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In his recent thesis [9], C.L. Kanes constructed an interesting new family of Fitting formations and proved that not all members of this family are saturated. His attempt to find a necessary and sufficient condition of saturation in this context was only partly successful. This note presents such a condition.

Standard terminology will be used: see for instance Gaschütz [5]. Let F be an algebraically closed field of prime characteristic q ; let π be a set of primes (with π' the complementary set), X a Fitting formation in the product class $S_\pi S_{\pi'}$, and $H_q^\pi(X)$ the class of all those groups G in $S_{q'} S_q X$ which satisfy the following condition: if U is an irreducible submodule of $F \otimes V$ for some q-chieffactor V of G , then U regarded as $O_{q'q\pi}(G)$-module is homogeneous (that is, a direct sum of pairwise isomorphic irreducible modules). By Theorem 5.2.2 of Kanes [9], each such class $H_q^\pi(X)$ is a Fitting formation. His Theorem 5.3.4 states in effect that when $q \notin \pi$, this formation is saturated if and only if X lies in the class $S_\pi \vee S_{\pi'}$ of all direct products of π-groups and π'-groups. The general version is the following.

THEOREM. *The formation* $H_q^{\pi}(X)$ *is saturated if and only if* $X \subseteq S_q(S_\pi \vee S_{\pi'})$ *, and in that case* $H_q^{\pi}(X) = S_{q'}S_q X$.

Thus when $H_q^{\pi}(X)$ is really a "new" Fitting formation (rather than an "old" one easier described as a product), it is never saturated. Note that in the case $q \notin \pi$, the assumption $X \subseteq S_\pi S_{\pi'}$ ensures that $X \subseteq S_q(S_\pi \vee S_{\pi'})$ is equivalent to $X \subseteq S_\pi \vee S_{\pi'}$, so this result is in agreement with the theorem of Kanes paraphrased above. The relevant special case of the proof given here is essentially his.

It is obvious that if $X \subseteq S_q(S_\pi \vee S_{\pi'})$ then $H_q^{\pi}(X)$ is the whole class $S_{q'}S_q X$ and of course this formation is always saturated. Suppose that $X \not\subseteq S_q(S_\pi \vee S_{\pi'})$ yet $H_q^{\pi}(X)$ is saturated: the Theorem will be proved by showing that this leads to a contradiction.

The argument is carried out in three steps. The central step is the construction of two groups, G_1 and G_2 say, with the following properties. First, $G_1 \in X \cap H_q^{\pi}(X)$, $O_{\pi'}(G_1) = 1$, and G_2 lies in the formation generated by G_1 . Second, there exists a faithful irreducible $FO_\pi(G_1)$-module W_1 which is G_1-invariant (that is, isomorphic to each of its conjugates by elements of G_1). Third, there is an irreducible $FO_\pi(G_2)$-module W_2 which is not G_2-invariant. Once we are in possession of these groups and modules, the final step goes easily, as follows. By Theorem 7.1 of Dade [4] (see also Isaacs [8]), W_1 is the restriction of some FG_1-module U_1 (which is faithful since

$O_{\pi'}(G_1) = 1$, and of course irreducible); in turn, U_1 is a submodule of $F \otimes V_1$ for some faithful irreducible F_qG_1-module V_1 . One readily sees that the semidirect product V_1G_1 lies in $\mathcal{H}_q^{\pi}(X)$. On the other hand, W_2 is a submodule of the restriction of some irreducible FG_2-module U_2 , and as W_2 is not G_2-invariant that restriction is certainly not homogeneous. Take an irreducible F_qG_2-module V_2 such that $F \otimes V_2$ contains U_2 ; then $V_2G_2 \notin \mathcal{H}_q^{\pi}(X)$. Since our formation is saturated, it has a "full integrated local definition" (see Carter and Hawkes [3]); the formation $\mathcal{F}$ corresponding to the prime q in that is such that $\mathcal{F} = \mathcal{S}_q\mathcal{F} \subseteq \mathcal{H}_q^{\pi}(X) \subseteq \mathcal{S}_{q'}\mathcal{F}$. As V_1 is faithful, $O_{q'}(V_1G_1) = 1$, so $V_1G_1 \in \mathcal{H}_q^{\pi}(X) \subseteq \mathcal{S}_{q'}\mathcal{F}$ implies that $V_1G_1 \in \mathcal{F}$. Because G_2 lies in the formation generated by G_1 , we get that $G_2 \in \mathcal{F}$; thus $V_2G_2 \in \mathcal{S}_q\mathcal{F} \subseteq \mathcal{H}_q^{\pi}(X)$, and we have the desired contradiction.

The first step is to consider a group H of minimal order among all groups contained in X but not in $\mathcal{S}_q(\mathcal{S}_{\pi} \vee \mathcal{S}_{\pi'})$. Such an H must clearly have a unique maximal normal subgroup, say M , and a unique minimal normal subgroup, say P . Let p^k denote the order of P , with p prime. Since H/P lies in $\mathcal{S}_q(\mathcal{S}_{\pi} \vee \mathcal{S}_{\pi'})$ but H does not, $p \neq q$; since $H \in X \subseteq \mathcal{S}_{\pi}\mathcal{S}_{\pi'}$ but $H \notin \mathcal{S}_{\pi'}$, we must have $P \leq O_{\pi}(H)$, so $p \in \pi$. Thus in turn we obtain that $O_q(H) = 1$, $O_q(M) = 1$, $M = O_{\pi}(M) \times O_{\pi'}(M)$, but also $O_{\pi'}(H) = 1$, $O_{\pi'}(M) = 1$; so M is a π-group. As $H \notin \mathcal{S}_{\pi}$, the prime index, say r , of M in

H must lie in π'. Thus H has no nontrivial π-quotient while $H/P \in S_q(S_\pi \vee S_{\pi'})$, so in fact $H/P \in S_q S_{\pi'}$. It follows that $M/P \in S_\pi \cap S_q S_{\pi'}$: in other words, M/P is a q-group which can only be nontrivial if $q \in \pi$.

If $M/P = 1$, we may now proceed with the central step as Kanes did in the case $q \notin \pi$. Set $G_2 = H$ and W_2 any nontrivial irreducible FP-module. For G_1 take a semidirect product of an extraspecial group of order p^{2k+1} and a group of order r, the latter acting on the Frattini quotient of the first as a Sylow r-subgroup of H acts on the direct sum of P with its contragredient, and so that G_1 have centre of order p. The Frattini quotient of G_1 is then a subdirect square of H and so lies in the metanilpotent Fitting formation $X \cap H_q^{\pi}(X) \cap S_p S_r$: by Hawkes [6] (see also Bryce and Cossey [2]), all such formations are saturated, so $G_1 \in X \cap H_q^{\pi}(X)$. Faithful irreducible modules of extraspecial groups are invariant under group automorphisms which act trivially on the centre (see Huppert [7], V.16.14): hence any such $FO_\pi(G_1)$-module will serve as W_1.

It is somewhat harder to deal with the case $M/P \neq 1$. As noted above, in this case $q \in \pi$; so $O_\pi(H) = M$ and $r \neq q$. Let Q be a Sylow q-subgroup of M and R a Sylow r-subgroup of H: by the Frattini argument, R can be chosen to normalize Q. As PQ is the only maximal normal subgroup, the mutual commutator subgroup $[Q, R]$ must then be Q. Note also that the centralizer $C_P(Q)$ must be trivial, else P would not be the only minimal

normal subgroup.

If $[P, R] < P$, set $G_1 = H$. Let P_0 be a maximal subgroup of P containing $[P, R]$; let Q_0 be the largest subgroup of Q which normalizes P_0 and acts trivially on P/P_0. By our choice of P_0 we know that P_0, and hence also Q_0, is normalized by R; also, P_0Q_0 is normal in PQ_0. Let W_1 be the FM-module induced from any 1-dimensional FPQ_0-module W with kernel P_0Q_0. The maximal choice of Q_0 ensures that PQ_0 is the "inertia subgroup" in M of the restriction of W to P; hence W_1 is irreducible. As R acts trivially on PQ_0/P_0Q_0 we know that W is R-invariant: hence so is W_1. It follows that the kernel of W_1 is normal in G; as it does not contain P, it must be trivial.

If $[P, R] = P$, we construct G_1 as a group which is like H in every relevant respect except this. (Of course G_1 will not necessarily retain the minimal property of H, either. The construction will make no use of $[P, R] = P$ and could be performed in any case; the only reason for handling $[P, R] < P$ above separately was to explain, before getting submerged in other complications, just how the property of G_1 corresponding to $[P, R] < P$ will be exploited.) Let $1 \neq h \in R$ so $R = \langle h \rangle$, and set $h_1 = (h^{-1}, h) \in QR \times QR$, $Q_1 = Q \times Q \leq QR \times QR$, $R_1 = \langle h_1 \rangle \leq QR \times QR$: then Q_1R_1 is normal in the direct product $QR \times QR$, and $[Q_1, R_1] = Q_1$ is easy to verify. Write $P \# P$ for $P \otimes P$ viewed as $QR \times QR$-module, and note the following facts. As $QR \times 1$ or $1 \times QR$

module, $P \# P$ is a direct sum of "isomorphic copies" of P. It follows from $C_P(Q) = 1$ and $Q \times 1 \leq Q_1$ that $C_{P\#P}(Q_1) = 1$. On the other hand, $C_{P\#P}(R_1) > 1$, for the Kronecker product of a matrix and its inverse must have at least one eigenvalue 1. Now consider the semidirect product $(P\#P)(QR{\times}QR)$. This is a product of two normal subgroups, $(P\#P)(QR{\times}1)$ and $(P\#P)(1{\times}QR)$, each of which is a subdirect power of H; therefore it lies in the Fitting formation $X \cap H_q^{\pi}(X)$. Consequently, so does its normal subgroup $(P\#P)Q_1R_1$. By Maschke's Theorem, $P \# P$ is completely reducible as Q_1R_1-module: so the observation $C_{P\#P}(R_1) > 1$ above yields that $P \# P$ has an irreducible Q_1R_1-submodule P_1 such that $C_{P_1}(R_1) > 1$. Let S be a Q_1R_1-submodule complementing P_1 in $P \# P$, and T a normal subgroup of $(P\#P)Q_1R_1$ maximal with respect to containing S but avoiding P_1. We set $G_1 = (P\#P)Q_1R_1/T$. Clearly, P_1T/T is operator-isomorphic to P_1 and is both the unique Sylow p-subgroup and the unique minimal normal subgroup of G_1. As $C_{P\#P}(Q_1) = 1$, we have $[P_1, Q_1] = P_1$, so $Q_1 \nleq T$; thus by $[Q_1, R_1] = Q_1$ we also have $R_1 \nleq T$. On the other hand, $C_{P_1}(R_1) > 1$ ensures that $[P_1T/T, R_1T/T] < P_1T/T$. These facts guarantee that the present G_1 has all the relevant properties of the G_1 of the previous paragraph: one may choose W_1 as before.

To simplify the description of G_2 and W_2, we *change notation* so from now on P, Q, R will stand for appropriate Sylow subgroups of G_1. If P has a maximal subgroup P_2 not

containing any conjugate of $[P, R]$, we can take $G_2 = G_1$. Indeed, in that case let W be a 1-dimensional FP-module with kernel P_2 , and W_2 any irreducible FPQ-module whose restriction to P contains W_2 (that is, any irreducible quotient of the PQ-module induced from W) . By its choice, W is not invariant under any conjugate of R : so the number of isomorphism types of G_2-conjugates of W is divisible by r . If W_2 were G_2-invariant, the set of isomorphism types of irreducible submodules of the restriction of W_2 to P would be G_2-invariant; in any case, by Clifford's Theorem, that set is a single Q-orbit: thus the cardinality of this set would have to be a power of q divisible by r . As this is impossible, W_2 is not G_2-invariant, and we are done.

If P does not have such a maximal subgroup P_2 , we choose G_2 differently: not as G_1 , the semidirect product of QR with the QR-module P , but as the semidirect product of QR with the k-fold direct power P^k of this QR-module. Then G_2 is a subdirect power of G_1 , so it lies in the formation generated by G_1 . If we can find a maximal subgroup P_2 in P^k not containing any conjugate of $[P^k, R]$, we can choose W_2 as before. Recall that k was chosen so that P itself is the k-fold direct power of a group C of order p ; if $\varphi_1, \ldots, \varphi_k$ are the corresponding coordinate-projections of P onto C , then the intersection of the kernels of these homomorphisms is trivial. Define a homomorphism φ from P^k to C by $(x_1, \ldots, x_k)\varphi = \prod x_i\varphi_i$; we claim the kernel of φ can serve as

P_2. As $[P, R] \neq 1$ (else the normal closure of R, which includes Q because $[Q, R] = Q$, would also act trivially on P, contrary to $C_P(Q) = 1$), to each element g of QR there is an index $i(g)$ such that $[P, R]^g \varphi_{i(g)} \neq 1$. Accordingly, there is an element y_g in P such that $\left[y_g, h^g\right]\varphi_{i(g)} \neq 1$ (here, as before $\langle h \rangle = R$). Let z_g be the element of P^k whose components are all trivial except the $i(g)$-component which is y_g: then all components of $\left[z_g, h^g\right]$ are trivial except the $i(g)$-component which is $\left[y_g, h^g\right]$, so $\left[z_g, h^g\right]\varphi = \left[y_g, h^g\right]\varphi_{i(g)} \neq 1$. This shows that $\left[P^k, R^g\right]$, that is, $\left[P^k, R\right]^g$, is not contained in P_2, and the proof of the Theorem is complete.

In conclusion, note that when $q = 2$, $\pi = \{2, 13\}$, and $X = S_{13}S_2S_3 \cap Y_{13}^{\;3}$ where $Y_{13}^{\;3}$ is as defined in the paper [1] of Berger and Cossey, the smallest group H in X but outside $S_q(S_\pi \vee S_{\pi'})$ has $H/P \cong SL(2, 3)$ and $[P, R] = P$: so the hard version of the case $M/P \neq 1$ genuinely does arise and must be coped with. On the other hand, when $q = 31$, $\pi = \{5, 31\}$, and $X = S_5S_{31}S_3 \cap Y_5^{\;3}$, we find that H/P is the nonabelian group of order 93 and each of the 31 maximal subgroups of the group P of order 5^3 is a conjugate of $[P, R]$: so the easy choice of G_2 is not always available, either.

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