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On the First Cohomology of a Group with Coefficients in a Simple Module

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The aim of this note is to establish the following.

THEOREM. *Let $\mathbb{F}$ be any field and N a subnormal subgroup in a (not necessarily finite) group G; let V be a finite dimensional simple $\mathbb{F}G$-module on which N acts nontrivially, and choose a nontrivial simple $\mathbb{F}N$-submodule U in V. Then* $\dim H^1(G, V) \leqslant \dim H^1(N, U)$.

Note that by repeated use of Clifford's theorem V is semi-simple as $\mathbb{F}N$-module, so if N acts nontrivially on V then such U always exist. Induction on the subnormal defect of N shows that it suffices to prove the theorem for normal N. For finite G with soluble G/N, this was done in 4.4 of Aschbacher and Scott [1]. The proof we give below is in a sense dual to theirs. This eliminates the solubility hypothesis from their 4.4, and so removes one of the two points where [1] depended on the Schreier conjecture and so indirectly on the classification of finite simple groups.

We mentioned above that we only have to prove the theorem for normal N. The fixed point space V^N of N in V is then an $\mathbb{F}G$-submodule, so since N acts nontrivially and V is simple as $\mathbb{F}G$-module, $V^N = 0$. The inflation-restriction exact sequence (Proposition 4 in Chap. VII of Serre [2]) now shows that restriction provides an embedding of $H^1(G, V)$ into $H^1(N, V)$. It is easy to see that restriction is a G-homomorphism and (cf. Proposition 3, *loc. cit.*) that G acts trivially on $H^1(G, V)$ so the embedding is into the fixed point space $H^1(N, V)^G$. Therefore our claim will follow once we show that $\dim H^1(N, V)^G \leqslant \dim H^1(N, U)$.

We take $H^1(N, V)$ as the quotient of the $\mathbb{F}$-space $\mathrm{Der}(N, V)$ of all derivations (1-cocycles) modulo the space of inner derivations (coboun-

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daries). If $V = \bigoplus V_i$ is a direct decomposition of V as $\mathbb{F}N$-module, there is a corresponding decomposition $H^1(N, V) = \bigoplus H^1(N, V_i)$ obtained via the obvious embeddings of the $\mathrm{Der}(N, V_i)$ into $\mathrm{Der}(N, V)$. For each g in G, also $V = \bigoplus V_i g$ and $H^1(N, V) = \bigoplus H^1(N, V_i g)$, and it is readily seen that the implied identifications are consistent with the action of G in the sense that $H^1(N, V_i g) = H^1(N, V_i)^g$.

If U is a simple $\mathbb{F}N$-submodule of V, there is an $\mathbb{F}N$-submodule W which complements it: $V = U \oplus W$. Since $\bigcap_{g \in G} Wg$ is an $\mathbb{F}G$-submodule in the simple module V, this intersection must be 0. As V is finite dimensional, there are finite subsets R in G such that $\bigcap_{r \in R} Wr = 0$; chose R minimal with respect to this propoerty. For each s in R, set $W_s = \bigcap_{r \neq s} Wr$; then $V = \bigoplus W_s$ and $Wr = \bigoplus_{s \neq r} W_s$ follow by elementary arguments familiar in the context of semi-simple modules. The comments made in the previous paragraph may then be extended to yield that $H^1(N, V) = \bigoplus H^1(N, W_s)$ and $\bigoplus_{s \neq r} H^1(N, W_s) = H^1(N, Wr) = H^1(N, W)^r$. Therefore $\bigcap H^1(N, W)^r = 0$, whence we conclude that $H^1(N, W)$ cannot contain any nonzero $\mathbb{F}G$-submodule of $H^1(N, V)$. In particular, $H^1(N, W) \cap H^1(N, V)^G = 0$, so $\dim H^1(N, V)^G \leqslant \operatorname{codim} H^1(N, W)$. On the other hand we also have $H^1(N, V) = H^1(N, U) \oplus H^1(N, W)$, so $\operatorname{codim} H^1(N, W) = \dim H^1(N, U)$. These results on dimensions combine to yield our claim.

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