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Periodicity of Weyl Modules for $SL(2, q)$

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Weyl modules play a central role in the representation theory of the general and special linear groups. There has been great interest recently in the modular case of these groups over fields of prime characteristic p. In particular, Glover [1] has proved some remarkable and comprehensive results about these modules for $SL(2, p)$. We shall extend one of these theorems to the case of $SL(2, q)$ where $q = p^e$.

First, let us fix some notation. We set $S = SL(2, k)$, k a field of q elements and $R = k[x, y]$ the polynomial algebra in two variables considered as a module for kS, the group algebra, in the usual way. Let V_n be the kS-module of dimension n consisting of the homogenous polynomials of degree $n - 1$. Hence, V_1 is the trivial kS-module and the V_n are the duals of the Weyl modules for S. Let $V_n = W_n + P_n$ be a direct decomposition where P_n is a projective kS-module and W_n has no non-zero projective direct summand. Our main result is as follows:

THEOREM. *The sequence of isomorphism classes of the modules $W_1, W_2,\ldots$ is periodic of period $q(q-1)$.*

This is a consequence of two preliminary results:

LEMMA 1. *V_{mq} is projective for all $m \geqslant 1$.*

LEMMA 2. *$V_{q(q-1)+1} \simeq V_1 \oplus P$, where P is a projective kS-module.*

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To see that these imply the theorem we shall use the fundamental exact sequence [1, 2]

$$0 \to V_{n-1} \otimes V_{m-1} \to V_n \otimes V_m \to V_{m+n-1} \to 0,$$

for all m, $n > 1$. Setting $m = q(q-1)+1$ we have

$$0 \to V_{n-1} \otimes V_{q(q-1)} \to V_n \otimes V_{q(q-1)+1} \to V_{n+q(q-1)} \to 0.$$

However, $V_{q(q-1)}$ is projective, by Lemma 1, so that so is $V_{n-1} \otimes V_{q(q-1)}$. But projective kS-modules are injective, since S is a finite group, so the last exact sequence splits and we obtain the isomorphism

$$V_n \otimes V_{q(q-1)+1} \simeq V_{n-1} \otimes V_{q(q-1)} \oplus V_{n+q(q-1)}.$$

But, by Lemma 2, we have

$$V_n \otimes V_{q(q-1)+1} \simeq V_n \oplus V_n \otimes P$$

and $V_n \otimes P$ is also projective. Hence, the two previous isomorphisms prove the theorem.

It remains to demonstrate the two lemmas and we begin by proving the first one by induction on m. We shall show that it holds for $m = 1$ by seeing that V_q is the Steinberg module, the tensor product of the e Galois conjugates of V_p. Indeed, V_p has a basis consisting of x^{p-1}, $x^{p-2}y, \ldots, xy^{p-1}$ and y^p. For each non-negative integer i there is a ring homomorphism, which is not a kS-module homomorphism, of R to itself which sends each element to its p^ith power. This gives a vector space isomorphism of V_p onto the kS-module $V_p^{(i)}$ of dimension p which has a basis consisting of $x^{p^{i+1}-p^i}$, $x^{p^{i+1}-2p^i}y^{p^i}, \ldots, y^{p^{i+1}-p^i}$. Hence, it suffices to show the isomorphism of kS-modules

$$V_p^{(0)} \otimes V_p^{(1)} \otimes \cdots \otimes V_p^{(e-1)} \simeq V_q.$$

However, multiplication together of the elements from the $V_p^{(i)}$, $0 \leqslant i < e$, gives a kS-module homomorphism of the tensor product to V_q; since the tensor product and V_q both have dimension q, it suffices to show that the map is a surjection. Let $x^i y^j \in V_q$, $i+j = q-1$; we need only see that this element is in the image. Let $i = a_0 + a_1 p + \cdots + a_{e-1} p^{e-1}$ be the p-adic expansion of i so $0 \leqslant a_i < p$ for all i. Hence, if we set $b_i = p-1-a_i$, then

$$(p-1) + (p-1)p + \cdots + (p-1)p^{e-1} = q-1$$

and $i+j = q-1$ yield that $j = b_0 + b_1 p + \cdots + b_{e-1} p^{e-1}$ is the p-adic expansion of j and so the factorization

$$(x^{a_0} y^{b_0})(x^{a_1 p} y^{b_1 p}) \cdots (x^{a_{e-1} p^{e-1}} y^{b_{e-1} p^{e-1}}) = x^i y^j$$

establishes the lemma for the case $m = 1$.

Suppose the lemma holds for m. Consider the exact sequence

$$0 \to V_{q-1} \otimes V_{mq} \to V_q \otimes V_{mq+1} \to V_{(m+1)q} \to 0.$$

However, the first two terms are projective, by induction, so the last is also.

We turn now to the second lemma. In view of Lemma 1, it is enough to demonstrate the existence of an exact sequence

$$0 \to V_{q^2-2q} \to V_{q(q-1)+1} \to V_1 \oplus V_q \to 0.$$

By an easy direct calculation, the element $x^q y - xy^q$ is fixed by S. Hence, multiplication by $x^q y - xy^q$ defines a kS homomorphism of V_{q^2-2q} to $V_{q(q-1)+1}$. It is a monomorphism since R is an integral domain. This gives the first part of the desired exact sequence.

Let F be the k-algebra of functions from k^2 to k. Identifying k^2 and V_2 we see that F has a natural kS-module structure. Moreover, the map of R to F, sending a polynomial to a polynomial function, is a kS homomorphism. Moreover, it is an epimorphism since k is finite. Let F_n, $n > 0$, be the image of V_n (so F_n is the space of homogenous polynomial fumnctions of degree $n-1$).

We claim that it suffice to prove that $F_{q(q-1)+1}$ and $V_1 \oplus V_q$ are isomorphic kS-modules. Indeed, in that case we can form the sequence

$$0 \to V_{q^2-2q} \to V_{q(q-1)+1} \to F_{q(q-1)+1} \to 0$$

by using the maps just decribed. The composite will be zero since $x^q y - xy^q$ goes to the zero function, the first map is a monomorphism, the second is an epimorphism and the dimension of $V_{q(q-1)+1}$ is the sum of the dimensions of V_{q^2-2q} and $F_{q(q-1)+1}$. Hence, we have the sequence we want.

Let X and Y be the images of x and y in F. Since F has dimension q^2 over k and since $X^q = X$, $Y^q = Y$, it follows that the functions $X^i Y^j$, $0 \leqslant i,j < q$ form a basis of F. In particular, $V_1 \simeq F_1$, $V_q \simeq F_q$ and $F_1 + F_q$ is a direct sum. Hence, it suffices to prove that $F_{q(q-1)+1} \cong F_1 + F_q$. The functions $X^i Y^j$, $0 \leqslant i,j$ and $i + j = q(q-1)$, span $F_{q(q-1)+1}$. First, suppose that neither i nor j is divisible by $q-1$ so they are congruent to r and s, respectively, modulo $q-1$ where $0 < r, s < q-1$. Hence, $q-1$ divides $r+s$, as it divides $i+j$, $X^i Y^j = X^r Y^s \in F_q$. Suppose that $q-1$ divides i so it also divides j; set $i = t(q-1)$, $j = u(q-1)$. It is easy to see that $X^i Y^j$ equals X^{q-1}, Y^{q-1} or $x^{q-1} Y^{q-1}$ as $u = 0$, $t = 0$ or neither t or u is zero. This proves that F_q has codimension 1 in $F_{q(q-1)+1}$; in view of Lemma 1, our claim now follows.

References

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2. G. E. Wall, unpublished.