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A long time ago, stimulated by a paper of D. W. Barnes : "On complemented chief factors of finite soluble groups" Bull. Austral. Math. Soc. 7 (1972) 101-104, we discussed sharpened forms of the Jordan-Hölder Theorem. The results of this discussion were written down in a letter by one of us to Barnes. The issues addressed have cropped up from time to time in conversation and correspondence with others. There has, by now, been sufficient interest in them to justify some archiving. So we reproduce the letter here in typed form (with only some minor cosmetic editing).

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Dear Don,

Mike and I have had some long discussions, provoked by the paper you are currently publishing in the Bulletin. Here is a description of some stray results we <u>think</u> we have : the proofs haven't been written down yet, and I would appreciate your comments on what if any of this may be worth writing down in full.

To begin with, let us consider a possibly infinite group G with a finite chief series $1 = H_0 < \ldots < H_n = G$. By the Jordan-Hölder Theorem, every other chief series has the same length; let $1 = K_0 < \ldots < K_n = G$ be one. Let $X/Y \in \Phi$ mean that X/Y is a Frattini chief factor, and $X/Y \notin \Phi$ that X/Y is a non-Frattini chief factor. Our first result is that one of your theorems, with suitable modification, holds in this generality, and that the correspondence between the non-Frattini chief factors is unique :

(1) There is a permutation β of $\{1,\ldots,n\}$ and to each i with $H_i/H_{i-1} \notin \Phi$ a maximal subgroup M_i such that M_i supplements H_i/H_{i-1} and $K_{i\beta}/K_{i\beta-1}$ (that is, $H_{i-1}K_{i\beta-1} \leq M_i$ but $H_iM_i = K_{i\beta}M_i = G$) . If β' also has this property, then

$$i\beta = i\beta' = \max\{j \mid H_iK_{j-1}/H_{i-1}K_{j-1} \notin \Phi\} \quad \text{whenever} \quad H_i/H_{i-1} \notin \Phi .$$

It follows that the number of Frattini chief factors in a chief series of G is independent of the chief series, for β must respect the Φ or non-Φ nature of chief factors. It would be nice to know that β can be chosen so as to also respect the G-isomorphism type of the chief factors : however, in general this is not possible. We have an example of a group whose normal subgroup lattice is finite and distributive, yet it has two chief series such that each has just one chief factor of a specific G-isomorphism type, in the first series this is Frattini, in the other it is non-Frattini (in fact, complemented by a maximal subgroup). What can be salvaged here? The permutation β can be chosen to behave perfectly for all finite or abelian chief factors, and such trouble is only possible in relation to infinite nonabelian Frattini chief factors. Also, β can be chosen so as to respect Frattini type <u>and</u> G-similarity : we call two chief

factors of G similar if the automorphism groups induced on them by G are isomorphic *quâ* automorphism groups. (If A is a group of automorphisms of X and B a group of automorphisms of Y, we say A and B are isomorphic *quâ* automorphism groups if there exist ordinary isomorphisms $\varphi : A \rightarrowtail\!\!\!\!\twoheadrightarrow B$, $\psi : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y$ such that

$$\forall x \in X \,.\, \forall a \in A \,.\, (x^a)\psi = (x\psi)^{(a\varphi)}.)$$

All this has also provoked a careful look at the Jordan-Hölder Theorem itself. The upshot is as follows. Let Λ be a modular lattice of finite length (say, the normal subgroup lattice of G above), and $H_0 < \ldots < H_n$, $K_0 < \ldots < K_n$ two maximal chains in Λ. If $X, Y \in \Lambda$, write $(X \wedge Y, X) \preccurlyeq (Y, X \vee Y)$ and say $(X \wedge Y, X)$ is perspective to $(Y, X \vee Y)$. On the set of "chief factors" or "prime intervals" of Λ, *i.e.* on

$$\{(U, V) \mid U, V \in \Lambda, U < V, \quad U \le Z \le V \Rightarrow Z = U \text{ or } V\},$$

the relation $\preccurlyeq$ is easily seen to be a partial order; call the equivalence relation generated by $\preccurlyeq$, projectivity. The usual statement asserts that there is a permutation γ on $\{1,\ldots,n\}$ such that (H_{i-1}, H_i) and $(K_{i\gamma-1}, K_{i\gamma})$ are projective for all i; and such a γ is always given by some equivalent of the definition $i\gamma = \max\{j \mid H_{i-1} \vee K_{j-1} < H_i \vee K_{j-1}\}$. The γ so defined is in fact distinguished in a number of ways: (2) it is the only permutation for which corresponding chief factors are projective within the sublattice generated by the H_i and K_j (by Theorem 5 on p. 72 of the 1948 edition of Birkhoff, this sublattice is finite and distributive); (3) it is the only permutation such that to each i there is an (X_i, Y_i) perspective to both (H_{i-1}, H_i) and $(K_{i\gamma-1}, K_{i\gamma})$; (4) it is the only permutation such that to each i there is a (U_i, V_i) to which both (H_{i-1}, H_i) and $(K_{i\gamma-1}, K_{i\gamma})$ are perspective.

Coming closer to the situation considered by you, assume G is finite but not necessarily soluble. It is easy to see that for two chief factors to have a common maximal supplement (as in (1) above) it is now necessary and sufficient that they be both perspective to a (common) non-Frattini chief factor. This opens up the way to dualization of your condition, and we have:-

(5) if G is finite, there is a unique permutation β such that

(5a) for each i with $H_i/H_{i-1} \notin \Phi$ there exists a non-Frattini X_i/Y_i to which both H_i/H_{i-1} and $K_{i\beta}/K_{i\beta-1}$ are perspective; and

(5b) to each i with $H_i/H_{i-1} \in \Phi$, there is a Frattini X_i/Y_i

perspective to both H_i/H_{i-1} and $K_{i\beta}/K_{i\beta-1}$.

Namely,

$$i\beta = \begin{cases} \max\,\{j \mid H_iK_{j-1}/H_{i-1}K_{j-1} \notin \Phi\} & \text{if } H_i/H_{i-1} \notin \Phi \;, \\ \min\,\{j \mid (H_i \cap K_j)/(H_{i-1} \cap K_j) \in \Phi\} & \text{if } H_i/H_{i-1} \in \Phi \;. \end{cases}$$

In this situation, of course, corresponding chief factors are G-isomorphic. (This definition of β could also be used in the case of infinite G , but there we have no convenient interpretation for the relationship set up by β between the Frattini chief factors of the two series.) Without the solubility assumption you had, one cannot quite expect common complements for corresponding non-Frattini chief factors: it is not hard to make an insoluble finite group with two chief series which have different numbers of complemented chief factors. On the other hand, a supplement of an abelian chief factor is always a complement, so (1) and (5) really generalize your Theorem 2.

Our proofs are based on the key idea in the proof of Lemma 2.6 in the Carter-Fischer-Hawkes paper, and are largely lattice theoretic. Using that, (5) is quickly translated to the context of a modular lattice Λ of finite length, with the set of "chief factors" partially ordered by perspectivity, and the set Φ of Frattini chief factors being an arbitrary subset subject to two axioms :

(A1) If $(X, Y) \in \Phi$ and $(X, Y) \leqslant (U, V)$ then $(U, V) \in \Phi$

(A2) If $(X, Y) \notin \Phi$, $(U, V) \in \Phi$, $(X, Y) \leqslant (U, V)$, and (X, U) is a chief factor, then $\exists\ F \in \Lambda\ .\ (X, F) \in \Phi$ and $(F, V) \notin \Phi$;

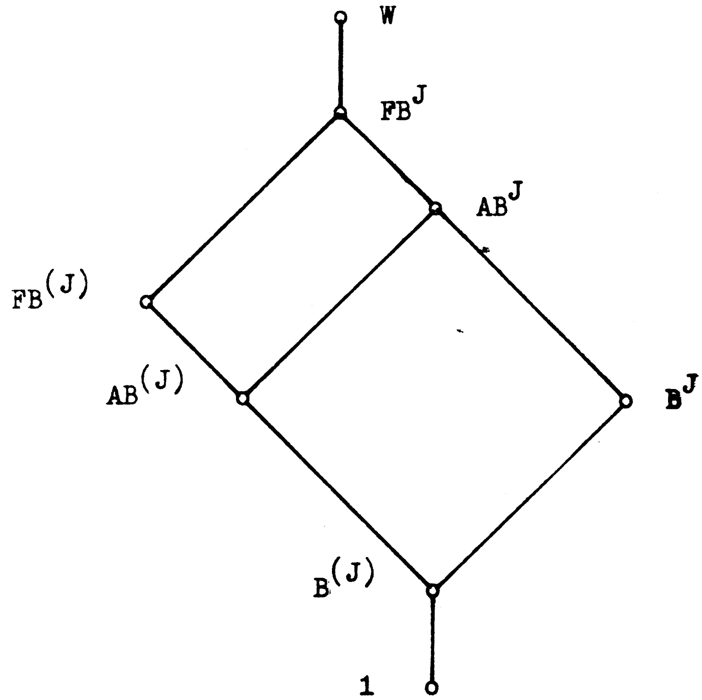
and it is this lattice version of (5) that we deal with. [Incidentally, the empty set and the set of all chief factors of Λ both satisfy the conditions (A1) and (A2) on Φ , (4) and (3) arise as special cases of the lattice version of (5).]

Before closing, perhaps I should describe the examples I mentioned. For the first one, let B be a nonabelian finite simple group, S the full

symmetric group on a countably infinite set J and W the unrestricted permutational wreath product :

$$W = B \,\mathrm{Wr}\, S = S B^J .$$

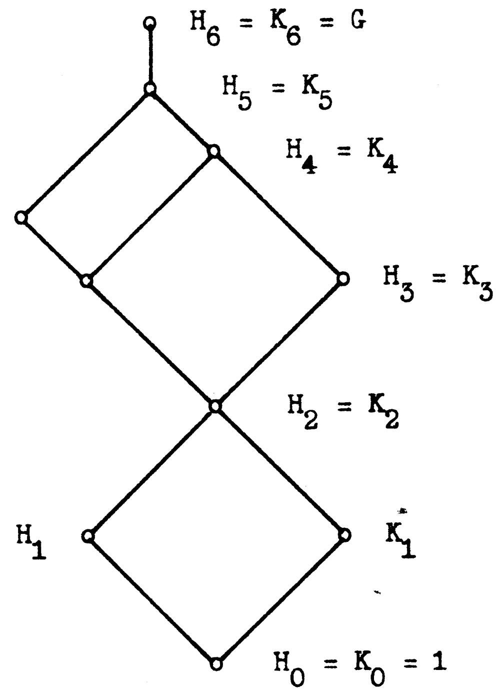
The normal subgroup lattice of W is fairly easily found to be the following, with F the set of finitary permutations on J and A the alternating part of F :



The next key fact is that, according to a result of B. H. Neumann (MZ 87 (1965)), $B^{(J)}$ is not complemented in B^J , from which one derives that $B^{(J)}$ has no proper supplement in W . Take now G to be the subgroup of the direct square $W \times W$ of W generated by the diagonal $W\delta$ and $B^{(J)} \times 1$; put

$H_1 = B^{(J)} \times 1$, $K_1 = 1 \times B^{(J)}$, $H_2 = K_2 = B^{(J)} \times B^{(J)}$, $H_3 = K_3 = H_2(B^J\delta)$,
$H_4 = K_4 = H_2(A\, B^J)\delta$, $H_5 = K_5 = H_2(F\, B^J)\delta$, $H_6 = K_6 = G$.

The only chief factor K_j/K_{j-1} which is G-isomorphic to H_1 is clearly K_2/K_1 . Now, $G = H_1(W\delta)$, $H_1 \cap W\delta = 1$, and it is easy to see that $W\delta$ is maximal in G ; so $H_1/1 \notin \Phi$. However, $K_2/K_1 \in \Phi$. For, if M is a maximal subgroup containing K_1 then M/K_1 is a maximal subgroup of $G/K_1 \cong W$; as in this isomorphism K_2/K_1 corresponds to $B^{(J)}$ which has no proper supplement in W , one must have $M/K_1 \geq K_2/K_1$. The diagram of the normal subgroup lattice of G is



The second example is much easier. Let $\mathfrak{A}$ be the automorphism group of the alternating group A_6. As is well known, $\mathfrak{A}/A_6 = C_2 \times C_2$ (we consider A_6, and also S_6, embedded in $\mathfrak{A}$), and - as is perhaps less well known - $\mathfrak{A}$ does not split over A_6. [Indeed, every element of $\mathfrak{A}$ but not of S_6 is known to interchange the two classes of odd involutions in S_6, and a complement of A_6 in $\mathfrak{A}$ would have to contain an odd involution and an element of $\mathfrak{A}$, outside S_6, centralizing that involution.] Let T denote a Sylow 2-subgroup of $\mathfrak{A}$, and let G be the subgroup of the direct square $\mathfrak{A} \times \mathfrak{A}$ of $\mathfrak{A}$ generated by $A_6 \times 1$ and the diagonal copy $T\delta$ of T. The complemented chief factors of G are the abelian non-Frattini factors and some of the nonabelian chief factors. Each chief series has just one nonabelian chief factor; a chief series through $A_6 \times 1$ has its nonabelian factor complemented (by $T\delta$, say). However, another chief series may be taken through $1 \times (T \cap A_6)$ and $A_6 \times (T \cap A_6)$, and here

$$(A_6 \times (T \cap A_6))/(1 \times (T \cap A_6))$$

is not complemented. Indeed, if M were a complement, its projection into the first direct factor $\mathfrak{A} \times 1$ of $\mathfrak{A} \times \mathfrak{A}$ would have to supplement $A_6 \times 1$, but as $A_6 \times 1$ has no complement in $\mathfrak{A} \times 1$, we would have to have an element $a \times b$ in M with $1 \neq a \in A_6$. As $G = (A_6 \times 1)(T\delta)$, we would have to have $a \times b = (c \times 1)(t \times t)$ with $c \in A_6$, $t \in T$; hence $a = ct$

would show $t \in A_6 \cap T$ and so $a \times b \in A_6 \times (T \cap A_6) \cap M = 1 \times (T \cap A_6)$, contrary to $a \neq 1$ above.

In the hope that I have not overstated any of our claims, and that you will excuse the hurried nature of this note, I am sending it on without risking further indefinite delays by attempting to polish it up.

All the best,

Laci.