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A NON-EXTENDIBLE ABSTRACT KERNEL

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The motivation for the theory of obstructions to group extensions is the observation in Reinhold Baer's classic Zeitschrift paper of 1934, that not all homomorphisms $C \to \mathrm{Out}\, A$ into outer automorphism class groups arise from conjugation action in group extensions $1 \to A \to B \to C \to 1$. However, the one explicit example of this phenomenon he wrote down, and to which current texts still refer, is incorrect.

What Baer wrote was based on the following criterion of his. A finitely generated abelian group D of exponent m is the central factor group of some other group if and only if D contains a direct product of two cyclic groups of order m. In effect, he asked: if A is a finite nilpotent group of class 2 and D an abelian subgroup of $\mathrm{Aut}\, A$ including all inner automorphisms and acting trivially on the centre Z of A, why should D have to pass this test? Once A and D are chosen so as to fail it, the inclusion of $C = D/\mathrm{Inn}\, A$ in $\mathrm{Out}\, A$ cannot be realized by any extension B: for, the centre of such a B would have to be Z, and one would have to have $B/Z \cong D$. For instance, take A to be the group defined on the generators a, b, c by the relations

$$a^{n^2} = b^{n^2} = c^{n^2} = 1 , \quad ba = ab , \quad ca = ac , \quad cb = a^n bc ,$$

and let D be generated by Inn A and the automorphism α which maps a to a, b to ab, and c to c. Then D is abelian, of order n^3 and exponent n^2, so we have our example.

The error lies in the claim that this D acts trivially on Z : clearly, α moves b^n. Indeed, the inclusion of C in Out A is realized in the extension B obtained by adjoining to A an element d such that $d^n = c$, $da = ad$, and $ba = abd$. Moreover, an easy answer to the general rhetorical question above is: because the exponent of such a D must be the same as the exponent of Inn A. [If δ $(\in D)$ maps x $(\in A)$ to xz_x (with $z_x \in Z$), then $x^n \in Z$ implies $z_x^n = 1$ and hence $\delta^n = 1$.]

To correct the example, change the action of α on c : let it map c to c^{n+1}. The required extension B would still have to be generated by A and an element d which conjugates A according to α, with d^n an element of A fixed by α and inducing the inner automorphism α^n. The elements inducing α^n are precisely the elements of the coset of c modulo Z, but now none of these is fixed by α : so no such B can exist.

This seems to be "the" simplest example. As it is also very close, in construction if not in justification, to what Baer wrote down, he presumably had had it in mind at some stage prior to actually writing the paper.

For the reader who does not wish to delve into obstruction theory, it may be worth emphasizing that the phenomenon does occur even with C cyclic (though of course not with C infinite cyclic). Another simple example, proved like the one above, shows that one can even keep both C and Z to order 2 : take A as the group defined on a, b, c by $a^{16} = b^2 = c^2 = 1$, $ba = a^{-1}b$, $ca = a^9c$, $cb = bc$, and let α map a to a^3 , b to b , and c to a^8c .

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