FREE GROUPS IN A DIHEDRAL VARIETY

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ABSTRACT

A formula is given for the orders of the finite rank free groups of the variety generated by an arbitrary finite dihedral group.

1. Fine in [2] and Hurley in his review of [2] (see *Mathematical Reviews* 87j:20049) expressed interest in the precise orders of the finite rank free groups of the variety generated by a finite dihedral group. The aim of this note is to derive this information from results on the classification of varieties of metabelian groups.

Theorem. *The free group $F_r(\mathrm{var}D)$ of rank r in the variety generated by the dihedral group D of order $2^{d+1}e$, with e odd, has order $2^{r+s}e^{r'}$ where $r' = (r+1)2^r + 1$ and*

replace (r + 1) by (r - 1)

$$s = \sum_{t=2}^{d} (d+1-t)(t-1)\binom{r+1}{t}.$$

The relevant conventions are that $s = 0$ when $d < 2$ and the binomial coefficient vanishes when $t > r + 1$. While s is a polynomial function of degree d in r for $r \geqslant d \geqslant 2$, for $2 \leqslant r \leqslant d$ it is exponential: it is easy to deduce from the formula above that in this range $s = [r(2d+1-r) - 2(d+1)]2^{r-1} + d + 1$. The proof of the theorem will include showing that, in the notation of Neumann [4],

$$\mathrm{var}D = \begin{cases} \mathfrak{A}_e\mathfrak{A}_2 & \text{when } d < 2, \\ \mathfrak{A}_e\mathfrak{A}_2 \vee (\mathfrak{A}_{2^{d-1}}\mathfrak{A}_2 \wedge \mathfrak{N}_d) & \text{when } d \geqslant 2. \end{cases}$$

2. Let a, b be involutions which together generate D, so the order of ab is $2^d e$. The subgroup generated by ab is the direct product of a group of order 2^d and a group of order e. Accordingly, D is a subdirect product of two dihedral groups: one of order $2e$, the other of 2^{d+1}. Let $\mathfrak{U}$ be the variety generated by the first subdirect factor, and $\mathfrak{V}$ that generated by the other: then $\mathrm{var}D = \mathfrak{U} \vee \mathfrak{V}$ and $\mathfrak{U} \wedge \mathfrak{V} = \mathfrak{A}_2$. Thus $F_r(\mathrm{var}D)$ is the subdirect

product of $F_r(\mathfrak{U})$ and $F_r(\mathfrak{B})$ amalgamating precisely $F_r(\mathfrak{A}_2)$. Consequently, it suffices to prove the theorem under the additional hypothesis that either $d = 0$ or $e = 1$.

3. Take first the case $d = 0$, $e > 1$. In this case obviously $\mathfrak{A}_{2e} \leqslant \mathrm{var}D \leqslant \mathfrak{A}_e\mathfrak{A}_2$. By a result of Houghton [4, 54.42], if $\mathrm{var}D$ were a proper subvariety of $\mathfrak{A}_e\mathfrak{A}_2$ then D would satisfy a law of the form $[x^2, y]^f$ with f a proper divisor of e. The substitution $x \mapsto ab$, $y \mapsto b$, shows that D satisfies no such law. Thus $\mathrm{var}D = \mathfrak{A}_e\mathfrak{A}_2$, and $|F_r(\mathrm{var}D)| = 2^r e^{r'}$ by 21.13 of [4]. (Alternatively, one might view $\mathrm{var}D = \mathfrak{A}_e\mathfrak{A}_2$ as a special case of the results in [1, chapter 4].)

4. Take next the case $e = 1$. If $d \leqslant 1$ then $\mathrm{var}D = \mathfrak{A}_2$ and the claim is obvious: so in the sequel it will also be assumed that $d \geqslant 2$.

The results needed here all come from [3], which is to be read with $p = 2$ and $\alpha = d - 1$. The notation of that paper will be used here without explanation. Since the exponent of D is 2^d, the invariant $\beta(\mathrm{var}D)$ is $\alpha + 1$: so the second term is redundant in the normal form of $\mathrm{var}D$ given in [3, 2.1]. To find the invariant $\nu(\tau, \mathrm{var}D)$, note that among the abelian normal subgroups of exponent dividing 2^τ there is a unique maximal one, and that the quotient modulo that is a dihedral group of nilpotency class precisely $d - \tau$. When $d - \tau$ is even and $d - \tau \geqslant 4$, we need to see whether that quotient satisfies the law

$$\prod_{i=2}^{d-\tau} [x_i, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{d-\tau}]:$$

substitute b for x_2 and a for all the other variables to see that the answer is negative. Thus $\nu(\tau, \mathrm{var}D) = d - \tau$ for all relevant τ. By [3, 4.03], the terms corresponding to $\tau = 1, \ldots, \alpha - 1$ are therefore redundant in the normal form of $\mathrm{var}D$ given by 2.1, so that the normal form is simply

$$\mathrm{var}D = \mathfrak{A}_{2^{d-1}}\mathfrak{A}_2 \wedge \mathfrak{N}_d .$$

Next we must count, for $t \geqslant 3$, the number of those elements of weight t in the set $\mathscr{B}$ which lie in the subgroup of H generated by $a_1, \ldots, a_r$, that is, which have weight 0 in each a_i with $i > r$. (The definition of $\mathscr{B}$ comes just before 4.05 in [3], the definition of weight and of weight in an a_i just before 4.06.) Such an element must have positive weight in either t or $t - 1$ of the generators; the relevant subset of $\{a_1, \ldots, a_r\}$ may of course be chosen in $\binom{r}{t}$ or $\binom{r}{t-1}$ different ways. Having chosen a t-element subset, one must put its first element as second commutator entry, can choose any of its remaining $t - 1$ elements as first commutator entry, and then the other elements must be used as commutator entries in their natural order: so one obtains precisely $t - 1$ elements of $\mathscr{B}$. When a $(t - 1)$-element subset is chosen, one can still choose which of the $t - 1$ to use twice, but

there is no freedom beyond that: so again one obtains precisely $t-1$ elements of $\mathfrak{B}$. As $\binom{r}{t}+\binom{r}{t-1}=\binom{r+1}{t}$, the conclusion of the count is that $\mathscr{B}$ has precisely $(t-1)\binom{r+1}{t}$ elements of weight t in the subgroup of H generated by $a_1, \ldots, a_r$. Obviously, it has $\binom{r}{2}$ such elements of weight 2.

That subgroup of H is of course a rank r free group of $\mathfrak{A}_{2^\alpha}\mathfrak{A}_2$, so $F_r(\mathrm{var}D)$ is just the quotient of that modulo the relevant term of its lower central series: that is, modulo its intersection with $N(0, d)$. The count performed above enables one to use [3, 4.06] for calculating the order of the quotient modulo that intersection, with the conclusion that

$$s = \alpha r + \alpha\binom{r}{2} + \sum_{t=3}^{\alpha+1}(\alpha+2-t)(t-1)\binom{r+1}{t}.$$

This completes the proof of the theorem.

REFERENCES

[1] COSSEY, P. J. 1966 On varieties of A-groups. Ph.D. thesis, Australian National University.
[2] FINE, B. 1986 The free groups in the dihedral variety. *Arch. Math.* **46**, 193–7.
[3] KOVÁCS, L. G. and NEWMAN, M. F. 1971 On non-Cross varieties of groups. *J. Austral. Math. Soc.* **12**, 129–44.
[4] NEUMANN, H. 1967 *Varieties of groups*. Berlin. Springer.