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# Distinguishing Eleven Crossing Knots

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## 1 Introduction

Work has been done on the tabulation of knots since the last century. Perko [**11**] presents 552 distinct knots with eleven crossings and also provides a list of knot tabulations. In 1979 Richard Hartley drew our attention to Perko's work and to seven pairs of eleven crossing knots which Perko had not succeeded in distinguishing at that time. We indicate how we distinguished these pairs using group-theoretic calculations, not routinely applied by knot theorists. These knot pairs have now also been distinguished by Perko in the cited work and by Thistlethwaite (unpublished) using more usual calculations.

## 2 The problems

At the beginning of 1979, the following seven pairs of knots (in Perko's notation, with the notation of Conway [**3**], indicated in parentheses) remained to be distinguished:

| | | | |
|---|---|---|---|
| 11–84 | (3, 3, 21, 2), | 11–357 | (3, 21, 3, 2); |
| 11–173 | (8 * 30.20), | 11–255 | (.21.21.20); |
| 11–220 | (21, 3, 21, 2), | 11–225 | (3, 21, 21, 2); |
| 11–427 | (3, 3, 21, 2–), | 11–428 | (3, 21, 3, 2–); |
| 11–429 | (3, 21, 21, 2–), | 11–430 | (21, 3, 21, 2–); |
| 11–433 | (3, 3, 21, 2––), | 11–434 | (3, 21, 3, 2––); |
| 11–475 | (.–(3, 2).20), | 11–476 | (.20.–(3, 2)). |

Of these, all except the pair 11–173 and 11–255 (see Fig. 1) are algebraic, and may be distinguished also by the work of Bonahon and Siebenmann [**1**], so we focus our attention on this pair. Fox [**4**] and Hartley [**5**] describe methods for deciding which groups from certain classes of metabelian groups are homomorphic images of a given knot group. In particular, those methods yield that the holomorph $H$ of the group of order 13, $\mathrm{Z}_{13} \oslash \mathrm{Z}_{12}$ in Hartley's nota-

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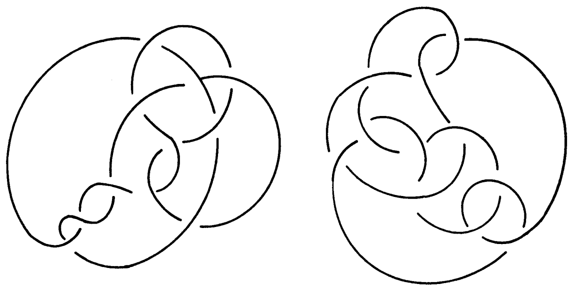

11-173 11-255

*Fig. 1*

tion, is a homomorphic image of the groups of 11–173 and 11–225. In view of the availability of various programs for computation in group theory, in particular the newly developed abelian decomposition program [**9**], Hartley suggested that we investigate subgroups $S$ of $G$ which arise as complete inverse images of index 13 subgroups of $H$, under homomorphisms of $G$ onto $H$. (For a knot-theoretic interpretation of such subgroups see, for example, [**6**].)

It is clear that, as any finitely generated group, $G$ has only finitely many such subgroups $S$, and that the family of the isomorphism types of their abelian quotients $S/S'$ is an invariant of $G$. (We speak of family, rather than of set or sequence, to indicate that the same isomorphism type may occur repeatedly and it is relevant to know the "multiplicity" showing just how often it does occur, but the particular order in which the isomorphism types happen to be listed is irrelevant.) It is also clear that these $S$ fall into conjugacy classes of 13 each, and that one may use instead just one representative of each conjugacy class.

## 3 The Approach and its Application to 11–173 and 11–255

The first task is to plan how to select a complete set of representatives of the conjugacy classes of the subgroups $S$. In fact, this is no harder—if anything, it is less laborious—than to obtain a complete, repetition-free listing of all the subgroups $S$.

Let us write $\Sigma$ for the set of these subgroups: thus $S \in \Sigma$ means that $S < G$, $|G:S| = 13$, and $G/\text{core } S \cong H$ (where core $S$ denotes the normal core of $S$ in $G$, that is, the intersection of the conjugates of $S$ in $G$). Also, let $\Phi$ stand for the set of all homomorphisms of $G$ *onto* $H$.

Without needing any special properties of $G$ or $H$, note that the composites of elements of $\Phi$ with automorphisms of $H$ all lie in $\Phi$, so the automorphism group Aut $H$ of $H$ acts on $\Phi$ by composition of maps; indeed, two elements of $\Phi$ are in the same orbit of this action if and only if their kernels coincide. From the fact that the subgroups of index 13 form a single conjugacy class in $H$, it follows that two members of $\Sigma$ are conjugate if and only if their normal cores coincide. By the definition of $\Sigma$, the normal core of a member of $\Sigma$ is the kernel of some elements of $\Phi$; conversely, if $\phi \in \Phi$ then the complete inverse image $A\phi^{-1}$ of any index 13 subgroup $A$ of $H$ is a member of $\Sigma$ whose normal core is just the kernel of $\phi$. Thus, there is an equivalence between the set of all conjugacy classes in $\Sigma$ and the set of all orbits in $\Phi$, a conjugacy class matching an orbit when the common normal core of the members of the former is the common kernel of the elements of the latter. It follows that a complete set of representatives of the relevant conjugacy classes may be envisaged as the set of the complete inverse images $A\phi^{-1}$ with $A$ fixed and $\phi$ ranging through a complete set of representatives of the orbits in $\Phi$.

In our calculations, $H$ will be taken as the subgroup generated in the symmetric group on the 13 symbols $1, 2, \ldots, 13$ by the permutations

$$a = (1,2,4,8,3,6,12,11,9,5,10,7) \text{ and}$$
$$b = (1,2,3,4,5,6,7,8,9,10,11,12,13);$$

we take $A$ to be the subgroup generated by $a$ alone. Note that $b$ generates the commutator subgroup $H'$, and the cyclic group $H/H'$ is generated by $H'a^i$ if and only if $i \equiv \pm 1, \pm 5 \bmod 12$. Each of these cosets is, of course, a single conjugacy class in $H$.

Each knot group $G$ will be written down in a 3-generator "over-presentation": that is, as $F/R$ where $F$ is free on, say, $\{x, y, z\}$, and $Rx$, $Ry$, $Rz$ are pairwise conjugate in $F/R$. A homomorphism of $G$ onto $H$ must map $Rx$, $Ry$, $Rz$ to a *conjugate triple*: this will be our name of convenience for 3-term (ordered) sequences of pairwise conjugate elements of $H$. Moreover, this conjugate triple will have to be *generating* in the sense that the set of its terms must generate $H$. Clearly, a generating conjugate triple cannot be a constant sequence (for $H$ is not cyclic), and its terms must come from a conjugacy class of $H$ whose image in $H/H'$ generates $H/H'$: as that image is a singleton, the conjugacy class in question must be an $H'a^i$ with $i = \pm 1, \pm 5$. Conversely, it is easy to see that each nonconstant 3-term sequence of elements from any one of these 4 cosets is a generating conjugate triple. One can now readily count that there are precisely $4 \cdot 13 \cdot (1 \cdot 12 + 12 \cdot 13)$, that is, $56|H|$ such triples (choose first one of 4 cosets, then one of the 13 elements of that coset as first term; repeat that as second term and choose one of the 12 others as last term, or choose one of 12 others as second term and any one of 13 as last term). The image of a generating conjugate triple under a nontrivial automorphism of $H$ is an *other* generating conjugate triple: thus the set of all such triples is permuted by Aut $H$ in orbits of size $|\text{Aut}\, H|$. Since $H$ has trivial centre and no outer automorphism, $|\text{Aut}\, H| = |H|$; hence we have precisely 56 orbits. If $\phi \in \Phi$ and $\alpha \in \text{Aut}\, H$, the image of $Rx$, $Ry$, $Rz$ under $\phi$ is mapped by $\alpha$ to the image of $Rx, Ry, Rz$ under the composite of $\phi$ and $\alpha$; thus the map from $\Phi$ to the set of all generating conjugate triples (which maps $\phi$ to the image of $Rx, Ry, Rz$ under $\phi$, and which is clearly one-to-one) is compatible with the action of Aut $H$ on these two sets: in particular, it takes orbits to orbits. We therefore plan to select representatives of the 56 orbits of generating conjugate triples, test which of these lie in the image of $\Phi$, and use those which pass the test to define the desired complete set of representatives of the orbits in $\Phi$. The test is simple: given a triple $u, v, w$, one has to check whether each defining relator of $G$ has trivial image under the homomorphism of $F$ to $H$ which takes $x$ to $u$, $y$ to $v$, $z$ to $w$. (We have had no occasion to mention these defining relators of $G$ so far: they are, of course the generators of $R$ as normal subgroup of $F$ listed in the presentation of $G$.)

It remains to choose a set of representatives of the 56 orbits of triples. We used the union of

$$\{a^i, a^i, a^ib \mid i = \pm 1, \pm 5\} \text{ and } \{a^i, a^ib, a^ib^j \mid i = \pm 1, \pm 5; j = 0, 1, \ldots, 12\}.$$

To see that this is indeed a complete set of representatives, note that it consists of the right number of triples (56), and verify that no two of these triples can lie in the same orbit. The verification is immediate from the following facts. All automorphisms of $H$ are inner. If $a^i$ and $a^{i'}$ are conjugate in $H$, they are certainly congruent modulo $H'$ so $i \equiv i' \bmod 12$, which is only possible within the given range of this parameter if $i = i'$. Finally, if $h \in H$, $(a^i)^h = a^i$, $(a^ib)^h = a^ib$, and $(a^ib^j)^h = a^ib^{j'}$, then $j = j'$ since the first two equations imply that $h$ is central in $H$.

## 4 Calculations for 11–173 and 11–255

Let $G_{173}$ and $G_{255}$ be the knot groups of 11–173 and 11–255 respectively. We proceeded to distinguish these groups in the following way.

First we obtained presentations for $G_{173}$ and $G_{255}$. In practice, we used the knot theory program described in [**8**] which provided both the presentations for $G_{173}$ and $G_{255}$ and, for confirmation, the Alexander polynomials. Next an application of the Tietze transformation program (see [**10**]) to the Wirtinger presentation produced by the knot theory program gave simplified presentations for these groups. It was important to reduce the number of generators in the presentations in order to simplify the further calculations. These presentations (with $\bar{x}, \bar{y}, \bar{z}$ denoting $x^{-1}, y^{-1}, z^{-1}$, respectively), are

$$\begin{aligned} G_{173} = \langle x, y, z \mid\, & xy\bar{x}zx\bar{y}\bar{x}y\bar{x}yx\bar{y}zy\bar{x}\bar{y}\bar{z}xy\bar{x}\bar{z}x\bar{y}\bar{x}zyx\bar{y}\bar{z}y\bar{x}\bar{y}x\bar{y}xy\bar{x}\bar{z} = \\ & xy\bar{x}zx\bar{y}\bar{x}y\bar{x}yx\bar{y}xy\bar{x}\bar{z}x\bar{y}\bar{x}zx\bar{y}\bar{x}y\bar{x}\bar{z}xy\bar{x}zx\bar{y}\bar{x}zy\bar{x}\bar{y}\bar{z}xy\bar{x}\bar{z}x\bar{y}\bar{x}zx\bar{y}xy\bar{x}\bar{z} = 1 \rangle, \end{aligned}$$

$$G_{255} = \langle x, y, z \mid xy\bar{x}yxz\bar{x}\bar{y}x\bar{y}\bar{x}yxz\bar{x}yx\bar{z}\bar{x}\bar{y}xy\bar{x}yx\bar{z}\bar{x}\bar{y}x\bar{y}\bar{x}yxz\bar{x}\bar{y}x\bar{y}\bar{x}y\bar{x}\bar{y}xy\bar{x}yx\bar{z}\bar{x}\bar{y} = xy\bar{x}yxz\bar{x}\bar{y}x\bar{y}\bar{x}yxz\bar{x}\bar{y}x\bar{z}\bar{y}xyz\bar{y}\bar{x}yz\bar{x}yx\bar{z}\bar{x}\bar{y}xy\bar{x}yx\bar{z}\bar{x}\bar{y}x\bar{y}\bar{x}yxz\bar{x}\bar{y}x\bar{z}\bar{y}\bar{x}yz\bar{x}yx\bar{z}\bar{x}\bar{y} = 1 \rangle.$$

Now a straightforward Cayley program (see [**2**]) implementing the algorithm of §3 provided the required complete sets of representatives of the conjugacy classes of desired subgroups of $G_{173}$ and $G_{255}$, via permutation representations. In this case each group had two subgroups in the representative set.

Using the Reidemeister-Schreier program (see [**7**]) we obtained presentations for these subgroups. These are presentations for index 13 subgroups of groups with 2 rather long relators, so the subgroup presentations are not particularly palatable. In each case, we initially obtained 27 generator 26 relator presentations, with lengthy relators.

Since the next step was to find the maximal abelian quotients of these subgroups, there was no need to simplify these presentations. They were used as input to the abelian decomposition program described in [**9**], where some more details of these calculations are presented. The abelian decomposition program revealed that the abelian groups concerned all have torsion free rank 2; the two corresponding to $G_{173}$ have torsion invariants 3 and 14, while for those belonging to $G_{255}$ the torsion invariants are 2 and 3, 3. This distinguishes the groups $G_{173}$ and $G_{255}$ and hence the knots 11–173 and 11–255.

## 5 The Other Six Pairs

Each of the other six pairs of knots was distinguished in a similar way. The smallest interesting metabelian quotient of the knot groups for all of these knots bar 11–475 and 11–476 is $Z_7 \rtimes Z_6$, while for this pair it is $Z_3{}^2 \rtimes Z_4$. Accordingly we investigated the corresponding characteristic classes of subgroups of index 7 (or 9) in these knot groups. The knots in each of these pairs are distinct because their groups have different families of isomorphism types of abelian quotients for the subgroups in these characteristic classes. We calculated these families using the principles exemplified above for $G_{173}$ and $G_{255}$.

For the first five pairs of knots, we obtained knot group presentations with 4 generators and 3 relators and the required index 7 subgroups came from testing 114 possibilities for each of the ten groups. In all cases, each group had sixteen subgroups in the representative set. All of the abelian groups concerned have torsion free rank 2, and we list their torsion invariants in Table 1. For the pair 11–475, 11–476, the knot groups were presented with 3 generators and 2 relators, and we tested 10 possibilities each for the requisite index 9 subgroups. In this case, each group had one subgroup in the

*Table 1*

| 11–84 | 11–357 | 11–220 | 11–225 | 11–427 | 11–428 | 11–429 | 11–430 | 11–433 | 11–434 |
|---|---|---|---|---|---|---|---|---|---|
| 7 | 14 | 14 | 7 | 7 | 7 | 14 | 7 | 14 | 7 |
| 7 | 14 | 14 | 7 | 7 | 7 | 14 | 7 | 14 | 7 |
| 7 | 14 | 14 | 7 | 21 | 7 | 14 | 7 | 28 | 7 |
| 7 | 14 | 14 | 7 | 21 | 7 | 14 | 7 | 28 | 7 |
| 1120 | 1120 | 154 | 798 | 84 | 42 | 252 | 595 | 28 | 21 |
| 1120 | 1120 | 154 | 798 | 84 | 42 | 252 | 595 | 28 | 21 |
| 2128 | 1120 | 329 | 952 | 504 | 84 | 840 | 595 | 28 | 21 |
| 2128 | 1120 | 329 | 952 | 504 | 84 | 840 | 595 | 28 | 21 |
| 2170 | 1855 | 329 | 2, 14 | 2, 14 | 84 | 2, 14 | 812 | 42 | 28 |
| 2170 | 1855 | 329 | 2, 14 | 2, 14 | 84 | 2, 14 | 812 | 42 | 28 |
| 2, 42 | 1855 | 2, 84 | 2, 168 | 2, 42 | 2, 28 | 2, 308 | 840 | 2, 14 | 28 |
| 2, 42 | 1855 | 2, 84 | 2, 168 | 2, 42 | 2, 28 | 2, 308 | 840 | 2, 14 | 28 |
| 2, 896 | 3542 | 7, 280 | 7, 280 | 7, 14 | 7, 14 | 2, 322 | 840 | 2, 14 | 336 |
| 2, 896 | 3542 | 7, 280 | 7, 280 | 7, 14 | 7, 14 | 2, 322 | 840 | 2, 14 | 336 |
| 2, 952 | 2, 56 | 7, 280 | 14, 28 | 14, 14 | 7, 14 | 7, 252 | 14, 14 | 2, 28 | 2, 14 |
| 2, 952 | 2, 56 | 7, 280 | 14, 28 | 14, 14 | 7, 14 | 7, 252 | 14, 14 | 2, 28 | 2, 14 |

representative set. The abelian groups concerned both have torsion free rank 3. The abelian group corresponding to $G_{475}$ has torsion invariants 2, 216 while that corresponding to $G_{476}$ has torsion invariants 2, 2, 72.

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