Arch. Math., Vol. 47, 309–311 (1986)

0003-889X/86/4704-0309 $ 2.10/0
© 1986 Birkhäuser Verlag, Basel

Two results on wreath products: corrigendum and addendum

By

L. G. Kovács

The original paper appeared in Arch. Math. **45**, 111–115 (1985). The author is indebted to Dr. E. A. Scott for discussions which identified the issues addressed here.

1. Corrigendum. The equation in (8) should have been

$$f(i) = f_i(0)\, a(x)\, f_x(0)^{-1} f_i((x\pi)\, 0)^{-1}.$$

In the next line, cf^{-1} should have been pf^{-1}. The (unnamed) statement displayed in the middle of p. 114 should have been established somewhat earlier, because it is needed in the claim (made three lines above it) that "(5) becomes (8)".

2. Addendum. The Centralizer Theorem has a somewhat stronger form and a more natural derivation along the following lines. The notation of the paper is maintained: $G \leqq W = A\, Wr\, (\mathrm{Sym}\, I)$; to each i in I, there is a g_i in G such that $(g_i \pi)\, 0 = i$; and $g_i = p_i f_i$ with $p_i \in \mathrm{Sym}\, I$, $f_i \in A^I$.

Form the (external) direct product $G \times A$ and denote by R the image of G_0 in $G \times A$ under the embedding $h \mapsto (h, h\pi_0)$. Note that $1 \times \mathbb{C}_A(G_0 \pi_0)$ avoids and normalizes R. For comparison only, note also that $(x, b) \in \mathbb{N}_{G \times A}(R)$ means precisely that $x \in C_0$ and (6) holds with $a(x) = b$. Define a map ψ: $\mathbb{N}_{G \times A}(R) \to W$, $(x, b) \mapsto fp^{-1}$ where p and f are given by

$$(7) \qquad pi = p_i(x\pi)\, 0,$$

$$(8') \qquad f(i) = f_i(0)\, b f_x(0)^{-1} f_i((x\pi)\, 0)^{-1}$$

for all i in I.

Centralizer Theorem. *The map ψ is independent of the choice of the g_i used in its definition. It is a homomorphism of $\mathbb{N}_{G \times A}(R)$ onto $\mathbb{C}_W(G)$, with kernel R. It maps $1 \times \mathbb{C}_A(G_0 \pi_0)$ isomorphically onto $A^I \cap \mathbb{C}_W(G)$, and the inverse isomorphism is $f \mapsto (1, f\pi_0)$.*

In preparation for the proof, we have to recall a number of facts on permutation groups.

The centralizer of a transitive permutation group is semi-regular (in the sense that nontrivial elements of it have no fixed points: because the fixed point set of such an element would have to be invariant under the transitive group). If two elements of a semiregular group agree on any one point, they must be equal.

Let $\varrho\colon Y \to \operatorname{Sym}\Omega$ be a transitive permutation representation, $\omega \in \Omega$, and Y_ω the stabilizer of ω in Y. For each element z in $\mathbf{N}_Y(Y_\omega)$, define a map $z\bar{\varrho}\colon \Omega \to \Omega$, $(y\varrho)\,\omega \mapsto ((yz^{-1})\,\varrho)\,\omega$. This definition is unambiguous because if $(y'\varrho)\,\omega = (y\varrho)\,\omega$ then $y^{-1}y' \in Y_\omega$ so $(yz^{-1})^{-1}(y'z^{-1}) \in Y_\omega$ and hence $((y'z^{-1})\,\varrho)\,\omega = ((yz^{-1})\,\varrho)\,\omega$. As map into the monoid of all self-maps of Ω, this $\bar{\varrho}$ is homomorphic: so it is really a group homomorphism into $\operatorname{Sym}\Omega$. Obviously, its image lies in the centralizer of $Y\varrho$. To see that the image is the whole centralizer, let β be any element of the latter. By the transitivity of ϱ, we have $\beta\omega = (z^{-1}\varrho)\,\omega$ for some z in Y, and then $(z\varrho)\,\omega = \beta^{-1}\omega$. Now $((zY_\omega z^{-1})\,\varrho)\,\omega = ((zY_\omega)\,\varrho)\,\beta\omega = \beta((zY_\omega)\,\varrho)\,\omega = \beta(z\varrho)\,\omega = \omega$ shows that $zY_\omega z^{-1} \leqq Y_\omega$. Similarly, $z^{-1}Y_\omega z \leqq Y_\omega$. Thus $z \in \mathbf{N}_Y(Y_\omega)$, and as β and $z\bar{\varrho}$ agree on ω they must be equal. It follows that the image cannot be a proper subgroup in the centralizer. The conclusion is that $\bar{\varrho}$ *is a homomorphism of* $\mathbf{N}_Y(Y_\omega)$ *onto* $\mathbf{C}_{\operatorname{Sym}\Omega}(Y\varrho)$, *and of course its kernel is just* Y_ω. We shall use this conclusion twice, in each case with $\Omega = A \times I$ and $\omega = (1, 0)$.

For the first application take $\varrho_1\colon W \to \operatorname{Sym}\Omega$ as the composite inclusion $W \to (\operatorname{Sym}A)\,Wr(\operatorname{Sym}I) \to \operatorname{Sym}(A \times I)$ obtained from the (left) regular representation of A. Explicitly,

$$(w\varrho_1)\,(a, i) = (f_w(i)\,a, (w\pi)\,i)$$

whenever $w \in W$ and $(a, i) \in A \times I$. It is easy to see that the stabilizer of $(1, 0)$ is complemented in its normalizer by the diagonal copy of A in A^I (that is, by the group of the constant functions). It follows that $\bar{\varrho}_1$ gives an isomorphism, $b \mapsto \bar{b}$ say, of A onto $\mathbf{C}_{\operatorname{Sym}\Omega}(W\varrho_1)$, such that

$$\bar{b}(a, i) = (ab^{-1}, i) \quad \text{for all } (a, i) \text{ in } \Omega.$$

Accordingly, write $\mathbf{C}_{\operatorname{Sym}\Omega}(W\varrho_1)$ simply as $\bar{A}$.

There is a converse to this: namely, $\mathbf{C}_{\operatorname{Sym}\Omega}(\bar{A}) = W\varrho_1$. Of course only the inclusion $\mathbf{C}(\bar{A}) \leqq W\varrho_1$ needs proof. Towards a sketch of that, recall first that the centralizer of any permutation group permutes the set of the orbits of that group. If $q \in \mathbf{C}(\bar{A})$ then there is an r in $\operatorname{Sym}I$ such that $r\varrho_1$ permutes these orbits exactly as q does. The quotient $q(r\varrho_1)^{-1}$ leaves each orbit invariant and still lies in $\mathbf{C}(\bar{A})$. On each orbit, $(A^I)\,\varrho_1$ and $\bar{A}$ act via the left and right regular representations of A, respectively: that is, as the centralizers of one another. It follows that $q(r\varrho_1)^{-1} \in (A^I)\,\varrho_1$, so $q \in W\varrho_1$.

For the second application of the preliminary step, take $Y = G \times A$, $\varrho_2\colon (x, b) \mapsto (x\varrho_1)\,\bar{b}$. This is a homomorphism, as $G\varrho_1$ and $\bar{A}$ commute elementwise. Its image is transitive because, given any (a, i), for instance $(g_i, a^{-1}f_i(0))\,\varrho_2$ will map $(1, 0)$ to (a, i). The stabilizer of $(1, 0)$ is now R, since $((x, b)\,\varrho_2)\,(1, 0) = (f_x(0)\,b^{-1}, (x\pi)\,0)$ so (x, b) lies in the stabilizer if and only if $x \in G_0$ and $b = x\pi_0$. As ϱ_1 is an embedding and $\mathbf{C}(\bar{A}) = W\varrho_1$, we have that $\mathbf{C}((G \times A)\,\varrho_2) = \mathbf{C}(G\varrho_1\bar{A}) = \mathbf{C}(G\varrho_1) \cap \mathbf{C}(\bar{A}) = \mathbf{C}(G\varrho_1) \cap W\varrho_1 = \mathbf{C}_W(G)\,\varrho_1$. The conclusion is that $\bar{\varrho}_2$ maps $\mathbf{N}_{G \times A}(R)$ homomorphically onto $\mathbf{C}_W(G)\,\varrho_1$, with kernel R. Of course neither ϱ_1 nor $\bar{\varrho}_2$ depend on the choice of the g_i.

For proving the first two sentences of (the present form of) the Centralizer Theorem, it now suffices to show that $(x^{-1}, b^{-1})\,\bar{\varrho}_2 = (pf^{-1})\,\varrho_1$ whenever $(x, b) \in \mathbf{N}_{G \times A}(R)$ and p, f are defined by (7), (8'). By the definition of $\bar{\varrho}_2$,

$$\begin{aligned}((x^{-1}, b^{-1})\,\bar{\varrho}_2)\,(a, i) &= (x^{-1}, b^{-1})\,\varrho_2[g_i, a^{-1} f_i(0))\,\varrho_2]\,(1, 0)\\ &= [(g_i x, a^{-1} f_i(0)\,b)\,\varrho_2]\,(1, 0)\\ &= (f_i^{x\pi}(0)\,f_x(0)\,b^{-1} f_i(0)^{-1}\,a, p_i(x\pi)\,0).\end{aligned}$$

Substituting (7) and (8') in the right hand side of

$$((pf^{-1})\,\varrho_1)\,(a, i) = (f(i)^{-1}\,a, p\,i)$$

directly yields the same result.

To prove the last sentence of the Centralizer Theorem, note first that (7) gives $p = 1$ if and only if $x \in G_0$, that is, when $(x, b) \in R(1 \times \mathbf{C}_A(G_0 \pi_0))$. Finally, if $(x, b) \in 1 \times \mathbf{C}_A(G_0 \pi_0)$ then b centralizes $f_0(0)$ and so (8') yields $f\pi_0 = f(0) = b$. This completes the proof.

Eingegangen am 28. 12. 1985

Anschrift des Autors:

L. G. Kovács
Department of Mathematics
Australian National University
GPO Box 4
Canberra, A.C.T. 2601
Australia