W_0 = S_{n-1}U^n$, and let π_0 be the projection to the first direct factor in the obvious direct decomposition $W_0 = U \times (S_{n-1}U^{n-1})$. The composite map $\pi_0\rho$ is then a transitive permutation representation of W_0 on Ω. Let $\mathbb{C}\Omega$ be the complex vector space with basis Ω regarded as a W_0-module via $\pi_0\rho$, and let $\mathbb{C}\Omega^n$ be the vector space with basis Ω^n regarded as a W-module via the product action of W on Ω^n. As a vector space, a tensor induced module $(\mathbb{C}\Omega)\!\uparrow^{\otimes W}$ is just the tensor power $\otimes^n\mathbb{C}\Omega$, and as such it has an obvious vector-space isomorphism to $\mathbb{C}\Omega^n$. It is a routine exercise to verify that that isomorphism intertwines the two actions of W, so $(\mathbb{C}\Omega)\!\uparrow^{\otimes W} \cong \mathbb{C}\Omega^n$ as W-modules. (As it stands, the last sentence is not quite right and not really fair to the reader. First, the action of W on $(\mathbb{C}\Omega)\!\uparrow^{\otimes W}$ has not been specified—it does depend on the choice of a transversal of W_0 in W. The claim that the obvious vector-space isomorphism

$(\mathbb{C}\Omega)\uparrow^{\otimes W} \cong \mathbb{C}\Omega^n$ intertwines the two actions cannot be expected to hold unless the transversal is well chosen. To name a good choice, let the S_n used to build W act on $\{1, \ldots, n\}$, and let the S_{n-1} used in defining W_0 be the stabilizer of n, say. For $i = 1, \ldots, n$, let τ_i be the element of S_n which swaps i with n and leaves all the other symbols fixed; thus $\tau_1, \ldots, \tau_{n-1}$ are transpositions and $\tau_m = 1$. As elements of W, the τ_i form a transversal for W_0, so we may let W, W_0, $\tau_1, \ldots, \tau_n$ play the role of G, H, $g_1, \ldots, g_n$ in [**5**, pp. 332–333]. Second, the reader should be warned that the routine exercise needs some care because of the dual role of S_n as top group in W and as top group in $W_0 \operatorname{Wr} S_n$.)

Of course the character of ρ is $1_T\uparrow^U$, so the character of W_0 afforded by $\mathbb{C}\Omega$ is the composite $\pi_0(1_T\uparrow^U)$; on the other hand, the character of W afforded by $\mathbb{C}\Omega^n$ is the character of the product action; so we have that

$$(5.1)\quad [\pi_0(1_T\uparrow^U)]\uparrow^{\otimes W} = 1_{S_n T^n}\uparrow^W.$$

We are now ready to proceed with the proof of Theorem 2. Recall that the notation used in Theorem 2 conflicts with that used in Theorems 1 and 4; accordingly, the conventions of our §§3, 4 no longer apply. From now on, G, A, B will denote an almost simple exception and M the socle of G, with $n > 1$ and V a large subgroup of $(G/M) \operatorname{Wr} S_n$. It will be convenient to set $W = G \operatorname{Wr} S_n$ and to think of $(G/M) \operatorname{Wr} S_n$ as W/M^n; then the definition of G^* amounts to $M^n \leqslant G^* \leqslant W$ and $G^*/M^n = V$. Recall also that $A^* = G^* \cap S_n A^n$ and $B^* = G^* \cap S_n B^n$.

We have seen that almost simple exceptions must be faithful; so $A \ngeqslant M$, and therefore $AM = G$. Moreover, $A \cap M > 1$ by [**1**, 6.3]. Thus if one thinks of G as a primitive permutation group on the set of its cosets modulo A, then [**14**, Theorem 1] becomes applicable; it yields that A^* is a corefree maximal subgroup of G^*.

As in the proof of (3.4) above, one sees that $AM = G$ implies that $BM = G$. By assumption, there is a subgroup D such that $B < D < G$. For such a D, we must now have that $B \cap M < D \cap M < M$. Set $D^* = G^* \cap S_n D^n$; as $G^* \geqslant M^n$, we get that $B^* \cap M^n = (B \cap M)^n$ and $D^* \cap M^n = (D \cap M)^n$. It follows that $B^* < D^* < G^*$, so B^* is not maximal in G^*.

It remains to prove that $1_{A^*}\uparrow^{G^*} = 1_{B^*}\uparrow^{G^*}$. To this end note first that $G^* \geqslant M^n$ and $BM = G$ yield that $(S_n B^n) G^* \supseteq S_n B^n M^n = S_n G^n = W$, so by Mackey's subgroup theorem

$$1_{S_n B^n}\uparrow^W\downarrow_{G^*} = 1_{B^*}\uparrow^{G^*}.$$

Apply (5.1) with $U = G$ and ρ the obvious permutation representation of G on the set of its cosets modulo B; one may then conclude that

$$[\pi_0(1_B\uparrow^G)]\uparrow^{\otimes W}\downarrow_{G^*} = 1_{B^*}\uparrow^{G^*}.$$

Similarly,

$$[\pi_0(1_A\uparrow^G)]\uparrow^{\otimes W}\downarrow_{G^*} = 1_{A^*}\uparrow^{G^*}.$$

The left-hand sides of the last two equations are equal (because $1_A\uparrow^G = 1_B\uparrow^G$ by assumption), hence so are the right-hand sides. This completes the proof of Theorem 2.

6. *Proof of Theorem* 3

The first steps towards the proof of Theorem 3 consist of some further analysis of G^*, A^*, B^*. We proceed with this, extending the notation of the previous section as follows. Let S_{n-1} be a point stabilizer in S_n,

$$W_0 = S_{n-1}G^n = G \times S_{n-1}G^{n-1} \geqslant M^n = M \times M^{n-1}$$

the corresponding direct decompositions, and $\pi_0 \colon W_0 \to G$ the projection of W_0 onto its first direct factor. Write K^* for the last direct factor M^{n-1}, put

$$N^* = G^* \cap W_0 = G^* \cap S_{n-1}G^n,$$
$$Z^* = G^* \cap \ker \pi_0 = G^* \cap S_{n-1}G^{n-1},$$

and let $\zeta \colon N^* \to G$ be the relevant restriction of π_0; so

(6.1) $\ker \zeta = Z^*$.

Since $G^* \geqslant M^n$, the second half of the assumption that V is large in $(G/M)\operatorname{Wr} S_n$ is that π_0 maps N^* *onto* G. It follows that π_0 maps onto A the intersection of N^* with the complete inverse image $A \times S_{n-1}G^{n-1}$ of A under π_0. In view of

$$\begin{aligned}
(A^* \cap N^*)K^* &= [(G^* \cap S_n A^n) \cap (G^* \cap S_{n-1}G^n)]M^{n-1}\\
&= [G^* \cap (S_n A^n \cap S_{n-1}G^n)]M^{n-1} = [G^* \cap S_{n-1}A^n]M^{n-1}\\
&= G^* \cap S_{n-1}A^n M^{n-1} \qquad (\text{since } G^* \geqslant M^{n-1})\\
&= G^* \cap (A \times S_{n-1}G^{n-1}) \qquad (\text{since } AM = G)\\
&= G^* \cap S_{n-1}G^n \cap (A \times S_{n-1}G^{n-1})\\
&= N^* \cap (A \times S_{n-1}G^{n-1}),
\end{aligned}$$

we have that

(6.2) $N^*\zeta = G$

and $[(A^* \cap N^*)K^*]\zeta = A$. As $Z^*\zeta = 1$ and $Z^* \geqslant K^*$, the latter equation may also be written as

(6.3) $[(A^* \cap N^*)Z^*]\zeta = A$.

Similarly,

(6.4) $[(B^* \cap N^*)K^*]\zeta = B$:

for, the only property of A used above was that $AM = G$, and we have seen in the proof of Theorem 2 that $BM = G$ also holds.

Now suppose that G_1, A_1, B_1 is another almost simple exception, write M_1 for the socle of G_1, let V_1 be a large subgroup of $(G_1/M_1)\operatorname{Wr} S_{n_1}$, and form G_1^*, A_1^*, B_1^*, K_1^*, N_1^*, Z_1^*, ζ_1 with reference to these data. Further, suppose that there is an isomorphism $\gamma \colon G^* \to G_1^*$ such that $A^*\gamma$ is conjugate to A_1^* and $B^*\gamma$ is conjugate to B_1^* (in G_1^*). By restriction, γ yields an isomorphism of the socle of G^* onto the socle of G_1^*, so the two socles must have the same number of simple direct factors: $n = n_1$. At the cost of replacing γ by a composite with an inner automorphism of G^* if necessary, we can arrange that $K^*\gamma = K_1^*$. Then $N^* = \mathrm{N}_{G^*}(K^*)$, $Z^* = \mathrm{C}_{G^*}(M^n/K^*)$, etc. ensure that $N^*\gamma = N_1^*$ and $Z^*\gamma = Z_1^*$, so γ induces an isomorphism, say δ, of N^*/Z^* onto N_1^*/Z_1^*. Let $\bar{\zeta}: N^*/Z^* \to G$ and $\bar{\zeta}_1: N_1^*/Z_1^* \to G_1$ denote the isomorphisms induced by ζ and ζ_1, respectively: the inverse of $\bar{\zeta}$ followed by δ and then by $\bar{\zeta}_1$ is then an isomorphism, γ^* say, of G onto G_1. By [**11**, Theorem 3.01], the assumption that $A^*\gamma$ is conjugate to A_1^* in G_1^* implies that $(A^*\gamma \cap N_1^*)K_1^*$ is conjugate to $(A_1^* \cap N_1^*)K_1^*$ in N_1^*; consequently $(A^*\gamma \cap N_1^*)Z_1^*$ is conjugate to $(A_1^* \cap N_1^*)Z_1^*$ in N_1^*. Of course, $(A^*\gamma \cap N_1^*)Z_1^* = [(A^* \cap N^*)Z^*]\gamma$, so we may conclude that $[(A^* \cap N^*)Z^*/Z^*]\delta$ is conjugate in N_1^*/Z_1^* to $(A_1^* \cap N_1^*)Z_1^*/Z_1^*$. As $[(A^* \cap N^*)Z^*/Z^*]\bar{\zeta} = A$ and $[(A_1^* \cap N_1^*)Z_1^*/Z_1^*]\bar{\zeta}_1 = A_1$, we have that $A\gamma^*$ is conjugate to A_1 in G_1. Similarly, $B\gamma^*$ is conjugate to B_1, for none of the arguments involves the maximality of A.

This proves that γ^* is an isomorphism of the almost simple extensions G, A, B and G_1, A_1, B_1. We now identify G with G_1 along γ^*; then A, A_1 become conjugate maximal subgroups and B, B_1 conjugate non-maximal subgroups in the one group G. Moreover, G^* and G_1^* become subgroups of the same wreath product $G \operatorname{Wr} S_n$, both containing M^n, with $G^*M^n = V$ and $G_1^*/M^n = V_1$. We shall complete the proof of the 'only if' part of Theorem 3 by showing that G^* and G_1^* are conjugate in $G \operatorname{Wr} S_n$.

To this end, we shall use the uniqueness theorem of [**10**] or its paraphrase from [**12**], comparing the inclusion ψ of G^* in $G \operatorname{Wr} S_n$ with the composite $\gamma\psi_1$ of γ and the inclusion ψ_1 of G_1^* in $G \operatorname{Wr} S_n$. First, use the projection $\pi: G \operatorname{Wr} S_n \to S_n$ (corresponding to the defining semidirect decomposition of this wreath product as $S_n G^n$). From $N^*\gamma = N_1^*$ we see that N^* is the stabilizer in G of the point o (whose stabilizer in S_n we took as S_{n-1}) with respect to both permutation representations $\psi\pi$, $\gamma\psi_1\pi$. Since V and V_1 are large, both permutation representations are transitive. It follows that there is an element τ in S_{n-1} such that $\psi\pi(\operatorname{inn}\tau) = \psi(\operatorname{inn}\tau)\pi = \gamma\psi_1\pi$, where the first $\operatorname{inn}\tau$ denotes the inner automorphism of S_n, and the second that of $G \operatorname{Wr} S_n$, induced by τ.

Consider an arbitrary element x of N^*. Since $\tau \in \ker \pi_o$,

$$x\psi(\operatorname{inn}\tau)\pi_o = x\psi\pi_o = x\zeta = (xZ^*)\bar{\zeta};$$

on the other hand,

$$x\gamma\psi_1\pi_o = x\gamma\zeta_1 = (x\zeta Z_1^*)\bar{\zeta}_1 = (xZ^*)\delta\bar{\zeta}_1.$$

By the definition of γ^* we have $(xZ^*)\bar{\zeta}\gamma^* = (xZ^*)\delta\bar{\zeta}_1$; as we have identified G and G_1 along γ^*, this means that $x\psi(\operatorname{inn}\tau)\pi_o = x\gamma\psi_1\pi_o$. The uniqueness theorem now gives that there is an element f in G^{n-1} such that $x\psi(\operatorname{inn}\tau)(\operatorname{inn}f) = x\gamma\psi_1$. Consequently, G^* is the conjugate of G by the element τf of $G \operatorname{Wr} S_n$.

This completes the proof of the 'only if' part of Theorem 3. The 'if' part is obvious.

REMARK. Theorem 3 does *not* mean that if one almost simple exception G, A, B is used with two non-conjugate large subgroups V, V_1 of a $G \operatorname{Wr} S_n$ then the exceptions G^*, A^*, B^* and G_1^*, A_1^*, B_1^* cannot be isomorphic. It is conceivable that G could have an automorphism stabilizing setwise both the conjugacy class of A and the conjugacy class of B, and being such that the corresponding automorphism of $G \operatorname{Wr} S_n$ maps V to V_1. (As we know no exceptions at all, of course we cannot possibly give a counterexample to that spurious consequence of Theorem 3; it could even be true, perhaps vacuously, but not simply as a consequence of our arguments.)

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