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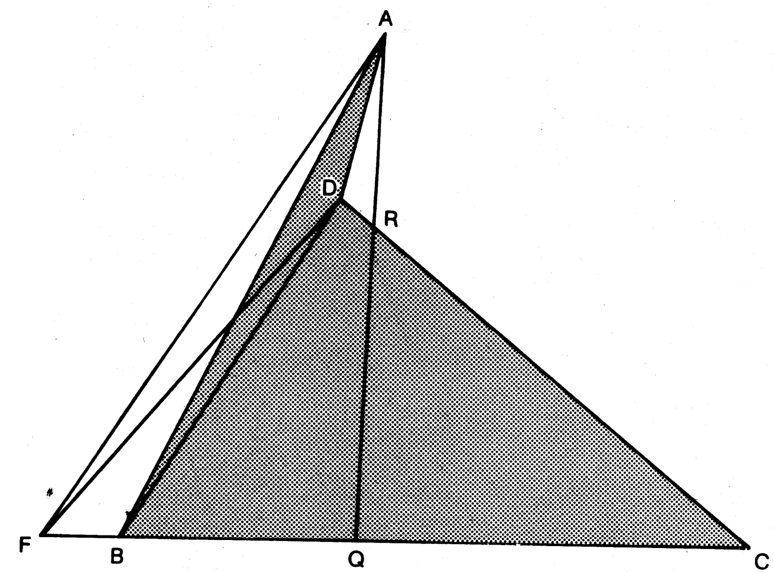

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BISECTION OF A QUADRILATERAL BY A LINE THROUGH A VERTEX

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One of the Hungarian mathematical competitions in 1978 included the following problem:

> Given a convex quadrilateral, construct a line through one of its vertices so as to cut the quadrilateral into two parts whose areas are equal.

This problem appears in textbooks on elementary Euclidean geometry (e.g. [1]) and, with the restriction that the quadrilateral be convex, is simple enough to solve.

However, as contestants in a mathematical competition are encouraged to generalize, one should inquire whether the solution can be adapted to deal with non-convex quadrilaterals. While some difficulty could be expected in ensuring that all cases are considered, it came as something of a surprise to us that in two cases a completely different approach was required.

J.C.Barton of the University of Melbourne has drawn our attention to [2] where, on page 77, under the general heading of "Area constructions by equivalent triangles or parallelograms" we find:

> The bisection of a triangle by a line drawn from a point in a side. The bisection of a quadrilateral by a line from a corner ...is a nice extension of this.

This reference to a "nice extension" may imply that the solution is straight-forward, as indeed it is when the quadrilateral is convex. On the other hand, the choice of the adjective "nice" rather than, say, "simple" or "routine", may have been intended as an indication that the extension is not without special interest of its own and as an invitation to the reader to pursue the matter further. If this was the case, the authors of the Report at least provided a clue, for, having referred as above to the bisection of the quadrilateral, they went on to remark that

> It is interesting to note that the obvious extension, the bisection of a triangle by a line through an external point, is specialist work, and hard at that, ...

It turns out that in order to provide a complete solution of the quadrilateral problem we need to be able to carry out the construction of a line through an arbitrary point to bisect a given triangle. Although a solution of this latter problem was evidently familiar to the authors of the Report, we have not been able to find one in print and the problem is discussed at some length in [3]. In the relevant part of [3], it is also required that a particular vertex of the given triangle lie in a triangular (rather than a quadrilateral) portion of the bisection, and that the bisector not go through this vertex. The number of such bisectors through a given point of the plane may be 0, 1, or 2, and [3] gives a construction for them.

Even if we assume the construction for bisecting the triangle, the original quadrilateral problem has more to it than appears at first sight and it seems worth while to offer a complete solution.

We shall begin with a construction which is adequate for a convex quadrilateral and then explore the possibilities of applying it to non-convex quadrilaterals. Let the quadrilateral be $ABCD$ with vertex C opposite vertex A and let A be the vertex through which the required line is to be drawn. In the first instance we distinguish four types of quadrilateral: (A) convex qudrilaterals and (B) non-convex quadrilaterals divided into three classes according to the position of the reflex angle relative to the vertex A: (B1) opposite A, (B2) at A, (B3) adjacent to A. Later it will be necessary to divide each of the classes (B2) and (B3) into three sub-classes.

We note first that if the diagonal AC bisects the diagonal BD, then AC bisects the quadrilateral. As shown in figure 1, this result holds for convex quadrilaterals and for non-convex quadrilaterals when the reflex angle is either opposite to A or at A. When the reflex angle is adjacent to A (B3), it is impossible for AC to bisect BD. In what follows, it will be assumed that the mid-point M of BD does not lie on AC.

The construction for the convex quadrilateral is of course well-known. It is set out now in a form which allows it to be used in other cases as well.

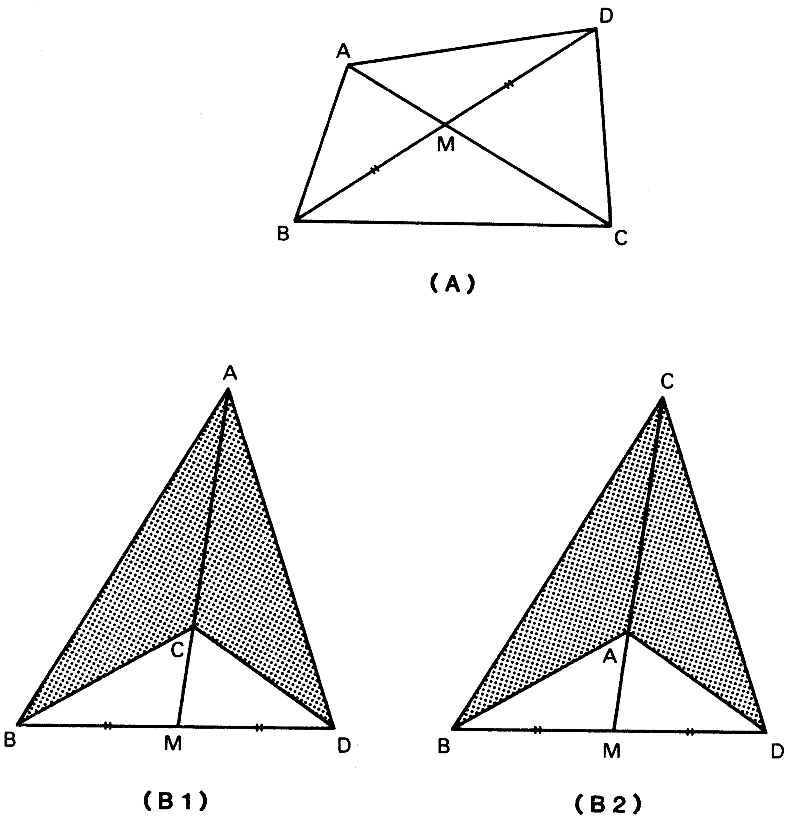


Figure 1.

Draw a line through the mid-point M of the diagonal BD parallel to the other diagonal AC to meet one (and, as M is not on AC, only one) of the segments BC, CD in a point P; and name the vertices in such a way that P lies on BC (in fact, strictly between B and C). Then the line AP is the required bisector provided the segment AP lies wholly within the quadrilateral $ABCD$ and is the only part of the line to do so. The construction is illustrated in figure 2 for cases (A), (B1) and (B2). Case (B2), in which the reflex angle is at the vertex A, is divided into three categories (B2a), (B2b), (B2c), according as BP is greater than, equal to, or less than BD' where D' is the point where DA cuts BC.

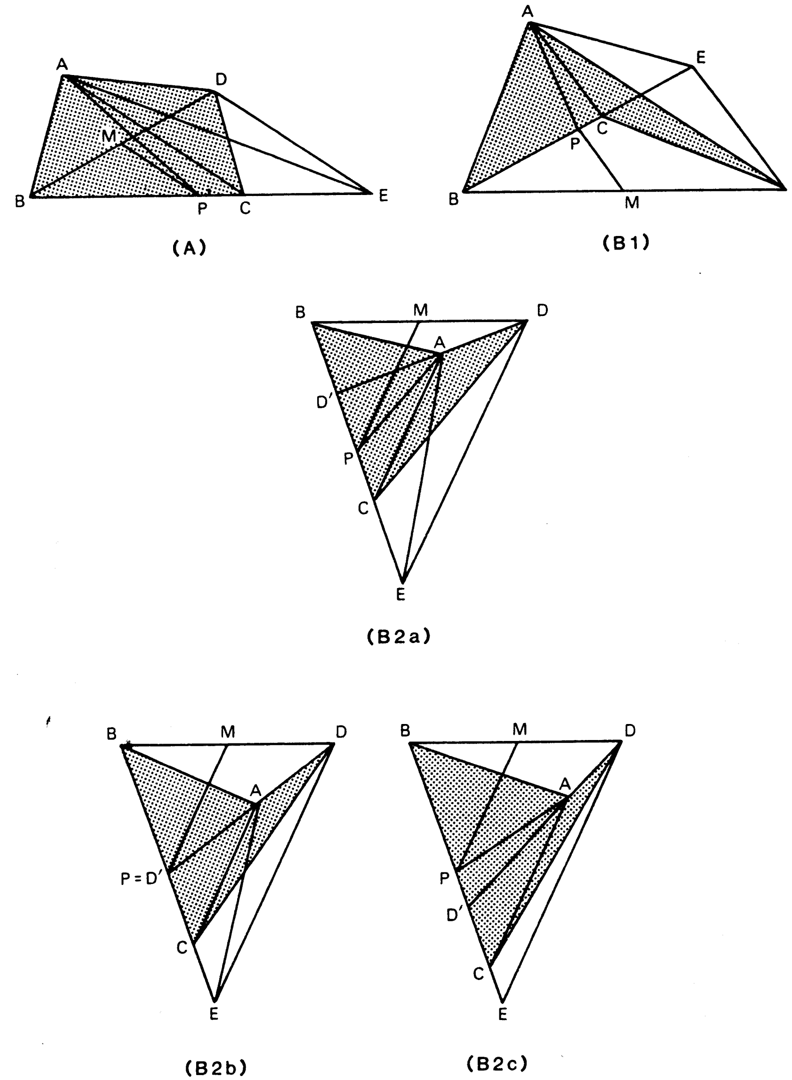


Figure 2

In all these cases, the construction produces a segment AP which lies wholly within the quadrilateral and, as we now show, divides the quadrilateral into two parts of equal area. Let the line through D parallel to AC meet the side BC produced beyond C in E. Since MP and DE are parallel and M is the mid-point of BD, $BP = PE$ and it follows that area ABP = area APE. Also area ACD = area ACE for the altitudes corresponding to the common side AC of these triangles are equal, being the distance between the parallel lines DE, AC. In all cases (A), (B1), (B2*a*), (B2*b*), (B2c) in figure 2 we now have

$$\begin{aligned} \text{area } ABP &= \text{area } APE \\ &= \text{area } APC + \text{area } ACE \\ &= \text{area } APC + \text{area } ACD \\ &= \text{area } APCD \,. \end{aligned}$$

Since area $ABCD$ = area ABP + area $APCD$, the segment AP bisects the quadrilateral.

It will be noted that in case (B2c), the distinction between the line AP and the segment AP becomes important. In the other four cases in figure 2, the only part of the line AP to lie within the quadrilateral is the segment AP so the construction has produced the required line through A. This is not so in (B2c) where the segment AP bisects the quadrilateral but the line evidently does not. Further constructon is needed in this case.

We note first that because area ABP = (1/2) area $ABCD$ > (1/2) area ABC, it follows that $BP > (1/2)BC$ so $BP > CP$. In figure 3, we choose Y on BC so $CY = BP$. Then $CY > CP$; thus the order of points on BC is $BYPD'C$.

Draw the line through A parallel to BC to cut CD in X; choose Z so that X is the mid-point of CZ. Three subcases arise according as CZ is greater than, equal to or less than CD; we picture in figure 3 only two as $Z = D$ can be handled as a degenerate version of, say, the first. Even for these two sub-cases, the arguments will not branch for a while.

The triangle CYZ has area double that of CYX which, because $CY = BP$ and AX is parallel to BC, has area equal to that of ABP and so to half that of the quadrilateral.

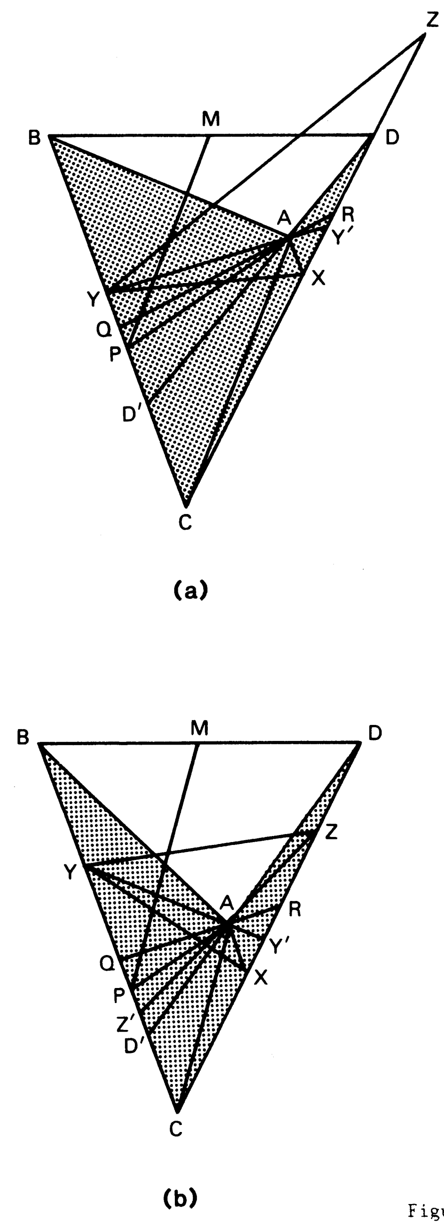


Figure 3.

We shall prove that there exists a line QAR, with Q on the segment YD' and R on the segment CD, which bisects the area of triangle CYZ. Then the area of CQR is half that of CYZ or of the quadrilateral so the line also bisects the quadrilateral; on the other hand, as a bisector of CYZ through A, the line (or perhaps two such lines) can be constructed by the method of [3].

The first point is to observe that A always lies inside the triangle CYZ, else this triangle would be a proper part of the quadrilateral in spite of their areas being equal. Let Y' be the intersection of YA and CZ; then Y' lies on the segment CZ, and also between C and D because D' is between C and Y. Moreover, area $CYY' >$ area CYA = area BPA (because $CY = BP$) = (1/2) area $ABCD$; thus area $CYY' >$ (1/2) area CYZ. Similarly, area $CDD' <$ area $PCDA$ (because D' lies between P and C) = (1/2) area $ABCD$; so area $CDD' <$ (1/2) area CYZ.

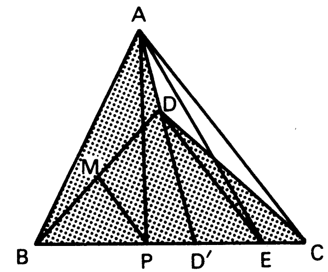
Continuity now guarantees the existence of a point Q between Y and D' such that QA cuts CD in R between Y' and D and area CQR = (1/2) area CYZ. When $CZ \geq CD$ (as in figure 3*a*), we have R on CZ so QAR bisects both the quadrilateral and triangle CYZ and our aim has been achieved.

It remains to consider the case in which $CZ < CD$ (figure 3*b*). We define Z' as the point where ZA cuts BC. Because $CZ < CD$ and A lies inside triangle CYZ, Z' lies between D' and Y, and Y' therefore lies between C and Z. Moreover, because X is the mid-point of CZ and AX is parallel to BC, A is the mid-point of ZZ'.

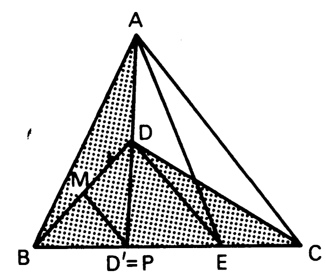
It is known [4], pp 89, 122, that of all triangles with vertex C, sides along CB and CD and base passing through A, the one with the smallest area is obtained when the base is bisected by A. Hence area $CDD' >$ area CZZ'.

We have already shown that 1/2 area $CYZ >$ area CDD' so we now have 1/2 area $CYZ >$ area CZZ'. As before, area $CYY' >$ 1/2 area CYZ so, by continuity, there is a suitable Q between Y and Z' and a matching point R between Y' and Z so that QAR bisects both the quadrilateral and the triangle as required. This completes the investigation of case (B2c).

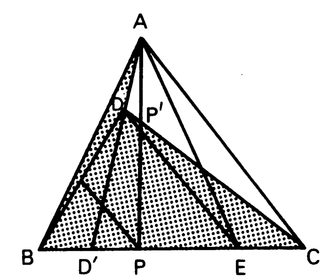
We turn now to case (B3) in which the reflex angle is adjacent to the vertex A. The construction is carried out exactly as before and is illustrated in figure 4. With D' defined as before as the intersection of AD and BC, we this time distinguish three cases (B3a), (B3b), (B3c) according as BP is less than, equal to, or greater than BD'.



(B3a)



(B3b)



(B3c)

Figure 4.

In all three cases,

$$\text{area } ABP = 1/2 \text{ area } ABE$$
$$= 1/2 \text{ (area } ABED + \text{area } DEA)$$
$$= 1/2 \text{ (area } ABED + \text{area } DEC)$$
$$= 1/2 \text{ area } ABCD .$$

Hence, in cases (B3a), (B3b), AP is the required bisector. In case (B3c) however, the construction is not achieved as the segment AP does not lie wholly within the quadrilateral.

In case (B3c) we see that

$$\text{area } ABD' < \text{area } ABP = 1/2 \text{ area } ABCD .$$

Thus area CDD' > 1/2 area $ABCD$; hence there is a line ARQ (figure 5) which bisects the quadrilateral, cutting CD in R and BC and Q . We proceed to construct such a line.

This is easily done if we first construct as in the front cover figure triangle DFC equal in area to the quadrilateral $ABCD$ by drawing AF parallel to DB and joining DF . Since A is outside CDF , using the construction discussed in [3], we obtain the unique line ARQ through A to bisect triangle DFC and to cut CD in R and BC in Q . The triangle CRQ thus produced has area equal to half that of triangle DFC and hence to half that of the quadrilateral. The line ARQ is accordingly the required bisector of the quadrilateral and the investigation of case (B3c) is complete.

Our conclusion is that the construction described initially for the case of a convex quadrilateral produces in all cases except (B2c) and (B3c) a line AP which bisects the quadrilateral and we have provided alternative constructions for the required line in each of the two exceptional cases.

Finally, we remark that it is not difficult to find examples of quadrilaterals $ABCD$ for which we can draw more than one line through the vertex A to bisect area $ABCD$. There is therefore scope for investigating the number of bisectors through A and the circumstances in which given numbers of bisectors occur.

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* * * * *

In his book [*The Psychology of Mathematical Invention* (1945), Jacques] Hadamard [a famous French mathematician] tried to find out how famous mathematicians and scientists actually thought while doing their work. Of those he contacted in an informal survey, he wrote "Practically all of them . . . avoid not only the use of mental words, but also . . . the mental use of algebraic or precise signs . . . they use vague images." (p.84) and ". . .the mental pictures of the mathematicians whose answers I have received are most frequently visual, but they may also be of another kind - for example kinetic." (p.85)

Albert Einstein wrote to Hadamard that "the words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. . . . The physical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined. . . . The above mentioned elements are, in my case, of visual and some of muscular type.Conventional words or other signs have to be sought for laboriously only in a secondary stage . . . "(p.142) Several recent studies on the way in which nonmathematical adults perform simple arithmetic seem to suggest the same is true for non-mathematicians as well."

Philip J. Davis and Reuben Hersh : *The Mathematical Experience*, Penguin, 1983.