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On Normal Subgroups Which Are Direct Products

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1. INTRODUCTION

Now that the classification of finite simple groups is complete, it is logical to look at the extension problem. An important special case to consider is when M is a minimal normal subgroup of G and both G/M and M are known groups. If M is abelian, various techniques have been used to derive information about G. Indeed, almost the entire theory of finite solvable groups can be said to rest upon these techniques.

The motivation behind the present paper was to develop techniques for dealing with the situation when M is not abelian. Specifically, we consider the following problems: (1) Determine the structure of G from the structure of G/M and some subgroup or subgroups of G. (2) Find subgroups H in G such that $G = HM$, and, in particular, determine whether M has a complement in G. (3) Determine when two subgroups H_1 and H_2 found in (2) are conjugate in G.

If M is a non-abelian minimal normal subgroup of a finite group G, then

$$M = S_1 \times S_2 \times \cdots \times S_n$$

where $\{S_i \mid 1 \leqslant i \leqslant n\}$ is a conjugacy class of subgroups of G. It turns out that G is completely determined by the groups G/M and $N_G(S_1)/(S_2 \times \cdots \times S_n)$. Now $N_G(S_1) = N_G(S_2 \times \cdots \times S_n)$ and it is more convenient to state our results in terms of the subgroups $K_i = \prod_{j \neq i} S_j$ rather

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than in terms of $\{S_i \mid 1 \leqslant i \leqslant n\}$. Then no finiteness assumptions are necessary and our main results are concerned with the following hypothesis:

M is a normal subgroup of the group G. For each $i \in I$, K_i is a normal subgroup of M and $\{K_i \mid i \in I\}$ is a conjugacy class of subgroups in G. Further, the natural homomorphism of M into the unrestricted direct product $\prod_{i \in I} (M/K_i)$ is an isomorphism onto.

(Note that we are no longer requiring G or I to be finite. Nor is M assumed to be a minimal normal subgroup of G. Finally, it does not matter whether or nor M is abelian.)

Our most fundamental result is that knowledge of the groups G/M and $N_G(K)/K$ (where K is K_i for some fixed i) together with a certain obvious homomorphism of $N_G(K)/K$ into G/M is sufficient to construct G. This construction is called the induced extension and is described in Section 3. With regard to the existence of complements, we show that G splits over M if, and only if, $N_G(K)/K$ splits over M/K. Further, there is a one-to-one correspondence between the conjugacy classes of complements of M in G and of M/K in $N_G(K)/K$. With regard to supplements, we show that if L/K is a subgroup of $N_G(K)/K$ such that $N_G(K)/K = (L/K)(M/K)$, then there is a subgroup H in G such that $G = HM$, $L = (H \cap N_G(K))\,K$, and $H \cap M$ is the direct product of the groups $\{(H \cap M)/(H \cap K_i) \mid i \in I\}$. Further, H is unique up to conjugacy by some element of K.

After proving the above results about complements and supplements in Section 4, we go on in Section 5 to extend these results to the situation where $\{K_i \mid i \in I\}$ is a union (not necessarily finite) of conjugacy classes in G. Finally, in Section 6, we construct some examples to illustrate why some of our theorems are the way they are.

2. Notation and Preliminary Results

We write $H \leqslant G$ to indicate that H is a subgroup of the group G while $H \trianglelefteq G$ means that H is a normal subgroup. $N_G(H)$ and $C_G(H)$ denote the normalizer and centralizer, respectively, of H in G. If α is a homomorphism defined on G and $H \leqslant G$, then α_H is the restriction of α to H. If I is any set, then $|I|$ denotes its cardinality. If P is a permutation group acting on I and $i \in I$, then P_i is the stabilizer of i in P.

If $K_i \trianglelefteq G$ for each $i \in I$, then we will write

$$G = \prod_{i \in I} (G/K_i)$$

if the natural homomorphism of G into the direct product $\prod_{i \in I} (G/K_i)$ is an isomorphism onto. An equivalent definition, which we will use throughout, is the following:

$G = \prod_{i \in I} (G/K_i)$ if whenever $\{x_i \mid i \in I\}$ is a subset of G, then $\bigcap_{i \in I} K_i x_i$ consists of a single element of G.

In particular, note that if $G = \prod_{i \in I} (G/K_i)$, then $\bigcap_{i \in I} K_i = 1$. All direct products in this paper are unrestricted direct products.

If H is a subgroup of G, then a right transversal T of H in G is a set consisting of exactly one element chosen from each right coset of H in G. We adopt the convention that each transversal contains the identity, i.e., the element chosen from the coset H is always 1. Of course, $|T|$ is the index $|G:H|$ of H in G. For the right transversal T, the function u_T is defined on G by the rule that $u_T(x)$ is that element of I belonging to Hx. Note that

$$u_T(x^{-1})\, x \in H$$

for all $x \in G$ and

$$u_T(h^{-1})\, h = h$$

for all $h \in H$. If M is a normal subgroup of G contained in H and if x and y are elements of G such that $x \equiv y (\mathrm{mod}\, M)$, then $u_T(x) = u_T(y)$.

If A is a group and I is a non-empty set, then A^I is the group of all functions defined on I with values in A and multiplication defined point-wise. For $i \in I$,

$$A[i] = \{a \in A^I \mid a(i) = 1\}.$$

Then $A[i] \trianglelefteq A^I$, $A^I/A[i]$ is isomorphic to A, and $A^I = \prod_{i \in I} A/A[i]$.

With A^I as above, assume that P is a permutation group acting on I. We have P operate on A^I as follows:

$$a^p(i) = a(ip^{-1})$$

where $a \in A^I$, $p \in P$, and $i \in I$. The semi-direct product PA^I is denoted by $A\,\mathrm{Wr}(P, I)$. (We use this notation to distinguish it from the so-called standard wreath product. In the standard wreath product, $I = P$ and P acts regularly on itself.) Every element of $W = A\,\mathrm{Wr}(P, I)$ has the form pa with $p \in P$ and $a \in A^I$. For all $i \in I$, $p \in P$, and $a \in A^I$, we find that

$$(pa)^{-1} A[i](pa) = A[ip].$$

Then $N_W(A[i]) = P_i A^I$. We will use e_i to denote the function defined on $P_i A^I$ by

$$e_i(pa) = a(i).$$

It is easily verified that e_i is a homomorphism of P_iA^I onto A with kernel $P_iA[i]$. Suppose now that α is a homomorphism of A into a group B. Then α induces a homomorphism $\bar{\alpha}$ of $A \operatorname{Wr}(P, I)$ into $B \operatorname{Wr}(P, I)$ where

$$\bar{\alpha}(pa) = p(\alpha a)$$

and αa is that element of B^I defined by

$$\alpha a(i) = \alpha(a(i)).$$

We will also use e_i to denote the homomorphism of P_iB^I onto B. Then $\bar{\alpha}$ maps P_iA^I into P_iB^I and

$$e_i\bar{\alpha}(x) = \alpha e_i(x)$$

for all $x \in P_iA^I$. Note that if α maps A onto B, then $\bar{\alpha}$ maps $A \operatorname{Wr}(P, I)$ onto $B \operatorname{Wr}(P, I)$. In any event, the kernel of $\bar{\alpha}$ is (kernel $(\alpha))^I$. (Here, (kernel $(\alpha))^I$ is to be regarded as a subgroup of A^I.)

Suppose now that G is a group, H is a subgroup of G, and T is a right transversal of H in G. Let I be the set of all right cosets of H in G and let ρ be the permutation representation of G on I. For $x \in G$, let x_T be that element of H^I defined by

$$x_T(Ht) = u_T(tx^{-1})\, xt^{-1}$$

for all $t \in T$. Let λ_T be the mapping from G into $H \operatorname{Wr}(\rho(G), I)$ defined by

$$\lambda_T(x) = \rho(x)\, x_T.$$

Then, as is well known (see [5, p. 413], for example), λ_T is a monomorphism. If T' is some other right transversal of H in G, then there is an $m \in H^I$ such that

$$\lambda_{T'}(x) = m^{-1}\lambda_T(x)\, m$$

for all $x \in G$. If T is understood, we write simply λ. With $W = H \operatorname{Wr}(\rho(G), I)$, note that $W = \lambda(G)\, H^I$. Since $\rho(G)$ acts transitively on I, $\{H[i] \mid i \in I\}$ is a conjugacy class of subgroups in W and $\{H[i] \mid i \in I\}$ are conjugate under $\lambda(G)$. If j denotes the coset H, then $\rho(H) = (\rho(G))_j$ and so

$$N_{\lambda(G)}(H[j]) = \lambda(H).$$

If $h \in H$, then

$$e_j\lambda(h) = h_T(j) = h_T(H) = u_T(h^{-1})\, h = h.$$

Thus $e_j\lambda_H = 1_H$, the identity mapping of H.

We now list some properties of groups with a normal subgroup which is a direct product.

2.1. LEMMA. *Assume* $M \trianglelefteq G$, $K_i \trianglelefteq M$ *for* $i \in I$, *and* $M = \prod_{i \in I} M/K_i$. *Then the following are true:*

(1) *Suppose* $x_i \in G$ *for* $i \in I$. *Then* $|\bigcap_{i \in I} K_i x_i| = 0$ *or* 1. *Further,* $|\bigcap_{i \in I} K_i x_i| = 1$ *if, and only if,* $x_i x_j^{-1} \in M$ *for all* $i, j \in I$.

(2) *Assume that* $H \leqslant G$. *Then the following are equivalent:*

(a) $H = \bigcap_{i \in I} K_i H$.

(b) $H \cap M = \bigcap_{i \in I} K_i (H \cap M)$.

(c) $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$.

Proof. (1) If $x, y \in \bigcap_{i \in I} K_i x_i$, then $xy^{-1} \in \bigcap_{i \in I} K_i = 1$ and so $x = y$. Hence,

$$\left| \bigcap K_i x_i \right| \leqslant 1.$$

If $x \in \bigcap_{i \in I} K_i x_i$, then

$$x_i x_j^{-1} = (x_i x^{-1})(x_j x^{-1})^{-1} \in K_i K_j \leqslant M.$$

Assume now that $x_i x_j^{-1} \in M$ for all $i, j \in I$. Fixing j, $\bigcap_{i \in I} K_i x_i x_j^{-1}$ must consist of a single element m. But then $m x_j \in \bigcap_{i \in I} K_i x_i$ and so $|\bigcap_{i \in I} K_i x_i| = 1$.

(2) Assume $H \leqslant G$. Now $K_i H$ need not be a subgroup of G but, as is easily checked,

$$K_i H \cap M = K_i (H \cap M).$$

Thus

$$\left(\bigcap_{i \in I} K_i H \right) \cap M = \bigcap_{i \in I} K_i (H \cap M).$$

It follows from this that (a) implies (b).

Assume now that (b) holds and that $x_i \in H \cap M$ for each $i \in I$. Now, since $x_i \in M$,

$$\bigcap_{i \in I} K_i x_i = \{m\}$$

for some $m \in M$. But then, using (b),

$$m \in \bigcap_{i \in I} K_i (H \cap M) = H \cap M.$$

It now follows that

$$\bigcap_{i\in I} (H\cap K_i)\,x_i = \{m\}$$

and hence

$$H\cap M = \prod_{i\in I} (H\cap M)/(H\cap K_i).$$

Finally, suppose (c) holds. Clearly

$$\bigcap_{i\in I} K_iH \supseteq H.$$

We need to prove the reverse inclusion. Let x be an arbitrary element of $\bigcap_{i\in I} K_iH$. Then $x = k_ih_i$ for some $k_i \in K_i$ and $h_i \in H$. Let j be some fixed element of I. Then $k_ih_i = k_jh_j$ and so

$$h_ih_j^{-1} = k_i^{-1}k_j \in H\cap M$$

for all $i\in I$. Then, by (c),

$$\bigcap_{i\in I} (H\cap K_i)\,h_ih_j^{-1} = \{m\}$$

for some $m\in H\cap M$. Then $m\in K_ih_ih_j^{-1}$ for all $i\in I$. Also $h_ih_j^{-1}k_j^{-1} = k_i^{-1}\in K_i$ for all $i\in I$. Hence m and k_j both belong to

$$\bigcap_{i\in I} K_ih_ih_j^{-1}.$$

It follows from part (1) that $k_j = m\in H\cap M$. But then

$$x = k_jh_j \in (H\cap M)\,H = H.$$

It follows that

$$\bigcap_{i\in I} K_iH = H.$$

Remark. Suppose H is a subgroup satisfying the conditions in part (2) of the above lemma. If we identify M with $\prod_{i\in I}(M/K_i)$, then it follows from (2)(b) that

$$H\cap M = \prod_{i\in I} ((H\cap M)\,K_i/K_i).$$

This says not only that $H\cap M$ is a direct product but that the direct decomposition of $H\cap M$ is compatible with the direct decomposition of M.

In the sequel, subgroups which satisfy the conditions in part (2) of the lemma play a special role. The effect of the next two results is that if $\{K_i \mid i \in I\}$ is a union of conjugacy classes in G and if A is any subgroup of G, then there is a unique subgroup B of the desired type containing A but such that both AM and $N_A(K_i)\,K_i$ stay the same when A is replaced by B.

2.2. LEMMA. *Assume $M \trianglelefteq G$, $K_i \trianglelefteq M$ for $i \in I$, and $M = \prod_{i \in I} M/K_i$. Assume that $\{K_i \mid i \in I\}$ is a union of conjugacy classes in G. Let A be a subgroup of G and let $B = \bigcap_{i \in I} K_iA$. Then B is a subgroup of G, $B \cap M = \bigcap_{i \in I} K_i(A \cap M)$, $B = (B \cap M)A = \bigcap_{i \in I} AK_i = \bigcap_{i \in I} K_iB$, and $B \cap M = \prod_{i \in I} (B \cap M)/(B \cap K_i)$.*

Proof. As in the previous lemma, K_iA need not be a subgroup of G. Since, however, $K_i \trianglelefteq M$, $K_i(A \cap M)$ is a subgroup. Then

$$B \cap M = \bigcap_{i \in I} (K_iA \cap M) = \bigcap_{i \in I} K_i(A \cap M)$$

is a subgroup. Conjugation by elements of A permute the members of $\{K_iA \mid i \in I\}$ among themselves. Therefore A must normalize both B and $B \cap M$. It follows that $(B \cap M)A$ is a subgroup of G. Clearly

$$(B \cap M)A \subseteq \bigcap_{i \in I} K_i(A \cap M)A = \bigcap_{i \in I} K_iA = B.$$

Suppose $x \in B$. Then $x \in K_ia_i$ for some $a_i \in A$. Then $|\bigcap_{i \in I} K_ia_i| > 0$ and so part (1) of the previous lemma yields

$$a_ia_j^{-1} \in M$$

for all $i, j \in I$. Then, since a_i and a_j belong to A,

$$xa_j^{-1} = (xa_i^{-1})(a_ia_j^{-1}) \in K_i(A \cap M).$$

Therefore,

$$xa_j^{-1} \in \bigcap_{i \in I} K_i(A \cap M) = B \cap M.$$

It follows from this that $x \in (B \cap M)A$. Thus

$$B = (B \cap M)A.$$

An argument similar to the above shows that

$$\bigcap_{i \in I} AK_i = A\left(\bigcap_{i \in I} (A \cap M)K_i\right).$$

But $A \cap M$ normalizes K_i, and $(B \cap M)$ and $(B \cap M)A$ are both subgroups of G. It now follows that

$$\bigcap_{i \in I} AK_i = A\left(\bigcap_{i \in I} (A \cap M)K_i\right) = A\left(\bigcap_{i \in I} K_i(A \cap M)\right) = A(B \cap M) = B.$$

Finally,

$$B \subseteq \bigcap_{i \in I} K_i B \subseteq \bigcap_{i \in I} K_i K_i A = \bigcap_{i \in I} K_i A = B.$$

This implies that $B = \bigcap_{i \in I} K_i B$. It follows from the previous lemma that

$$B \cap M = \prod_{i \in I} (B \cap M)/(B \cap K_i).$$

2.3. THEOREM. *Assume $M \trianglelefteq G$, $K_i \trianglelefteq M$ for $i \in I$, $M = \prod_{i \in I} (M/K_i)$, and $\{K_i \mid i \in I\}$ is a union of conjugacy classes in G. Let A be any subgroup of G. Then the following are true.*

(1) *G contains one and only one subgroup B such that $A \leqslant B$, $B \cap M = \prod_{i \in I} (B \cap M)/(B \cap K_i)$, $AM = BM$, and $N_A(K_i) K_i = N_B(K_i) K_i$ for all $i \in I$.*

(2) *$B = \bigcap_{i \in I} K_i A = A(B \cap K_i)$ for all $i \in I$.*

(3) *If $A \leqslant C \leqslant G$ and $C \cap M = \prod_{i \in I} (C \cap M)/(C \cap K_i)$, then $C \geqslant B$.*

(4) *If $A \leqslant C \leqslant AM$ and $N_A(K_i) K_i = N_C(K_i) K_i$ for all $i \in I$, then $C \leqslant B$.*

Proof. Let $B = \bigcap_{i \in I} K_i A$. By Lemma 2.2, B is a subgroup of G and $B \cap M = \prod_{i \in I} (B \cap M)/(B \cap K_i)$. Clearly B contains A, and since $K_i \leqslant M$, B is contained in MA. Certainly, then $AM = BM$. Now let $i \in I$. Then $B \subseteq K_i A$ and so, since $B \geqslant A$, $B = (B \cap K_i)A$. Setting $N_i = N_G(K_i)$, we have

$$N_B(K_i) = B \cap N_i \subseteq K_i A \cap N_i = K_i(A \cap N_i) = K_i N_A(K_i).$$

This implies that $N_B(K_i) K_i = N_A(K_i) K_i$.

We now have verified (2) and all but the uniqueness in (1). The uniqueness is an immediate consequence of (3) and (4). Thus we will be finished once we prove (3) and (4).

Suppose then that $A \leqslant C \leqslant G$ and $C \cap M = \prod_{i \in I} (C \cap M)/(C \cap K_i)$. From Lemma 2.1(2), we obtain

$$C = \bigcap_{i \in I} K_i C \supseteq \bigcap_{i \in I} K_i A = B.$$

Thus (3) is proved.

Finally, suppose $A \leqslant C \leqslant AM$ and $N_C(K_i)\,K_i = N_A(K_i)\,K_i$ for all $i \in I$. By taking intersections with M and using $K_i \leqslant M \leqslant N_G(K_i)$, we obtain

$$K_i(A \cap M) = K_i(C \cap M).$$

From $A \leqslant C \leqslant AM$, we must have $C = (C \cap M)\,A$ and so

$$K_i C = K_i(C \cap M)\,A = K_i(A \cap M)\,A = K_i A.$$

But then

$$C \subseteq \bigcap_{i \in I} K_i C = \bigcap_{i \in I} K_i A = B.$$

With this, the proof of the theorem is complete.

Remark. Assuming the same hypothesis as in the theorem, suppose $\mathscr{S}$ is the set of subgroups H in G which satisfy

$$H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i).$$

Then it follows from part (3) of the theorem that $\mathscr{S}$ is closed under arbitrary intersections. For suppose $H_j \in \mathscr{S}$ for all $j \in J$. Applying the theorem with $A = \bigcap_{i \in I} H_j$, we find that $H_j \geqslant B$ for all $j \in J$. But then this implies that $A = B \in \mathscr{S}$.

3. The Induced Extension

Assume that α is a homomorphism of a group A into a group B. We proceed to describe how to construct a group G which we will call the induced extension. This may be compared with the semi-direct product construction in which we start with a homomorphism of one group into the automorphism group of another group.

Let $C = \alpha(A)$, let I be the set of right cosets of C in B, let ρ be the permutation representation of B on I, let $P = \rho(B)$, and let T be a fixed right transversal of C in B. Then, as was shown earlier, there is a monomorphism $\lambda = \lambda_T$ of B into $C\,\mathrm{Wr}(P, I)$. Let $\bar{\alpha}$ be the epimorphism of $A\,\mathrm{Wr}(P, I)$ onto $C\,\mathrm{Wr}(P, I)$ induced by α. If M is the kernel of $\bar{\alpha}$, then $M = (\mathrm{kernel}(\alpha))^I$ and we have an exact sequence

$$E : 1 \to M \to A\,\mathrm{Wr}(P, I) \xrightarrow{\bar{\alpha}} C\,\mathrm{Wr}(P, I) \to 1.$$

Let G be the middle term in the pull-back exact sequence $E\lambda$. Then we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M & \longrightarrow & G & \xrightarrow{\sigma} & B & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\lambda} & & \\ 1 & \longrightarrow & M & \longrightarrow & A\ \mathrm{Wr}(P,I) & \xrightarrow{\bar{\alpha}} & C\ \mathrm{Wr}(P,I) & \longrightarrow & 1 \end{array}$$

(This diagram serves to define σ.) Since λ is one-to-one, the 5 lemma implies that the middle vertical arrow is a monomorphism. Thus we may consider G as a subgroup of $W = A\ \mathrm{Wr}(P, I)$. Then, as is easily verified,

$$G = \{x \in W \mid \bar{\alpha}(x) = \lambda(b) \text{ for some } b \in B\},$$

and $\sigma = \lambda^{-1}(\bar{\alpha})_G$. If T' is another right transversal of C in B, then $\lambda_{T'}(B) = m^{-1}\lambda(B)\, m$ for some $m \in C^I$. Then $m = \bar{\alpha}(n)$ for some $n \in A^I$. Then

$$\{x \in W \mid \bar{\alpha}(x) = \lambda_{T'}(b) \text{ for some } b \in B\} = n^{-1}Gn.$$

Thus, up to conjugacy in W and certainly up to isomorphism, G is independent of the choice of T.

We will say that G is the induced extension defined by $\alpha : A \to B$. Strictly speaking, we should say that

$$1 \to M \to G \xrightarrow{\sigma} B \to 1 \tag{G}$$

is the extension-by-B induced from the extension-by-C

$$1 \to \mathrm{kernel}(\alpha) \to A \xrightarrow{\alpha} C \to 1. \tag{A}$$

The origin of the terminology lies in a special case. If kernel(α) is abelian, one may regard it as a C-module and (A) as an extension of this module by C. Then M is the induced B-module and (G) is the extension of M by B corresponding to (A) in the natural isomorphism of the cohomology groups $H^2(C, \mathrm{kernel}(\alpha))$ and $H^2(B, M)$ given by Shapiro's Lemma [4, p. 29]. Indeed, much of this paper may be thought of as a non-abelian generalization of Shapiro's Lemma for the first and second cohomology groups, but space does not allow us to pursue that view here.

We now derive some properties of G. Clearly $M \trianglelefteq G$ and G/M is isomorphic to $\sigma(G) = B$. Also $M = (\mathrm{kernel}(\alpha))^I \leqslant A^I \trianglelefteq W$. For each $i \in I$, let $K_i = M \cap A[i]$. Then $K_i \trianglelefteq M$, M/K_i is isomorphic to kernel(α), and

$M = \prod_{i \in I} (M/K_i)$. If $p \in P$, then $p = \rho(b)$ and $\lambda(b) = \bar{\alpha}(x)$ for some $b \in B$ and $x \in G$. Then

$$x^{-1} K_i x = K_{ip}.$$

It now follows that $\{K_i \mid i \in I\}$ is a class of conjugate subgroups of G.

Let j be that element of I corresponding to the coset C, let $K = K_j$, and $N = N_G(K)$. Since $P_j = \rho(C)$, we find that

$$N = \{x \in W \mid \bar{\alpha}(x) = \lambda(c) \text{ for some } c \in C\}.$$

Then $\sigma(N) = C$. Let e be the restriction of e_j (e_j is defined in Section 2) to N. If $x \in N$ and $\bar{\alpha}(x) = \lambda(c)$ with $c \in C$, then

$$\alpha e(x) = e_j \bar{\alpha}(x) = e_j \lambda(c) = c = \sigma(x).$$

Thus $\alpha e(N) = C = \alpha(A)$ and so $e(N)(\text{kernel}(\alpha)) = A$. But $N \geqslant M$ and $e_j(M) = (\text{kernel}(\alpha))$ (this follows since $M = (\text{kernel}(\alpha))^I$). We now see that $e(N) = A$. Suppose $x \in \text{kernel}(e)$. Then $\bar{\alpha}(x) = \lambda(c)$ for some $c \in C$ and $c = \alpha e(x) = \alpha(1) = 1$. Then $\bar{\alpha}(x) = 1$ and so $x \in N \cap \text{kernel}(\bar{\alpha}) = N \cap M = M$. But

$$M \cap \text{kernel}(e_j) = M \cap A[j] = K.$$

It now follows that the kernel of e is K.

We now have shown the following:

(1) $M \trianglelefteq G$, $K \trianglelefteq M$, $M = \prod_{i \in I} (M/K_i)$ where $\{K_i \mid i \in I\}$ are all the conjugates of K in G, and $N = N_G(K)$.

(2) σ is a homomorphism of G onto B, the kernel of σ is M, and $\sigma(N) = \alpha(A)$.

(3) e is a homomorphism of N onto A, the kernel of e is K, $e(M) = \text{kernel}(\alpha)$, and $\alpha e = \sigma_N$.

Since G is defined by a pull-back diagram, it is natural to expect G to satisfy some type of universal mapping property. Thus we obtain the following result.

3.1. THEOREM. *Let the notation be as above. Suppose τ is a homomorphism of a group G^* onto B and let $N^* = \{x \in G^* \mid \tau(x) \in \alpha(A)\}$. Assume that β is a homomorphism of N^* into A such that $\tau_{N^*} = \alpha\beta$. Then there is a homomorphism γ of G^* into G such that $\tau = \sigma\gamma$ and $\beta = e\gamma_{N^*}$. Suppose further that γ' is also a homomorphism of G^* into G such that $\tau = \sigma\gamma'$ and $\beta = e(\gamma')_{N^*}$. Then there exists $k \in K$ such that $\gamma'(x) = k^{-1}\gamma(x)\, k$ for all $x \in G^*$.*

Proof. For each $t \in T$, there is a $t^* \in G^*$ such that $\tau(t^*) = t$. Then if $T^* = \{t^* \mid t \in T\}$, T^* is a right transversal of N^* in G^*. (Of course, we choose $1^* = 1$.) We may identify I with the set of right cosets of N^* in G^* so that $\rho^* = \rho\tau$ where ρ^* is the permutation representation of G^* on the cosets of N^*. Let λ^* be the monomorphism λ_{T^*} of G^* into $N^* \operatorname{Wr}(P, I)$. (Obviously, $\rho^*(G^*) = \rho\tau(G^*) = \rho(B) = P$.) Let $\bar{\beta}$ be the homomorphism of $N^* \operatorname{Wr}(P, I)$ into $A \operatorname{Wr}(P, I)$ induced by β. Let $\bar{\tau}$ be the homomorphism induced by τ_{N^*} of $N^* \operatorname{Wr}(P, I)$ onto $C \operatorname{Wr}(P, I)$. Then, since $\tau_{N^*} = \alpha\beta$, $\bar{\tau} = \bar{\alpha}\bar{\beta}$. If $x \in G^*$, then

$$\begin{aligned}\bar{\tau}\lambda^*(x) = \bar{\tau}(\rho^*(x)\, x_{T^*}) &= \rho^*(x)(\tau(x))_T \\ &= \rho(\tau(x))(\tau(x))_T \\ &= \lambda\tau(x).\end{aligned}$$

Hence, $\bar{\tau}\lambda^* = \lambda\tau$. But then

$$\bar{\alpha}\bar{\beta}\lambda^* = \bar{\tau}\lambda^* = \lambda\tau$$

and we have the commutative diagram

$$\begin{tikzcd} G^* \arrow[r, "\tau"] \arrow[d, "\bar{\beta}\lambda^*"'] & B \arrow[d, "\lambda"] \\ A \operatorname{Wr}(P, I) \arrow[r, "\bar{\alpha}"] & C \operatorname{Wr}(P, I) \end{tikzcd}$$

Since G is the pull-back of

$$\begin{tikzcd} & B \arrow[d, "\lambda"] \\ A \operatorname{Wr}(P, I) \arrow[r, "\bar{\alpha}"] & C \operatorname{Wr}(P, I) \end{tikzcd}$$

it follows that there must be a homomorphism γ of G^* into G such that the following diagram commutes.

$$\begin{tikzcd} G^* \arrow[rr, "\tau"] \arrow[dr, "\gamma"] \arrow[dd, "\bar{\beta}\lambda^*"'] & & B \arrow[dd, "\lambda"] \\ & G \arrow[ur, "\sigma"'] \arrow[dl] & \\ A \operatorname{Wr}(P, I) \arrow[rr, "\bar{\alpha}"] & & C \operatorname{Wr}(P, I) \end{tikzcd}$$

(Recall that the map from G to $A\,\mathrm{Wr}(P, I)$ is an inclusion map.) Then $\gamma = \bar{\beta}\lambda^*$ and $\tau = \sigma\gamma$. Now

$$\sigma(N) = \alpha(A) = C = \tau(N^*) = \sigma\gamma(N^*).$$

This forces $\gamma(N^*) \leqslant N$ (since $N \geqslant M = \mathrm{kernel}(\sigma)$). Then

$$e\gamma_{N^*} = e_j\bar{\beta}(\lambda^*)_{N^*} = \beta e_j(\lambda^*)_{N^*} = \beta$$

where we have used the fact that $e_j(\lambda^*)_{N^*}$ is the identity. We now see that γ has the desired property.

Suppose now that γ' is another homomorphism of G^* into G such that $\tau = \sigma\gamma'$ and $\beta = e(\gamma')_{N^*}$. Define the function f on G^* by $f(x) = \gamma(x)^{-1}\,\gamma'(x)$. Then since $\sigma f(x) = \tau(x)^{-1}\,\tau(x) = 1$, we see that $f(x) \in \mathrm{kernel}(\sigma) = M$ for all $x \in G^*$. If $x \in N^*$, then $ef(x) = 1$ and so $f(x) \in \mathrm{kernel}(e) = K$ for all $x \in N^*$.

Now $\tau(T^*) = T$ which is a right transversal of C in B. Also σ maps G onto B and N is the inverse image of C under σ. Since $\sigma\gamma = \tau$, we conclude that $\gamma(T^*)$ is a right transversal of N in G. It now follows that the distinct conjugates of K in G are $\{\gamma(t^*)^{-1}\,K\gamma(t^*) \mid t^* \in T^*\}$. Since $f(t^*) \in M$ and since $M = \prod_{i \in I} (M/K_i)$, we see that

$$\bigcap_{t^* \in T^*} \gamma(t^*)^{-1}\,K\gamma(t^*)f(t^*)$$

must consist of a single element $k \in M$. Since $1^* = 1$, we conclude that $f(1^*) = 1$ and that $k \in K$. To finish the proof of the theorem we will show that $\gamma'(x) = k^{-1}\gamma(x)\,k$ for all $x \in G^*$. Equivalently, if we set $\gamma''(x) = k\gamma'(x)\,k^{-1}$ for all $x \in G^*$, we need to show that $\gamma'' = \gamma$.

Clearly, γ'' is a homomorphism of G^* into G. We assert that γ'' satisfies the following three conditions:

(1) $\sigma\gamma'' = \tau$;

(2) $e(\gamma'')_{N^*} = \beta$;

(3) $\bigcap_{t^* \in T^*} \gamma(t^*)^{-1}\,K\gamma''(t^*) = \{1\}$.

The validity of (1) and (2) follows from the similar equations satisfied by γ' and from

$$k \in K = \mathrm{kernel}(e) \leqslant M = \mathrm{kernel}(\sigma).$$

If $t^* \in T^*$, then

$$K\gamma''(t^*) = Kk\gamma'(t^*)\,k^{-1} = K\gamma'(t^*)\,k^{-1} = K\gamma(t^*)f(t^*)\,k^{-1}.$$

This implies that

$$\bigcap_{t^*\in T^*} \gamma(t^*)^{-1} K\gamma''(t^*) = \left(\bigcap_{t^*\in T^*} \gamma(t^*)^{-1} K\gamma(t^*)f(t^*)\right) k^{-1} = \{1\}$$

and so (3) is verified. Next we show that (1), (2), and (3) imply that $\gamma'' = \gamma$.

Note first that

$$\gamma(t^*)^{-1} K\gamma''(t^*) = (\gamma(t^*)^{-1} K\gamma(t^*))\, \gamma(t^*)^{-1} \gamma''(t^*)$$

and so it follows from (3) that

$$\gamma(t^*)^{-1} \gamma''(t^*) \in \gamma(t^*)^{-1} K\gamma(t^*).$$

Hence $\gamma''(t^*) \in K\gamma(t^*)$ for all $t^* \in T^*$. From (2), we obtain $\gamma''(y) \in K\gamma(y)$ for all $y \in N^*$.

If $x \in G^*$, then $x = yt^*$ with $y \in N^*$ and $t^* \in T^*$. Since $\gamma(y)$ normalizes K, we have

$$\begin{aligned} K\gamma''(x) &= K\gamma''(y)\,\gamma''(t^*) = K\gamma(y)\,\gamma''(t^*) = \gamma(y)\,K\gamma''(t^*) \\ &= \gamma(y)\,K\gamma(t^*) = K\gamma(y)\,\gamma(t^*) = K\gamma(x). \end{aligned}$$

Thus we have shown that $K\gamma''(x) = K\gamma(x)$ for all $x \in G^*$. Consequently, whenever u, $x \in G^*$, we must have

$$K\gamma(u)\,\gamma(x) = K\gamma(ux) = K\gamma''(ux) = K\gamma''(u)\,\gamma''(x) = K\gamma(u)\,\gamma''(x).$$

Since $\{\gamma(t^*)^{-1} K\gamma(t^*) \mid t^* \in T^*\}$ are all the conjugates of K in G, Lemma 2.1 implies that

$$\bigcap_{t^*\in T^*} \gamma(t^*)^{-1} K\gamma(t^*)\, g = \{g\}$$

for all $g \in G$. We now see that

$$\begin{aligned} \{\gamma(x)\} &= \bigcap_{t^*\in T^*} \gamma(t^*)^{-1} K\gamma(t^*)\,\gamma(x) \\ &= \bigcap_{t^*\in T^*} \gamma(t^*)^{-1} K\gamma(t^*)\,\gamma''(x) = \{\gamma''(x)\} \end{aligned}$$

for all $x \in G^*$. This shows that $\gamma'' = \gamma$ and the theorem is proved.

3.2. COROLLARY. *Let G, M, N, and K be as in the theorem. Then the automorphisms of G which leave M and K invariant as sets and act identically on both G/M and N/K are precisely the inner automorphisms of G induced by the elements of K.*

Proof. Set $G^* = G$, $\tau = \sigma$, $\beta = e$, and $\gamma =$ the identity automorphism of G. If γ' is any automorphism of G which fixes M and K as sets and which acts identically on both G/M and N/K, then Theorem 3.1 implies that γ' is an inner automorphism of G induced by some element of K.

Keeping the same notation as in Theorem 3.1, suppose we have a homomorphism φ of a group A^* into A such that $\alpha\varphi(A^*) = C$. Then, setting $\alpha^* = \alpha\varphi$, let G^* be the induced extension defined by $\alpha^* : A^* \to B$ and let M^*, K^*, N^*, σ^*, and e^* be defined by analogy with M, K, N, σ, and e, respectively. Then

$$N^* = \{x \in G^* \mid \sigma^*(x) \in \alpha^*(A^*) = C = \alpha(A)\}$$

and

$$\alpha(\varphi e^*) = (\alpha\varphi)\, e^* = \alpha^* e^* = \sigma^*_{N^*}.$$

Setting $\tau = \sigma^*$ and $\beta = \varphi e^*$, Theorem 3.1 is applicable with the result that there is an essentially unique homomorphism γ of G^* into G such that $\sigma^* = \sigma\gamma$ and $\varphi e^* = e\gamma_{N^*}$.

Similarly, if ψ is a homomorphism of a group A^{**} into A^* such that $\alpha^*\psi(A^{**}) = C$, then with $\alpha^{**} = \alpha^*\psi$ and G^{**} the induced extension defined by $\alpha^{**} : A^{**} \to B$, there is a homomorphism δ of G^{**} into G^* such that $\sigma^{**} = \sigma^*\delta$ and $\psi e^{**} = e^*\delta_{N^{**}}$. (Here σ^{**}, e^{**}, and N^{**} are defined by analogy with σ^*, e^*, and N^*, respectively.) Moreover, the composite map $\gamma\delta : G^{**} \to G$ is essentially unique with respect to the conditions $\sigma^{**} = \sigma(\gamma\delta)$ and $\varphi\psi e^{**} = e(\gamma\delta)_{N^{**}}$ in the sense that any other homomorphism of G^{**} into G satisfying these conditions must be a composite of $\gamma\delta$ with an inner automorphism of G induced by some element of K. This shows that we are dealing with a functorial phenomenon.

In particular, if φ is an isomorphism of A^* onto A, $A^{**} = A$, and if $\psi = \varphi^{-1}$, then the preceding shows that $\gamma\delta$ is an inner automorphism of G. Similarly, $\delta\gamma$ is an automorphism of G^*. This implies that both γ and δ are isomorphisms. We set out this conclusion in the following brief form.

3.3. COROLLARY. *Suppose $\alpha : A \to B$ and $\alpha^* : A^* \to B$ are group homomorphisms such that $\alpha^* = \alpha\varphi$ where φ is an isomorphism of A^* onto A. Then the induced extensions defined by $\alpha : A \to B$ and $\alpha^* : A^* \to B$ are isomorphic.*

An interesting particular case of this corollary concerns twisted wreath products (see [6] or [5, pp. 99–100] for the definition, but note that we always mean the unrestricted product). It will follow from our Theorem 4.1 that a twisted wreath product is precisely an induced extension defined by a homomorphism $\alpha : A \to B$ such that A splits over the kernel of α. In this situation, we may consider A to be the semi-direct product CS where $S =$ kernel(α), $C = \alpha(A)$, and α is the natural projection. To complete the specification of A, we need a homomorphism of C into the automorphism group of S. Suppose χ is a homomorphism of C into S and we define $\rho(c)$, for $c \in C$, to be the inner automorphism of S induced by $\chi(c)$. Then $A = CS$ is isomorphic to the direct product $A^* = C \times S$ with $\varphi : (c, s) \rightarrow c(\chi(c^{-1})\, s)$ being an isomorphism of A^* onto A and having the property that $\alpha^* = \alpha\varphi$ (where α^* is the natural projection of $A^* = C \times S$ onto its first direct factor). Corollary 3.3 now asserts that the induced extension G defined by $\alpha : A \rightarrow B$ is isomorphic to the induced extension G^* defined by $\alpha^* : A^* \rightarrow B$. When C contains no non-trivial normal subgroup of B (so that B is faithfully represented by a permutation group P acting on I, the set of all right cosets of C in B), G^* is easily seen to be the wreath product $S\,\mathrm{Wr}(P, I)$. If $\chi(C)$ is not contained in the center of S, then C (in A) acts non-trivially on S but the twisted wreath product G is still isomorphic to the ordinary wreath product G^*. As a special case, we have the following result.

3.4. Corollary. *Any two twisted wreath products of the alternating groups A_m and A_n in which A_m is twisted by the point stabilizer A_{n-1} are isomorphic.*

Proof. Here A is the semi-direct product $A_m A_{n-1}$, $C = A_{n-1}$, $S = A_m$, and $B = A_n$. Any homomorphism of A_{n-1} into the automorphism group of A_m must map A_{n-1} into the group of inner automorphisms. (If this were not the case, then A_{n-1} would have to have a proper normal subgroup whose index is 2 or 4.) It now follows from our previous discussion that any twisted wreath product of A_m and A_n with the twisting being done by A_{n-1} is isomorphic to the ordinary wreath product of A_m and A_n. The corollary is an immediate consequence.

Further properties of induced extensions will be proved in the next section. (In particular, see Sections 4.1 and 4.6.)

4. The Main Results

Throughout this section, we will be dealing with the following hypothesis:

$M \trianglelefteq G$, $K \trianglelefteq M$, $M = \prod_{i \in I} (M/K_i)$ where $\{K_i \mid i \in I\}$ are the distinct conjugates of K in G, and $N = N_G(K)$. $(*)$

In the previous section, it was shown that every induced extension satisfies $(*)$. Now we show the converse.

4.1. THEOREM. *Assume* $(*)$. *Let* $\alpha : N/K \to G/M$ *be defined by* $Kx \to Mx$*; let* $\sigma : G \to G/M$ *and* $e : N \to N/K$ *be the natural epimorphisms. Then the following are true.*

(1) *G is the induced extension defined by* $\alpha : N/K \to G/M$.

(2) *Suppose τ is a homomorphism of a group G^* onto G/M and let* $N^* = \{x \in G^* \mid \tau(x) \in N/M\}$. *Assume that β is a homomorphism of N^* into N/K such that* $\tau_{N^*} = \alpha\beta$. *Then there is a homomorphism γ of G^* into G such that $\tau = \sigma\gamma$ and* $\beta = e\gamma_{N^*}$. *If γ' is also a homomorphism of G^* into G such that $\tau = \sigma\gamma'$ and* $\beta = e(\gamma')_{N^*}$, *then there is a $k \in K$ such that* $\gamma'(x) = k^{-1}\gamma(x)\,k$ *for all* $x \in G^*$.

Proof. Let T be a right transversal of N in G. Then $\sigma(T)$ is a right transversal of N/M in G/M. We may identify I with the set of right cosets of N in G and then with the set of right cosets of N/M in G/M. If ρ is the permutation representation of G/M on the right cosets of N/M, then we may regard $\rho(G/M)$ as acting on I. Now $\rho\sigma$ is the permutation representation of G on the cosets of N. Let $P = \rho(G/M)$ and let $\bar{\lambda} = \lambda_T$ be the embedding of G into $N\,\mathrm{Wr}(P, I)$. Let $\lambda = \lambda_{\sigma(T)}$ be the embedding of G/M into $(N/M)\,\mathrm{Wr}(P, I)$. If $\bar{\sigma}$ denotes the homomorphism induced by σ of $N\,\mathrm{Wr}(P, I)$ onto $(N/M)\,\mathrm{Wr}(P, I)$, then we have the following commutative diagram.

$$\begin{tikzcd}
G \arrow[r, "\sigma"] \arrow[d, "\bar{\lambda}"'] & G/M \arrow[d, "\lambda"] \\
N\,\mathrm{Wr}(P, I) \arrow[r, "\bar{\sigma}"] & (N/M)\,\mathrm{Wr}(P, I)
\end{tikzcd}$$

Next, let $\bar{e}$ be the homomorphism of $N\,\mathrm{Wr}(P, I)$ onto $(N/K)\,\mathrm{Wr}(P, I)$ induced by e and let $\bar{\alpha}$ be the homomorphism of $(N/K)\,\mathrm{Wr}(P, I)$ onto $(N/M)\,\mathrm{Wr}(P, I)$ induced by α. Since $\alpha e = \sigma_N$, we find that $\bar{\alpha}\bar{e} = \bar{\sigma}$. Therefore, the following diagram is commutative.

$$\begin{tikzcd}
G \arrow[r, "\sigma"] \arrow[d, "\bar{e}\bar{\lambda}"'] & G/M \arrow[d, "\lambda"] \\
(N/K)\,\mathrm{Wr}(P, I) \arrow[r, "\bar{\alpha}"] & (N/M)\,\mathrm{Wr}(P, I)
\end{tikzcd}$$

The commutativity of this shows that $\bar{e}\bar{\lambda}(M) \leqslant \mathrm{kernel}(\bar{\alpha})$. We in fact assert that $\bar{e}\bar{\lambda}$ maps M isomorphically onto kernel $(\bar{\alpha})$. If $m \in M$, then, as $M \leqslant \mathrm{kernel}(\rho\sigma)$, $\bar{\lambda}(m) = m_T$ for all $t \in T$. Since M is normal in G and contained in N,

$$m_T(Nt) = u_T(tm^{-1})\,mt^{-1} = tmt^{-1}.$$

Suppose now that $x \in \text{kernel}(\bar{\alpha})$. Then $x \in (N/K)^I$ and

$$x(Nt) = Kx_t \in M/K$$

for some $x_t \in M$. There must be a unique $m \in M$ such that

$$\bigcap_{t \in T} (t^{-1}Kt)(t^{-1}x_t t) = \{m\}.$$

But this implies that

$$m_T(Nt) = tmt^{-1} \in Kx_t = x(Nt)$$

for all $t \in T$. It now follows that $\bar{e}\bar{\lambda}(m) = \bar{e}(m_T) = x$. Hence $\bar{e}\bar{\lambda}(M) = \text{kernel}(\bar{\alpha})$ and it only remains to show that $M \cap \text{kernel}(\bar{e}\bar{\lambda}) = 1$. Suppose, therefore, that $m \in M$ and $\bar{e}\bar{\lambda}(m) = 1$. Then

$$1 = \bar{e}(m_T(Nt)) = Ktmt^{-1}$$

for all $t \in T$. But then

$$m \in \bigcap_{t \in T} t^{-1}Kt = \{1\}.$$

Thus we have shown that $\bar{e}\bar{\lambda}$ maps M isomorphically onto kernel$(\bar{\alpha})$.

It now follows that the following diagram commutes and has exact rows.

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r] \arrow[d, equal] & G \arrow[r, "\alpha"] \arrow[d, "\bar{e}\bar{\lambda}"] & G/M \arrow[r] \arrow[d, "\lambda"] & 1 \\
1 \arrow[r] & M \arrow[r] & (N/K)\,\mathrm{Wr}(P, I) \arrow[r, "\bar{\alpha}"] & (N/M)\,\mathrm{Wr}(P, I) \arrow[r] & 1
\end{tikzcd}$$

But then the top row is a pull-back of the bottom row. This implies that G is the induced extension defined by $\alpha : N/K \to G/M$. This proves (1), and (2) now follows from Theorem 3.1.

One consequence of the above theorem is that induced extensions are quite common. For example, if M is a minimal normal subgrpoup of a finite group G and if M is not abelian, then M is a direct product $S_1 \times S_2 \times \cdots \times S_n$ where $\{S_1, ..., S_n\}$ is a conjugacy class of subgroups in G. The first part of Theorem 4.1 now shows that G is an induced extension. Theorem 4.1 also implies that any wreath product $A\ \mathrm{Wr}(P, I)$ in which P acts transitively on I is an induced extension. Similarly, as pointed out in the previous section, it follows from our theorem that any twisted wreath product is an induced extension.

Our original proof of the next theorem was based upon the induced extension construction, but there seems to be some merit in having a direct simple proof without bringing in any extra notation. The motivation behind our present proof, however, comes directly from induced extensions.

4.2. THEOREM. *Assume* $(*)$. *Let L be a subgroup of N such that $K \leqslant L$ and $N = LM$. Then G has a subgroup H satisfying the following:*

(1) *$G = HM$, $L = (H \cap N) K$, and $H \cap M = \prod_{i \in I} (H \cap M/(H \cap K_i)$.*

(2) *Suppose $H_1 \leqslant G$, $G = H_1 M$, and $H_1 \cap N \leqslant L$. Then there is an element $k \in K$ such that $k^{-1} H_1 k \leqslant H$. Further, $k^{-1} H_1 k = H$ if, and only if, $L = (H_1 \cap N) K$ and $H_1 \cap M = \prod_{i \in I} (H_1 \cap M)/(H_1 \cap K_i)$.*

Proof. Let T be a right transversal of N in G. Then $\{K_i \mid i \in I\} = \{t^{-1} K t \mid t \in T\}$. Let $u = u_T$ (recall that $u_T(x) \in T$ and $Nx = N u_T(x)$ for $x \in G$). Then define H by

$$H = \{x \in G \mid u(tx)\, x^{-1} t^{-1} \in L \text{ for all } t \in T\}.$$

If λ_T is the embedding of G into $N \operatorname{Wr}(P, I)$ and regarding $L \operatorname{Wr}(P, I)$ as a subgroup of $N \operatorname{Wr}(P, I)$ in the obvious way, it is not difficult to show that H is the inverse image under λ_T of the subgroup $\lambda_T(G) \cap L \operatorname{Wr}(P, I)$. From this it certainly follows that H is a subgroup of G. This also can be verified by a direct calculation without reference to wreath products. For suppose $x, y \in H$ and $t \in T$. Setting $t_1 = u(tx)$ and $t_2 = u(t_1 y^{-1})$, we have

$$u(txy^{-1})(xy^{-1})^{-1} t^{-1} = t_2 y x^{-1} t^{-1} = (t_1 y^{-1} t_2^{-1})^{-1} (t_1 x^{-1} t^{-1}).$$

Since $t_1 y^{-1} t_2^{-1} = u(t_2 y)\, y^{-1} t_2^{-1}$ and $t_1 x^{-1} t^{-1} = u(tx)\, x^{-1} t^{-1}$ both must belong to L, we conclude that $xy^{-1} \in H$. We now proceed to show that H satisfies the conditions in the conclusion of the theorem.

Let x be some fixed element of G and let $t \in T$. Then

$$u(tx)\, x^{-1} t^{-1} \in N = LM.$$

Hence, for each $t \in T$, there is an $m_t \in M$ such that

$$u(tx)\, x^{-1} t^{-1} \in L m_t.$$

If $x \in L$, then $u(x) = 1$ and so we may choose $m_1 = 1$. Certainly $t^{-1} m_t t \in M$ and so

$$\bigcap_{t \in T} (t^{-1} K t)(t^{-1} m_t t) = \{m\}$$

for some $m \in M$. If $x \in L$, then $m_1 = 1$ and so $m \in K$. In any event

$$m_t t m^{-1} t^{-1} \in K$$

for all $t \in T$. Since $tmx \equiv tx (\operatorname{mod} M)$, $u(tmx) = u(tx)$. Then

$$\begin{aligned} u(tmx)(mx)^{-1} t^{-1} &= u(tx) x^{-1} m^{-1} t^{-1} \\ &= (u(tx) x^{-1} t^{-1} m_t^{-1})(m_t t m^{-1} t^{-1}) \in LK = L. \end{aligned}$$

This implies that $mx \in H$ and so $G = MH$. If $x \in L$, then $m \in K \leqslant L$ and so $mx \in H \cap L \leqslant H \cap N$. Then $x \in K(H \cap N)$ and so $L \leqslant K(H \cap N)$. On the other hand $K \leqslant L$ and if $h \in H \cap N$, then $u(h) = 1$ (since $h \in N$) and (since $h \in H$)

$$u(1h) h^{-1} 1^{-1} \in L.$$

It now follows that $H \cap N \leqslant L$. Therefore, $L = K(H \cap N)$.

We now claim that $\bigcap_{i \in I} K_i(H \cap M) \leqslant H$. If so, then it will follow that $\bigcap_{i \in I} K_i(H \cap M) = H \cap M$ and then, using Lemma 2.1, that $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$. Suppose now that $x \in \bigcap_{i \in I} K_i(H \cap M)$. Then $x \in M$ and so $u(tx) = t$ for all $t \in T$. If $t \in T$, then

$$x \in (t^{-1}Kt)(H \cap M)$$

and so $x = t^{-1}kth$ for some $k \in K$ and $h \in H \cap M$. Since $h \in H \cap M$, L must contain

$$u(th) h^{-1} t^{-1} = th^{-1} t^{-1}.$$

But then

$$u(tx) x^{-1} t^{-1} = tx^{-1} t^{-1} = t(t^{-1}kth)^{-1} t^{-1} = (th^{-1}t^{-1}) k^{-1} \in LK = L.$$

It follows from this that $x \in H$. We now have proved the first part of the theorem.

Assume now that H_1 is a subgroup of G such that $G = MH_1$ and $H_1 \cap N \leqslant L$. For each $t \in T$, there must be an $m_t \in M$ such that $t \in m_t H_1$. Clearly we may choose $m_1 = 1$. Now

$$\bigcap_{t \in T} (t^{-1}Kt)(t^{-1}m_t t) = \{k\}$$

for some $k \in M$. Since $m_1 = 1$, we see that $k \in K$. Since $t^{-1}Kt \trianglelefteq M$,

$$kt^{-1}Kt = (t^{-1}m_t t)(t^{-1}Kt)$$

and so

$$m_t^{-1} tkt^{-1} \in K$$

for all $t \in T$. We now show that $k^{-1} H_1 k \leqslant H$.

Let $x \in H_1$ and $t \in T$. Since

$$tk^{-1}xk \equiv tx (\operatorname{mod} M),$$

$u(tk^{-1}xk) = u(tx)$. Let $t_1 = u(tx)$. Then

$$\begin{aligned} u(tk^{-1}xk)(k^{-1}xk)^{-1}\, t^{-1} &= t_1 k^{-1} x^{-1} k t^{-1} \\ &= (t_1 k^{-1} t_1^{-1} m_{t_1})(m_{t_1}^{-1} t_1 x^{-1} t^{-1} m_t)(m_t^{-1} tkt^{-1}) \\ &= (m_{t_1}^{-1} t_1 k t_1^{-1})^{-1}\, (m_{t_1}^{-1} t_1 x^{-1} t^{-1} m_t)(m_t^{-1} tkt^{-1}) \\ &\in K(m_{t_1}^{-1} t_1 x^{-1} t^{-1} m_t)\, K. \end{aligned}$$

Now $t_1 x^{-1} t^{-1} = u(tx)(tx)^{-1} \in N$ and so

$$m_{t_1}^{-1} t_1 x^{-1} t^{-1} m_t \in MNM = N.$$

Also

$$m_{t_1}^{-1} t_1 x^{-1} t^{-1} m_t = (m_{t_1}^{-1} t_1)\, x^{-1} (m_t^{-1} t)^{-1} \in H_1.$$

It follows from all this that

$$u(tk^{-1}xk)(k^{-1}xk)^{-1}\, t^{-1} \in K(N \cap H_1)\, K \subseteq KLK = L.$$

This implies that $k^{-1}xk \in H$ and so $k^{-1}H_1 k \leqslant H$.

Now suppose $H_1 \cap M = \prod_{i \in I} (H_1 \cap M)/(H_1 \cap K_i)$ and $(H_1 \cap N)\, K = L$. If $k^{-1}H_1 k = H_2$, then we must have $H_2 \cap M = \prod_{i \in I} (H_2 \cap M)/(H_2 \cap K_i)$ and $L = N_{H_2}(K)\, K = N_H(K)\, K$. The conjugates of K are transitively permuted by H_2 (since $G = H_2 M$) and H_2 normalizes H (since $H_2 \leqslant H$). It now follows that

$$N_{H_2}(K_i)\, K_i = N_H(K_i)\, K_i$$

for all $i \in I$. If we set $A = H_2$ in Theorem 2.3, then both H_2 and H satisfy the requirements for B in that theorem. It now follows from Theorem 2.3 that $H_2 = H$.

If on the other hand, $k^{-1}H_1 k = H$, then it is immediate that $(H_1 \cap M) = \prod_{i \in I} (H_1 \cap M)/(H_1 \cap K_i)$ while, since $k \in K \leqslant L$,

$$L = k(L)\, k^{-1} = k(H \cap N)\, Kk^{-1} = (H_1 \cap N)\, K.$$

The proof of the theorem is complete.

4.3. Corollary. *Let G, M, K, K_i, N, L, and H be the same as in the theorem. Then the following are true.*

(1) *$H \cap M$ is isomorphic to $((L \cap M)/K)^I$.*

(2) *H is a complement of M in G if, and only if, L/K is a complement of M/K in N/K.*

(3) *If $R \leqslant G$ and $G = RM$, then R is conjugate to a subgroup of H in G if, and only if, $R \cap N$ is conjugate to a subgroup of L in N.*

(4) *Let $R \leqslant G$. Then R is conjugate to H in G if, and only if, $G = RM$, $R \cap M = \prod_{i \in I} (R \cap M)/(R \cap K_i)$, and L is conjugate to $(R \cap N) K$ in N.*

Proof. (1) Since $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$ and since $\{H \cap K_i \mid i \in I\}$ is a conjugacy class in H (since $G = HM$), $H \cap M$ is isomorphic to $((H \cap M)/(H \cap K))^I$. From

$$K \leqslant L \cap M \leqslant L = (H \cap N) K$$

we obtain

$$L \cap M = K(H \cap N \cap L \cap M) = K(H \cap M).$$

Thus $(L \cap M)/K$ is isomorphic to $(H \cap M)/(H \cap K)$. It now follows that $H \cap M$ is isomorphic to

$$((L \cap M)/K)^I.$$

(2) From (1), $H \cap M = 1$ if, and only if, $(L/K) \cap (M/K) = 1$.

(3) Suppose $G = RM$ and $R \cap N$ is conjugate to a subgroup of L in N. Since $M \leqslant N \leqslant MR$, we must have $N = M(R \cap N)$. Then there is an $m \in M$ such that

$$m^{-1}(R \cap N) m \leqslant L.$$

If $H_1 = m^{-1}Rm$, then $G = H_1 M$ and $H_1 \cap N \leqslant L$. It follows from the theorem that H_1 is conjugate to a subgroup of H. Hence R is conjugate to a subgroup of H. Conversely, if R is conjugate to a subgroup of H, then, since $G = RM$, $m^{-1}Rm \leqslant H$ for some $m \in M$. Then

$$m^{-1}(R \cap N) m = m^{-1}Rm \cap N \leqslant H \cap N \leqslant L$$

(4) As in (3), if $G = RM$ and if $(R \cap N) K$ is conjugate to L in N, then there is an $m \in M$ such that

$$L = m^{-1}(R \cap N) Km = m^{-1}(R \cap N) mK.$$

Hence, if $H_1 = m^{-1}Rm$, then $G = H_1 M$, $H_1 \cap M = \prod_{i \in I} (H_1 \cap M)/(H_1 \cap K_i))$ and $L = (H_1 \cap N) K$. The theorem implies that H_1 is conjugate to H. The rest of (4) follows easily.

Although in Theorem 4.2 we start with the subgroup L and proceed to construct the subgroup H, we may reverse the procedure. Suppose, for example, that H is a subgroup of G such that (1) $G = HM$, and (2) $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$. Then if we set $L = (H \cap N) K$, we find that L and H satisfy the conditions of Theorem 4.2. Thus the theorem and its corollary can be applied to obtain, for example, necessary and sufficient conditions for a subgroup of G to be conjugate to H. In particular, if $\mathscr{S}$ denotes the conjugacy classes in G of subgroups H satisfying (1) and (2), then the mapping $H \to (H \cap N) K/K$ induces a bijection between $\mathscr{S}$ and the conjugacy classes in N/K of subgroups L/K such that $N/K = (M/K)(L/K)$. Further, if H is any complement to M in G, then H satisfies (1) and (2) (the latter because $H \cap M = 1$) and it follows from Corollary 4.3 that we have a bijection between the conjugacy classes in G of complements of M, if any, and the conjugacy classes in N/K of complements of M/K. Thus we have proved the following:

4.4. COROLLARY. *Assume* (*). *Then there is a bijection between, on the one hand, conjugacy classes in* G *of subgroups* H *satisfying* $G = HM$ *and* $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$, *and, on the other hand, the conjugacy classes in* N/K *of subgroups* L/K *satisfying* $N/K = (M/K)(L/K)$. *Moreover, under this bijection, the conjugacy classes in* G *of complements of* M, *if any, are in one-to-one correspondence with the conjugacy classes in* N/K *of complements of* M/K.

Because of the importance of the case when I is a finite set (this certainly happens if G is a finite group), we reformulate our results in this special case.

4.5. THEOREM. *Assume that* $M \trianglelefteq G$ *and that*

$$M = S_1 \times S_2 \times \cdots \times S_n$$

where $\{S_1, \ldots, S_n\}$ *is a conjugacy class of subgroups in* G. *Let* $N = N_G(S_1)$ *and* $K = S_2 \times \cdots \times S_n$. *Then the following are true.*

(1) *If* $K \leqslant L \leqslant N$ *and* $N = LS_1$, *then there is a subgroup* H *in* G *such that* $G = HM$, $L = (H \cap N) K$,

$$H \cap M = (H \cap S_1) \times (H \cap S_2) \times \cdots \times (H \cap S_n),$$

$\{H \cap S_1, \ldots, H \cap S_n\}$ *is a conjugacy class in* H, *and* $H \cap S_1 = L \cap S_1$. *Further,* H *is unique up to conjugacy under* K.

(2) *Suppose* $H \leqslant G$, $G = HM$, *and*

$$H \cap M = (H \cap S_1) \times (H \cap S_2) \times \cdots \times (H \cap S_n).$$

Assume further that $H_1 \leqslant G$ *and* $G = H_1 M$. *Then* H_1 *is conjugate in* G *to a subgroup of* H *if, and only if,* $H_1 \cap N$ *is conjugate in* N *to a subgroup of* $(H \cap N) K$. *Further,* H_1 *is conjugate to* H *in* G *if, and only if,* $(H_1 \cap N) K$ *is conjugate to* $(H \cap N) K$ *in* N *and also*

$$H_1 \cap M = (H_1 \cap S_1) \times (H_1 \cap S_2) \times \cdots \times (H_1 \cap S_n).$$

Proof. This follows directly from Theorem 4.2 and its corollaries.

The next result about splitting in an induced extension merely restates earlier results in the language of induced extensions.

4.6. THEOREM. *Let* G *be the induced extension defined by* $\alpha : A \to B$. *Let* S *be the kernel of* α *and let* M *be the normal subgroup of* G *which is the direct product of* $|B : \alpha(A)|$ *copies of* S. *Then* G *splits over* M *if, and only if,* A *splits over* S. *Further, there is a one-to-one correspondence between classes in* G *of complements of* M *and conjugacy classes in* A *of complements of* S.

Proof. This follows directly from the description of G given in Section 3 together with Corollaries 4.3 and 4.4.

To illustrate that the induced extension procedure can be used to construct groups which are not semi-direct products, we offer the following.

4.7. THEOREM. *Let* B *be any finite simple non-abelian group. Then there is a finite group* G *with a minimal normal subgroup* M *such that* M *is the direct product of copies of* A_6, G/M *is isomorphic to* B, *and* G *does not split over* M.

Proof. Let A be the automorphism group of A_6, the alternating group of degree 6. Since a Sylow 2-subgroup of B can be neither cyclic nor quaternion (this follows from the Feit–Thompson Theorem, the Burnside Transfer Theorem, and a theorem of Brauer and Suzuki), B must contain a subgroup C which is elementary abelian of order 4. Then there is a homomorphism α of A into B such that $\alpha(A) = C$ and the kernel of α is A_6. Now let G be the induced extension defined by $\alpha : A \to B$. It is well known that A does not split over A_6 and so it is easily verified that G has the required properties.

5. UNION OF CONJUGACY CLASSES

We now wish to consider our results when $\{K_i \mid i \in I\}$ is a union of conjugacy classes. First we look at the special case when $K_i \trianglelefteq G$ for all $i \in I$. Thus we consider the following hypothesis:

$M \trianglelefteq G$; for each $i \in I$, K_i is a normal subgroup of G contained in M; and $M = \prod_{i \in I} (M/K_i)$. $\quad (**)$

5.1. LEMMA. *Assume* (**). *Let* $\sigma : G \to G/M$, $e_i : G \to G/K_i$, *and* $\alpha_i : G/K_i \to G/M$ *be the natural epimorphisms. Assume that* τ *is a homomorphism of a group* G^* *onto* G/M. *For each* $i \in I$, *assume that* β_i *is a homomorphism of* G^* *into* G/K_i *such that* $\alpha_i \beta_i = \tau$ *for all* $i \in I$. *Then there is one and only one homomorphism* γ *of* G^* *into* G *such that* $\tau = \sigma\gamma$ *and* $\beta_i = e_i \gamma$ *for each* $i \in I$.

Proof. Let $x \in G^*$. For each $i \in I$,

$$\beta_i(x) = K_i y_i$$

for some $y_i \in G$. If $i, j \in I$,

$$My_i = \alpha_i \beta_i(x) = \tau(x) = \alpha_j \beta_j(x) = My_j.$$

Hence, $y_i y_j^{-1} \in M$. Lemma 2.1 now implies that

$$\left| \bigcap_{i \in I} K_i y_i \right| = 1.$$

Define $\gamma(x)$ by

$$\{\gamma(x)\} = \bigcap_{i \in I} K_i y_i.$$

It is easily verified that γ is a homomorphism and that

$$e_i \gamma(x) = K_i \gamma(x) = K_i y_i = \beta_i(x).$$

Hence $e_i \gamma = \beta_i$ and then

$$\tau = \alpha_i \beta_i = (\alpha_i e_i)\gamma = \sigma\gamma.$$

Finally, suppose γ' is a homomorphism of G^* into G such that $\tau = \sigma\gamma'$ and $\beta_i = e_i \gamma'$ for each $i \in I$. Then, for $x \in G^*$ and $i \in I$,

$$K_i \gamma'(x) = e_i \gamma'(x) = \beta_i(x) = e_i \gamma(x) = K_i \gamma(x).$$

But then $\gamma'(x)\gamma(x)^{-1} \in \bigcap_{i \in I} K_i = 1$. Thus $\gamma' = \gamma$.

A special case of the next result was proved by Gaschütz in [3].

5.2. THEOREM. *Assume* (**). *For each* $i \in I$, *let* L_i *be a subgroup of* G *such that* $G = L_i M$ *and* $K_i \leqslant L_i$. *Let* $H = \bigcap_{i \in I} L_i$. *Then the following hold.*

(1) $G = HM$ *and* $L_i = HK_i$ *for each* $i \in I$.

(2) $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$.

(3) H *is a complement of* M *in* G *if, and only if,* L_i/K_i *is a complement of* M/K_i *in* G/K_i *for each* $i \in I$.

(4) *Let R be a subgroup of G such that $G = RM$. Then R is conjugate to a subgroup of H if, and only if, R is conjugate to a subgroup of L_i for each $i \in I$. Moreover, R is conjugate to H if, and only if, $R \cap M = \prod_{i \in I} (R \cap M)/(R \cap K_i)$, and RK_i is conjugate to L_i for each $i \in I$.*

Proof. If $x \in G$, then $x = m_i l_i$ for each $i \in I$ with $m_i \in M$ and $l_i \in L_i$. Then, for $i, j \in I$,

$$l_i l_j^{-1} = m_i^{-1} m_j \in M.$$

By Lemma 2.1,

$$\bigcap_{i \in I} K_i l_i = \{y\}$$

for some $y \in G$. Then, since $K_i \leqslant L_i$,

$$y \in \bigcap_{i \in I} L_i = H.$$

Also

$$xy^{-1} = xl_i^{-1} l_i y^{-1} = m_i(l_i y^{-1}) \in MK_i = M.$$

Hehce, $x \in MH$ and so $G = MH$.

Clearly $L_i \geqslant HK_i$. Suppose $x \in L_i$ and let m_j, l_j, and y have the same meaning as above. Since $x \in L_i$, we may choose $m_i = 1$. Then

$$xy^{-1} = l_i y^{-1} \in K_i.$$

Since $y \in H$, we conclude that $L_i = HK_i$.

(2) Since

$$\bigcap_{i \in I} HK_i = \bigcap_{i \in I} L_i = H,$$

Lemma 2.1 implies that

$$H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i).$$

(3) $L_i \cap M = HK_i \cap M = (H \cap M) K_i$. It follows that $(H \cap M)/(H \cap K_i)$ is isomorphic to $(L_i \cap M)/K_i$. Hence, using (2), $H \cap M = 1$ if, and only if, $L_i \cap M = K_i$ for each $i \in I$.

(4) Suppose $G = RM$ and R is conjugate to a subgroup of L_i for each $i \in I$. Then there is an $m_i \in M$ such that

$$m_i^{-1} R m_i \leqslant L_i$$

for each $i \in I$. Then

$$\bigcap_{i \in I} K_i m_i = \{m\}$$

for some $m \in M$. Then $m_i^{-1} m \in K_i \leqslant L_i$ and so

$$m^{-1} R m = (m_i^{-1} m)^{-1} (m_i^{-1} R m_i)(m_i^{-1} m) \leqslant (m_i^{-1} m)^{-1} L_i (m_i^{-1} m) \leqslant L_i$$

for each $i \in I$. It follows from this that

$$m^{-1} R m \leqslant \bigcap_{i \in I} L_i = H.$$

Suppose next that $R \cap M = \prod_{i \in I} (R \cap M)/(R \cap K_i)$ and that RK_i is conjugate to L_i for each $i \in I$. Then, as before, M contains elements m *and* m_i such that

$$m_i^{-1} R K_i m_i = L_i \qquad \text{for all} \quad i \in I$$

and

$$\bigcap_{i \in I} K_i m_i = \{m\}.$$

Then, since $m_i^{-1} m \in K_i \leqslant L_i$,

$$L_i = (m_i^{-1} m)^{-1} L_i (m_i^{-1} m) = m^{-1} R K_i m.$$

Lemma 2.1 implies that

$$R = \bigcap_{i \in I} R K_i.$$

Then

$$m^{-1} R m = \bigcap_{i \in I} m^{-1} R K_i m = \bigcap_{i \in I} L_i = H.$$

We now have proved (4) in one direction. The other direction is obvious and so the theorem is proved.

For completeness, we include the following consequence of Lemma 2.1.

5.3. LEMMA. *Assume* $(**)$. *Suppose H is a subgroup of G such that $G = HM$ and $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$. Let $L_i = HK_i$ for each $i \in I$. Then $G = L_i M$, $K_i \leqslant L_i$, and $\bigcap_{i \in I} L_i = H_i$.*

Proof. It follows from Lemma 2.1 that

$$\bigcap_{i\in I} L_i = \bigcap_{i\in I} K_i H = H.$$

The rest of the result is obvious.

5.4. THEOREM. *Assume* $(**)$. *For each* $i \in I$, *let* $\mathscr{A}_i$ *denote the set of all complements (if any) of* M/K_i *in* G/K_i. *Let* $\mathscr{B}_i$ *denote the set of conjugacy classes in* $\mathscr{A}_i$ *under* G/K_i. *Let* $\mathscr{A}$ *be the set of all complements of* M *in* G *and let* $\mathscr{B}$ *be the set of all conjugacy classes in* $\mathscr{A}$ *under* G. *Then* $|\mathscr{A}| = |\prod_{i\in I} \mathscr{A}_i|$ *and* $|\mathscr{B}| = |\prod_{i\in I} \mathscr{B}_i|$.

Proof. Suppose $H \in \mathscr{A}$. Then $HK_i/K_i \in \mathscr{A}_i$ for all $i \in \mathscr{A}_i$. Let f_H be the element of the cartesian product $\prod_{i\in I} \mathscr{A}_i$ defined by

$$f_H(i) = HK_i/K_i.$$

On the other hand, if $f \in \prod_{i\in I} \mathscr{A}_i$, then $f(i) = L_i/K_i$ where L_i/K_i is a complement to M/K_i in G. Then $f = f_H$ where

$$H = \bigcap_{i\in I} L_i.$$

It now follows that $|\mathscr{A}| = |\prod_{i\in I} \mathscr{A}_i|$. If H_1 also belongs to $\mathscr{A}$, then H and H_1 are conjugate if, and only if, $f_H(i)$ and $f_{H_1}(i)$ are conjugate in G/K_i for each $i \in I$. This implies that

$$|\mathscr{B}| = \left| \prod_{i\in I} \mathscr{B}_i \right|.$$

We now give our theorem covering the situation when $\{K_i \mid i \in I\}$ is a union of conjugacy classes. We first fix some notation.

> $M \trianglelefteq G$ and $M = \prod_{i\in I} M/K_i$ where $K_i \trianglelefteq M$ for each $i \in I$ and $\{K_i \mid i \in I\}$ is a union of conjugacy classes in G. The subset J of I is chosen so that $\{K_j \mid j \in J\}$ contains exactly one subgroup from each conjugacy class in $\{K_i \mid i \in I\}$. The natural epimorphism of G onto G/M is denoted by σ. For each $i \in I$, $N_i = N_G(K_i)$ while $e_i : N_i \to N_i/K_i$ and $\alpha_i : N_i/K_i \to G/M$ are the natural homomorphisms. $(***)$

The following omnibus theorem is really just a combination of several of our earlier results.

5.5. THEOREM. *Assume* $(***)$. *Then the following are true.*

Suppose τ is a homomorphism of a group G^* onto G/M. For each $j \in J$, let $N_j^* = \{x \in G^* \mid \tau(x) \in \sigma(N_j)\}$ and assume that β_j is a homomorphism of N_j^* into N_j/K_j such that $(\tau)_{N_j^*} = \alpha_j\beta_j$. Then there is a homomorphism γ of G^* into G such that $\tau = \sigma\gamma$ and $\beta_j = e_j(\gamma)_{N_j^*}$ for each $j \in J$. Further, γ is unique up to composition with an inner automorphism of G induced by some element of $\bigcap_{j \in J} K_j$.

(2) *Suppose that for each $j \in J$, L_j is a subgroup of N_j such that $K_j \leqslant L_j$ and $N_j = L_j M$. Then G contains a subgroup H such that $G = HM$, $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$, and $L_j = (H \cap N) K_j$ for each $j \in J$. The subgroup H is unique up to conjugation by an element of $\bigcap_{i \in J} K_j$. Further, $H \cap M = 1$ if, and only if, $(L_j/K_j) \cap (M/K_j) = 1$ for all $j \in J$.*

(3) *Suppose $H \leqslant G$, $G = HM$, and $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$. Let R be a subgroup of G such that $G = RM$. Then R is conjugate in G to a subgroup of H if, and only if, $R \cap N_j$ is conjugate in N_j to a subgroup of $(H \cap N_j) K_j$ for each $j \in J$. Moreover, R is conjugate in G to H if, and only if, $R \cap M = \prod_{i \in I} (R \cap M)/(R \cap K_i)$ and $(R \cap N_j) K_j$ is conjugate in N_j to $(H \cap N_j) K_j$ for each $j \in J$.*

(4) *The following are equivalent:*

- (a) *G splits over M_j;*
- (b) *N_j splits over M for each $j \in J$;*
- (c) *N_j/K_j splits over M/K_j for each $j \in J$.*

(5) *For each $j \in J$, let $\mathscr{C}_j$ denote the set of conjugacy classes of complements of M/K_j in N/K_j. Let $\mathscr{C}$ denote the set of conjugacy classes of complements of M in G. Then $|\mathscr{C}| = |\prod_{j \in J} \mathscr{C}_j|$.*

Proof. For each $j \in J$, let $\Delta(j)$ denote the subset of I consisting of all i such that K_i is conjugate in G to K_j. Define B_j by

$$B_j = \bigcap_{i \in \Delta(j)} K_i.$$

Then $B_j \trianglelefteq G$, $B_j \leqslant K_j \leqslant M$, and $M = \prod_{j \in J} M/B_j$. Also $M/B_j = \prod_{i \in \Delta(j)} (M/B_j)/(K_i/B_j)$ and $\{K_i/B_j \mid i \in \Delta(J)\}$ is the set of all conjugates of K_j/B_j in G/B_j. Certainly

$$N_{G/B_j}(K_j/B_j) = N_j/B_j.$$

Let $b_j : G \to G/B_j$, $a_j : G/B_j \to G/M$, and $c_j : N_j/B_j \to N_j/K_j$ be the natural epimorphisms. We now consider the various parts of our theorem.

(1) Applying Theorem 4.1(2) with G, M, K, N, I, σ, e, α, and β replaced by G/B_j, M/B_j, K/B_j, N_j/B_j, $\Delta(j)$, a_j, c_j, α_j, and β_j, respectively, we find that there is a homomorphism γ_j of G^* into G/B_j such that $\tau = a_j\gamma_j$

and $\beta_j = c_j(\gamma_j)_{N_j^*}$. Since this is true for each $j \in J$, we may apply Theorem 5.1 with the result that there is a homomorphism γ of G^* into G such that $\tau = \sigma\gamma$ and $\gamma_j = b_j\gamma$ for all $j \in J$. Then

$$\beta_j = c_j(\gamma_j)_{N_j^*} = c_j(b_j\gamma)_{N_j^*} = c_j(b_j)_{N_j}(\gamma)_{N_j^*} = e_j(\gamma)_{N_j^*}.$$

Thus γ has the desired properties.

Suppose γ' is another homomorphism of G^* in G and $\tau = \sigma\gamma'$ and $\beta_j = e_j(\gamma')_{N_j^*}$ for all $j \in J$. Set $\gamma_j' = b_j\gamma'$. Then

$$\tau = \sigma\gamma' = (a_jb_j)\,\gamma' = a_j\gamma_j'$$

and

$$\beta_j = e_j(\gamma')_{N_j^*} = c_j(b_j)_{N_j}(\gamma')_{N_j^*} = c_j(\gamma')_{N_j^*}.$$

It follows from Theorem 4.1 that for each $j \in J$, there is a $k_j \in K_j$ such that

$$\gamma_j'(x) = b_j(k_j)^{-1}\,\gamma_j(x)\,b_j(k_j).$$

Now

$$\bigcap_{j \in J} B_jk_j = \{m\}$$

for some $m \in M$. Actually

$$m \in \bigcap_{j \in J} B_jK_j = \bigcap_{j \in J} K_j.$$

Since B_j is the kernel of b_j, we conclude that

$$\gamma_j'(x) = b_j(m)^{-1}\,\gamma_j(x)\,b_j(m)$$

for all $j \in J$. Now define γ'' on G^* by

$$\gamma''(x) = m\gamma'(x)\,m^{-1}.$$

Then γ'' is a homomorphism of G^* into G and

$$\sigma\gamma''(x) = \sigma(m)\,\sigma\gamma'(x)\,\sigma(m)^{-1} = \sigma\gamma'(x) = \tau(x)$$

while

$$b_j\gamma''(x) = b_j(m)\,b_j\gamma'(x)\,b_j(m)^{-1} = b_j(m)\,\gamma_j'(x)\,b_j(m)^{-1} = \gamma_j(x)$$

for all $x \in G^*$. Hence, $\sigma\gamma'' = \tau$ and $b_j\gamma'' = \gamma_j$. Theorem 5.1 now implies that $\gamma'' = \gamma$ and it follows at once that $\gamma'(x) = m^{-1}\gamma(x)\,m$ for all $x \in G^*$.

(2) Applying Theorem 4.2 with G, M, K, L, N, and I replaced by G/B_j, M/B_j, K_j/B_j, L_j/B_j, N_j/B_j, and $\Delta(j)$, respectively, we find that G/B_j contains a subgroup H_j/B_j such that $G = MH_j$, $L_j = (H_j \cap N_j) K_j$, and

$$(H_j \cap M)/B_j = \prod_{i \in \Delta(j)} ((H_j \cap M)/B_j)/((H_j \cap K_i)/B_j).$$

It follows from Theorem 5.2 that if $H = \bigcap_{j \in J} H_j$, then $G = HM$, $H \cap M = \prod_{j \in J} (H \cap M)/(H \cap B_j)$, and $H_j = HB_j$ for each $j \in J$. But then

$$L_j = (HB_j \cap N_j) K_j = (H \cap N_j) B_j K_j = (H \cap N_j) K_j.$$

It follows from Lemma 2.1 that

$$H_j/B_j = \bigcap_{i \in \Delta(j)} (K_i/B_j)(H_j/B_j).$$

Hence,

$$H_j = \bigcap_{i \in \Delta(j)} K_i H_j = \bigcap_{i \in \Delta(j)} K_i B_j H = \bigcap_{i \in \Delta(j)} K_i H.$$

This implies that

$$H = \bigcap_{j \in J} H_j = \bigcap_{j \in J} \bigcap_{i \in \Delta(j)} K_i H = \bigcap_{i \in I} K_i H.$$

Using Lemma 2.1 again, we obtain

$$H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i).$$

By Theorem 5.2, $H \cap M = 1$ if, and only if,

$$(H_j/B_j) \cap (M/B_j) = 1$$

for all $j \in J$. Corollary 4.3 implies that this happens if, and only if,

$$((L_j/B_j)/(K_j/B_j)) \cap ((M/B_j)/(K_j/B_j)) = 1$$

for all $j \in J$. Hence $H \cap M = 1$ if, and only if,

$$(L_j/K_j) \cap (M/K_j) = 1$$

for all $j \in J$.

To finish the proof of (2), we need to show that H is unique up to conjugation by an element of $\bigcap_{j \in J} K_j$. Suppose then that H_1 is a subgroup of G such that $G = H_1 M$, $L_j = (H_1 \cap N) K_j$ for each $j \in J$ and

$$H_1 \cap M = \prod_{i \in I} (H_1 \cap M)/(H_1 \cap K_i).$$

Certainly $G = (H_1 B_j) M$ and

$$(H_1 B_j \cap N_j) K_j = (H_1 \cap N_j) B_j K_j = (H_1 \cap N_j) K_j = L_j.$$

By Theorem 4.2, $H_1 B_j/B_j$ must be contained in some conjugate of HB_j/B_j under an element of K_j/B_j. Thus there is an element k_j in K_j such that

$$H_1 \leqslant k_j^{-1} HB_j k_j.$$

Then, since $M = \prod_{j \in J} M/B_j$,

$$\bigcap_{j \in J} B_j k_j = \{k\}$$

for some $k \in M$. Certainly, since $B_j k_j \subseteq K_j$

$$k \in \bigcap_{j \in J} B_j k_j \leqslant \bigcap_{j \in J} K_j.$$

Since $k_j k^{-1} \in B_j \leqslant N_G(HB_j)$, we must have

$$H_1 \leqslant k_j^{-1} HB_j k_j = k^{-1} HB_j k$$

for all $j \in J$. This implies that

$$H_1 \leqslant k^{-1} \left(\bigcap_{j \in J} HB_j \right) k = k^{-1} Hk.$$

Also, since $k \in \bigcap_{j \in J} K_j \leqslant L_j \leqslant N_j$ for all $j \in J$,

$$(H_1 \cap N_j) K_j = L_j = k^{-1} L_j k = k^{-1} (H \cap N_j) K_j k = (k^{-1} Hk \cap N_j) K_j$$

for all $j \in J$. Since for each $i \in I$, K_i is a conjugate of some K_j with $j \in J$ and since $G = H_1 M$ and since every element of H_1 must normalize $k^{-1} Hk$, we obtain

$$(H_1 \cap N_i) K_i = (k^{-1} Hk \cap N_i) K_i$$

for all $i \in I$. The uniqueness portion of Theorem 2.3(1) (with $A = H_1$) now yields $H_1 = k^{-1} Hk$.

(3) We assume $H \leqslant G$, $R \leqslant G$, $G = HM = RM$, and $H \cap M = \prod_{i \in I} (H \cap M)/(H \cap K_i)$. Then $H \cap M = \prod_{j \in J} (H \cap M)/(H \cap B_j)$ and so, by Lemma 2.1, $H = \bigcap_{j \in J} B_j H$. By Theorem 5.2, R is conjugate to a subgroup of H if, and only if, R is conjugate to a subgroup of $B_j H$ for each $j \in J$. Corollary 4.3 applied to G/B_j yields that R is conjugate to a subgroup of $B_j H$ if, and only if, $(R \cap N_j)$ is conjugate in N_j to a subgroup of

$$(HB_j \cap N_j) K_j = (H \cap N_j) B_j K_j = (H \cap N_j) K_j.$$

Therefore, R is conjugate in G to a subgroup of G if, and only if, for each $j \in J$, $R \cap N_j$ is conjugate in N_j to a subgroup of $(H \cap N_j)K_j$. The proof of the rest of (3) is similar.

(4) It is straightforward that (a) implies (b) and that (b) implies (c). Using (2) we see that (c) implies (a).

(5) This follows directly from (2) and (3).

The following corollary probably is known but is included as an easy application of our theorem.

5.6. COROLLARY. *Let P be a permutation group acting on the set I. Let A be any group and let $W = A\ \mathrm{Wr}(P, I)$. If $M = A^I$, then the following are equivalent.*

(a) *All complements of M in W are conjugate.*

(b) *For each $i \in I$, the only homomorphism of P_i into A is the trivial homomorphism, i.e., every element of P_i is mapped onto the identity of A.*

Proof. $N_w(A[i])/A[i]$ is isomorphic to the direct product $A \times P_i$. It follows from the theorem then that (a) is equivalent to the following statement:

For each $i \in I$, P_i is the only complement of A in $A \times P_i$. Since it is immediate that this is equivalent to (b), the corollary follows.

Note that the corollary includes P. Neumann's theorem about standard wreath products [7, Theorem 10.1] as well as Dixon's generalization [2, Lemma 2], since P_i is always 1 in both of these results. (Another generalization of Neumann's theorem to twisted wreath products is presented as Theorem 10.7 on page 271 of [8]. This result follows directly from our Theorem 4.2.)

6. EXAMPLES

In the results assuming (*), it is necessary to assume that $\{K_i \mid i \in I\}$ is a full conjugacy class rather than just a collection of conjugate subgroups. For, if p is an odd prime, let G be the group with generators x and y and relations

$$x^{p^2} = y^p = x^{-1}yx^{p+1}y^{-1} = 1.$$

Let $M = \langle x^p, y\rangle$, $K = K_1 = \langle y\rangle$, and $K_2 = x^{-1}Kx$. Then G is the nonabelian group of order p^3 and exponent p^2, $M \lhd G$, $M = \prod_{i=1,2}(M/K_i)$, and $N_G(K) = M$. Thus $N_G(K)/K$ splits over M/K. Since M contains all elements of order p in G, G cannot split over M. The point is that K has other conjugates besides K_1 and K_2.

Suppose G, M, K and N are as in $(*)$. If R is a complement to M in N, then RK/K is a complement to M/K in N/K. It follows from Theorem 4.2 that G contains a complement H to M such that $RK = (H \cap N)K$. It is reasonable to ask whether H may be chosen so that $R \leqslant H$. In a similar vein, we might ask whether there is a one-to-one correspondence between conjugacy classes of complements of M in N and conjugacy classes of complements of M in G. Or perhaps the correspondence should be between the complements themselves rather than their conjugacy classes.

However, consider the following example. Let S be a finite group of even order. Let $M = S \times S$ and let G be the semi-direct product $G = M\langle x\rangle$ where x has order 4 and operates on M according to the rule

$$(s_1, s_2)^x = (s_2, s_1).$$

Let $K = \{(s, 1) \mid s \in S\}$. This satisfies $(*)$. Assume that S has a total of m involutions which are distributed into n conjugacy classes.

The complements of M in G are precisely the subgroups $\langle x(s_1, s_2)\rangle$ with $(s_1 s_2)^2 = 1$. Two such complements $\langle x(s_1, s_2)\rangle$ and $\langle x(s_1', s_2')\rangle$ are conjugate if, and only if, $s_1 s_2$ and $s_1' s_2'$ are conjugate in S. Hence, M has $(m+1)|S|$ distinct complements in G which belong to $(n+1)$ distinct conjugacy classes.

The complements of M in N are all the subgroups of the form $\langle x^2(s_1, s_2)\rangle$ with $s_1^2 = s_2^2 = 1$. It now follows that the number of complements of M/K in N/K is $(m+1)$, the number of complements of M in N is $(m+1)^2$, and the number of conjugacy classes in N of complements of M in N is $(n+1)^2$.

It is easy to choose S such that the numbers $(m+1)|S|$, $(n+1)$, $(m+1)$, $(m+1)^2$, and $(n+1)^2$ are all distinct. (For example, let S be the symmetric group of degree 4.) Suppose, finally, that $R = \langle y\rangle$ where $y = x^2(u, 1)$ with u some involution in S. Then R is a complement of M in N but R cannot be contained in any complement of M in G. For if y were contained in some complement of M in G, y would be the square of some element of the form $x(s_1, s_2)$. Since, however,

$$(x(s_1, s_2))^2 = x^2(s_2 s_1, s_1 s_2),$$

this is impossible.

Now suppose M is a non-abelian minimal normal subgroup of the finite group G. Then

$$M = S_1 \times S_2 \times \cdots \times S_n$$

where $\{S_1, S_2, \ldots, S_n\}$ is a conjugacy class in G and $S = S_1$ is a nonabelian simple group. If

$$K = S_2 \times \cdots \times S_n,$$

then it follows from Section 4 that M has a complement in G if, and only if, M/K has a complement in $N_G(S)/K$. Our final two examples deal with other possible necessary and sufficient conditions.

In [1], Bercov showed that G must split over M provided that the automorphism group of S splits over the subgroup of inner automorphisms. A slight modification of Bercov's argument proves the stronger result that G splits over M if $N_G(S)/C_G(S)$ splits over $SC_G(S)/C_G(S)$. Since $K = C_G(S) \cap M$, if $N_G(S)/C_G(S)$ splits over $SC_G(S)/C_G(S)$, then $N_G(S)/K$ splits over M/K. Thus, certainly G must split over M by our results. However, G may split over M even though $N_G(S)/C_G(S)$ does not split over $SC_G(S)/C_G(S)$. To see this, let $M = S = A_6$, the alternating group of degree 6, and let G be the semi-direct product $G = AM$ with A the automorphism group of A_6. Then G splits over M but $N_G(S)/C_G(S)$ is isomorphic to A. Since A does not split over the group of inner automorphisms, $N_G(S)/C_G(S)$ does not split over $SC_G(S)/C_G(S)$.

If M/K has a complement in $N_G(S)/K$, then S has a complement in $N_G(S)$. It is not enough though to simply assume that S has a complement in $N_G(S)$. This is shown by the following example.

Let $S = A_6$, let L be the automorphism group of S, and let N be the semi-direct product

$$N = LS = \{(l, s) \mid l \in L, s \in S\}.$$

Let x be an element of order 2 which operates on N according to the rule

$$(l, s)^x = (li(s), s^{-1})$$

where $i(s)$ is the inner automorphism

$$y \to s^{-1}ys.$$

Let G be the semi-direct product $N\langle x\rangle$ and let

$$M = \{(i(s_1), s_2) \mid s_1, s_2 \in S\}.$$

Then M is a minimal normal subgroup of G, M is the direct product $S \times C_N(S)$, and $C_N(S)$ is the only other conjugate of S in G. Further, $N_G(S) = N$ and L is a complement of S in N. However, there is no complement of M in N. For if N splits over M (which would have to happen if M had a complement in G), then N/S would split over M/S. This would imply that L splits over the group of all inner automorphisms. But this is not the case.

Note added in proof. We have recently learned that those portions of Corollary 4.4 and Theorem 4.5 which are concerned with the existence and conjugacy of complements in finite groups were also obtained by M. Aschbacher and L. Scott [9, Theorem 2].

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