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SOME INDECOMPOSABLES FOR SL_2

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Let p be a prime, $\mathbb{F}$ the algebraic closure of the field of order p, and G the group $SL(2, \mathbb{F})$ of all 2-by-2 matrices over $\mathbb{F}$ with determinant 1, acting naturally (on the right) on a 2-dimensional $\mathbb{F}$-space V with basis $\{x, y\}$. This paper* is concerned with the smallest class M of (finite dimensional, right) G-modules (over $\mathbb{F}$) which contains V and is closed under taking isomorphic copies, (finite) direct sums and tensor products, composition factors, and direct summands. In other words, we are interested in the composition factors of the tensor powers of V, and in the indecomposable direct summands of the tensor products of these irreducibles (M consists of the direct sums of these indecomposables). The information obtained is then used to describe the indecomposable direct summands of the tensor products of all the irreducible modules (over $\mathbb{F}$) for the finite groups $SL(2, p^n)$.

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1.

Since $f \mapsto f^p$ is an automorphism of $\mathbb{F}$, the map

$$g = (g_{ij}) \mapsto g^{(p)} = \left(g_{ij}^p\right)$$

is an automorphism of G . If W is any G-module, let $W^{(p)}$ be $\{w^{(p)} \mid w \in W\}$, another copy of the vectorspace W written so that $w \mapsto w^{(p)}$ is a vectorspace isomorphism $W \to W^{(p)}$, and define a G-action on $W^{(p)}$ by setting

$$w^{(p)}g = (wg^{(p)})^{(p)} .$$

The resulting G-module $W^{(p)}$ is said to be a *twisted version* of W . Repeated twisting leads to the G-modules $W^{(p^k)}$, one for each positive integer k ; add to this definition the convention that $W^{(p^0)} = W$. It will follow from our discussion that all twisted versions of modules in $\mathcal{M}$ are in $\mathcal{M}$.

There are $2p - 1$ distinguished modules, labelled $U_0, U_1, \ldots, U_{2p-2}$, which serve as the building blocks of all modules in $\mathcal{M}$. The following information can serve as their definition here:

$U_0 \otimes V = U_1 = V$ (so U_0 is the trivial module),

$U_a \otimes V = U_{a-1} \oplus U_{a+1}$ if $1 \leq a \leq p - 2$,

$U_{p-1} \otimes V = U_p$,

$U_p \otimes V = U_{p-1} \oplus U_{p-1} \oplus U_{p+1}$ if $p > 2$,

$$U_a \otimes V = U_{a-1} \oplus U_{a+1} \qquad \text{if } \ p + 1 \le a \le 2p - 3 \ .$$

Also, $U_{2p-2} \otimes V = U_{2p-3} \oplus (U_{p-1} \otimes U_1^{(p)})$ if p > 2, while if p = 2 then U_2 \otimes V is U_1 \oplus U_1 \oplus (U_1 \otimes (U_1)^{(2)}).

If $0 \le a \le p - 1$, then $\dim U_a = a + 1$ and U_a is irreducible. If $p \le a \le 2p - 2$, then $\dim U_a = 2p$ and U_a has a unique composition series, with composition factors isomorphic to U_{2p-2-a}, $U_{a-p} \otimes U_1^{(p)}$, U_{2p-2-a} (in that order). We shall also need the twisted versions of these modules: to ease notation, write $U(a, k)$ for $U_a^{(p^k)}$.

Consider all sequences $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_i, \ldots)$ of integers such that $0 \le \alpha_i \le 2p - 2$ for all i and $\alpha_i = 0$ for almost all i. For each α, set $U_\alpha = \otimes U(\alpha_i, i)$: note this is really a finite product, for the factors $U(0, i)$ are 1-dimensional trivials and therefore irrelevant. Call U_α *thin* if $\alpha_i \le p - 1$ for all i, and *fat* if there is an m such that $\alpha_i \ge p - 1$ for $i < m$ and $\alpha_i = 0$ for $i \ge m$. It is clear from the description of the composition structure of the U_a above that each U_α contains an isomorphic copy of a particular thin one, namely of $U_{\alpha-}$ where the sequence $\alpha- = (\alpha_0^-, \alpha_1^-, \ldots)$ is defined by

$$\alpha_i^- = \begin{cases} \alpha_i & \text{if } \ \alpha_i \le p - 1 \ , \\ 2p - 2 - \alpha_i & \text{if } \ \alpha_i \ge p - 1 \ . \end{cases}$$

Similarly, one sees that U_α maps homomorphically onto $U_{\alpha-}$, and that each U_α is contained in (infinitely) many fat ones.

The impetus for the final (and still somewhat tentative) phase of the work here reported came from a recent preprint* by Andersen, Jørgensen, and Landrock [3]. They considered the fat U_α and determined the Loewy structure of these modules. What we need here is that a fat U_α has no other minimal submodule than the copy of $U_{\alpha-}$ we already observed. As each U_α is contained in some fat one, this must then hold for all U_α; in particular, *each* U_α *is indecomposable*. (The irreducibility of the thin U_α is of course a very special case of Steinberg's celebrated "twisted tensor product" theorem, but we do not need to appeal to that.) From the defining description above, one can also see that each composition factor of each U_α is a thin one, and can actually calculate the Jordan-Hölder multiplicities: how many factors of a composition series of U_α are isomorphic to a particular thin module. (Indeed, the methods of [3] should yield the full Loewy structure of each U_α.) Moreover, the indecomposable direct summands of $U_\alpha \otimes U_\beta$ are isomorphic to various U_γ, and one can calculate the Krull-Schmidt multiplicities: how many summands of an unrefinable direct decomposition of $U_\alpha \otimes U_\beta$ are isomorphic to a particular U_γ. Clearly, the indecomposables in M

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* I am grateful to Dr R.W. Richardson for drawing it to my attention.

are precisely the U_α . (An interesting intermediate result of [3] asserts that the fat U_α with a fixed m are injective in the category of those G-modules whose composition factors are all thin U_β with $\sum \beta_i p^i < 2p^m - 1$.) Finally, note that the U_a , and hence also the U_α , are all self-contragradient.

2.

For the application which motivated my interest in the subject, the U_α which matter are the *almost fat* ones: these correspond to the sequences α such that for some m $(= m(\alpha))$,

$$\alpha_i \geq p - 1 \qquad \text{for } i < m ,$$

$$\alpha_m < p - 1 , \qquad \text{and}$$

$$\alpha_i = 0 \qquad \text{for } i > m .$$

The almost fat modules can be conveniently indexed by the nonnegative integers a obtained as $a = \bar{\alpha} = \sum \alpha_i p^i$. Indeed, if a is any nonnegative integer, choose m $(= m(a))$ so that $p^m \leq a + 1 < p^{m+1}$ and write $a' = a + 1 - p^m = \sum a'_i p^i$ with $0 \leq a'_i \leq p - 1$ for all i ; then

$$\alpha = (p-1+a'_0, \ldots, p-1+a'_{m-1}, a'_m, 0, 0, \ldots)$$

is the unique almost fat sequence with $\bar{\alpha} = a$. (For $a \leq 2p - 2$ this is consistent with our notation so far. The analogue of the intermediate result of [3] is that the U_a with $m(a) = m$ fixed are

injective and projective in the category of those G-modules whose composition factors are all thin U_β with $\beta_i = 0$ for $i > m$.)

The relevance of the (almost fat) U_a is that they are precisely the indecomposable direct summands of the tensor powers of V itself: indeed,

$$\text{(K)} \qquad V^{\otimes a} \cong \oplus\, U_b^{\oplus K(a,b)} \quad \text{with} \quad K(a,\, b) = \begin{cases} 1 & \text{if } b = a \,, \\ 0 & \text{if } b > a \,. \end{cases}$$

Write a to base p , as $a = \sum a_i p^i$ with $0 \le a_i \le p - 1$ for all i , and set $a* = (a_0, a_1, \ldots)$; clearly, each thin U_α is a U_{a*} (with $a = \sum \alpha_i p^i$) . We also have that

each U_b has the same Jordan-Hölder multiplicities as a suitable

$$\text{(J)} \qquad \oplus\, U_{c*}^{\oplus J(b,c)} \qquad \text{with} \quad J(b,\, c) = \begin{cases} 1 & \text{if } c = b \,, \\ 0 & \text{if } c > b \,. \end{cases}$$

The results illustrate that while the calculation of multiplicities in M does not seem capable of being expressed in closed formulas, its algorithmic nature is such that some overall conclusions can still be drawn. The fact that the infinite matrix J is unitriangular means not only that it has an inverse (with integer entries), but also that its inverse is readily accessible to algorithmic calculation. Let me put this yet another way. Write L for the class of direct sums of direct summands of tensor powers of V . *If two modules in* L *have the same Jordan-Hölder multiplicities, they must be isomorphic* (this is clearly false in the larger class M) , *and their* (common) *Krull-Schmidt multiplicities can be actually*

calculated, with the help of J^{-1} , *from the Jordan-Hölder multiplicities.* This was conjectured by Schooneveldt in his recent thesis [8], and proved under the restriction $a < p^2$, but he missed the twisted tensor factorization of the relevant indecomposables, which is the key to the proof in general. I shall say more about his work in the last section.

Before moving on, let me indicate the proofs of (K) and (J) . For the first, it is clearly sufficient to show that

$$U_a \otimes V = \oplus U_b^{\oplus k(a,b)} \quad \text{with} \quad k(a,\, b) = \begin{cases} 1 & \text{if } b = a+1\,, \\ 0 & \text{if } b > a+1\,. \end{cases}$$

This is done by induction on $m(a)$, the case $m(a) = 0$ being part of the definition of the U_a . So suppose $m(a) > 0$, and define the integer d by

$$(d+1-p^{m-1})p = a' - a_0' \;;$$

equivalently,

$$pd + (p-1+a_0') = a \;,$$

$$p(d+1) = a + \dot{1} - a_0' \;.$$

Clearly, $m(d) = m(a) - 1$, and $U_a = U(p-1+a_0',\, 0) \otimes U_d^{(p)}$. If $a_0' < p - 1$, we have

$$U_a \otimes V = U(p-1+a_0',\, 0) \otimes V \otimes U_d^{(p)}$$

$$= [U(p-1+a_0'-1,\, 0)^{\oplus e} \oplus U(p-1+a_0'+1,\, 0)] \otimes U_d^{(p)}$$

$$= U_{a-1}^{\oplus e} \oplus U_{a+1}$$

where

$$e = \begin{cases} 0 & \text{if } a_0' = 0 , \\ 2 & \text{if } a_0' = 1 \quad (\text{so } p > 2) , \\ 1 & \text{if } 2 \le a_0' \le p - 2 , \end{cases}$$

and we are done. The inductive hypothesis (applied to U_d) is needed only when $a_0' = p - 1$; in that case,

$$\begin{aligned} U_a \otimes V &= U(2p-2, 0) \otimes V \otimes U_d^{(p)} \\ &= \{U(2p-3, 0) \oplus [U(p-1, 0) \otimes V^{(p)}]\} \otimes U_d^{(p)} \\ &= U_{a-1} \oplus [U(p-1, 0) \otimes (U_d \otimes V)^{(p)}] \\ &= U_{a-1} \oplus \bigoplus_c [U(p-1, 0) \otimes U_c^{(p)}]^{\oplus k(d,c)} \\ &= U_{a-1} \oplus \bigoplus_c U_{p-1+pc}^{\oplus k(d,c)} . \end{aligned}$$

From the last equivalent of the definition of d , read with $a_0' = p - 1$, we see that $p - 1 + pc \ge a + 1$ if and only if $c \ge d + 1$, so the inductive hypothesis does indeed give what we need.

The proof of (J) follows the same pattern: one shows by induction on c that the Jordan-Hölder multiplicity $j(c, d)$ of U_{d*} in $U_{c*} \otimes V$ is 1 if $d = c + 1$ and 0 if $d > c + 1$. Details are left to the reader.

3.

Next, set $H = SL(2, p^n) < G$. We shall be concerned with the $\mathbb{F}H$-modules obtained from the restriction $V{\downarrow_H}$ in the same way as the U_α were obtained from V . Twisting is still available, but it is now an operation with period n . It is therefore convenient to consider finite sequences $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$: continuing such a sequence with all further terms 0 provides the connection with the foregoing. For $0 \le a \le 2p - 2$ and $0 \le i \le n - 1$, set $V(a, i) = U(a, i){\downarrow_H}$, and $V_\alpha = \otimes V(\alpha_i, i)$ except that $V_{(2p-2,\ldots,2p-2)}$ is defined instead by

$$\otimes V(2p-2, i) = V_{(2p-2,\ldots,2p-2)} \oplus (\otimes V(p-1, i)) .$$

Fat shall now mean $\alpha_i \ge p - 1$ for all i . The thin V_α form a complete set of representatives of the isomorphism types of the p^n irreducible $\mathbb{F}H$-modules. Similarly, the fat V_α are the principal indecomposables. Each V_α has a unique minimal submodule, isomorphic to $V_{\alpha-}$; in particular, all the V_α are indecomposables. Their Jordan-Hölder multiplicities can be readily calculated from the information given above (those of the fat V_α form the Cartan matrix). Indeed, the full Loewy structure of the fat V_α is determined in [3], and the same should be doable for all the V_α . The indecomposable direct summands of the $V_\alpha \otimes V_\beta$ are isomorphic to various V_γ , and the Krull-Schmidt multiplicities can also be conveniently calculated. Thus the V_α are precisely the $(2p-1)^n$ "irreducibly generated" indecomposables: the indecomposable direct summands of the tensor

products of the irreducibles.

For $p = 2$, these results go back to Alperin [1], [2]. In that case $2p - 2 = 2$, so the building blocks are just the trivial $V(0, 0)$, the natural $V(1, 0)$, and the tensor square $V(2, 0)$ of the latter, so in fact all the irreducibly generated indecomposables are tensor products of twisted forms of the natural module. The maximum of the Loewy lengths of these indecomposables was determined by Alperin, but for their detailed Loewy structure one has to go to [3] even when $p = 2$. Alperin conjectured that, for any p and for any finite group H with abelian Sylow p-subgroups, there are only finitely many irreducibly generated indecomposables over $\mathbb{F}$: this is now confirmed for $H = SL(2, p^n)$.

4.

The algorithms for calculating multiplicities consist essentially of multiplying polynomials (with integer coefficients) and changing bases in free abelian groups. Much of this can be given a conceptually pleasing expression, as was done by Alperin for $p = 2$. We shall consider two commutative rings. The first is the subring of the Green ring of $\mathbb{F}H$ generated by (the isomorphism types of) the V_α. Additively, this is a free abelian group with the set of the V_α as a basis; multiplication is defined by

$$V_\alpha V_\beta = \sum m(\alpha, \beta, \gamma)V_\gamma$$

where the $m(\alpha, \beta, \gamma)$ are the Krull-Schmidt multiplicities:

$$V_\alpha \otimes V_\beta \cong \oplus V_\gamma^{\oplus m(\alpha,\beta,\gamma)} .$$

The ring can also be generated (as ring) by the $V(1, i)$ with $0 \leq i \leq n - 1$, which we denote by x_i. In terms of these, the ring may be defined (as quotient of the polynomial ring in the x_i over $\mathbb{Z}$) by the following n relations (one for each value of i):

$$\left\{x_{i+1} - \sum (-1)^j \left[\binom{p-j}{j} + \binom{p-1-j}{j-1}\right] x_i^{p-2j}\right\} \sum (-1)^k \binom{p-1-k}{k} x_i^{p-1-2k} = 0$$

where x_n is to be read as x_0. In particular, the ring has as another additive basis the set of (the cosets of) the monomials whose degree in each indeterminate is at most $2p - 2$. The change of basis in one direction is given by

$$V(a, i) = \sum (-1)^k \binom{a-k}{k} x_i^{a-2k} \qquad \text{and}$$

$$V(p-1+a, i) = \sum (-1)^j \left[\binom{a-j}{j} + \binom{a-j-1}{j-1}\right] x_i^{a-2j} V(p-1, i)$$

here \otimes should be \Pi

whenever $0 \leq a \leq p - 1$, and $V_\alpha = \otimes V(\alpha_i, i)$ unless $\alpha_i = 2p - 2$ for all i in which case $V_\alpha = \Pi V(2p-2, i) - \Pi V(p-1, i)$. In the opposite direction, we have

$$x_i^a = \sum \left[\binom{a}{j} - \binom{a}{j-1}\right] V(a-2j, i) + \sum \left[\binom{a}{j+p} - \binom{a}{j+p-1}\right] V(a-2j, i)$$

whenever $0 \leq a \leq 2p - 2$; the first sum being over the j such that $0 \leq a - 2j \leq a$ and the second over the j such that $2p - 2 - a \leq a - 2j \leq p - 2$. To calculate $m(\alpha, \beta, \gamma)$, first one expresses V_α and V_β in terms of the x_i, then one calculates the product $V_\alpha V_\beta$ using the defining relations given in terms of the x_i

to express the result so as to involve only monomials of degree at most $2p-2$ in each indeterminate, and finally one converts the result back to the basis consisting of the V_γ. Similarly, one can express $U_\alpha{\downarrow}_H$ (for each infinite sequence α) in terms of the V_γ by using that $U(a,\ i){\downarrow}_H \cong V(a,\ i')$ where i' is the remainder of i upon division by n.

The second ring has an additive basis the set of (the isomorphism types of) the *thin* V_α, and its multiplication is defined by

$$V_\alpha V_\beta = \sum h(\alpha,\ \beta,\ \gamma)V_\gamma$$

where $h(\alpha,\ \beta,\ \gamma)$ is the Jordan-Hölder multiplicity of V_γ in $V_\alpha \otimes V_\beta$. This ring is also generated (as ring) by the $V(1,\ i)$, which we now denote by y_i. In terms of these, the ring may be defined by the n relations

$$y_{i+1} = \sum (-1)^j \left[\binom{p-j}{j} + \binom{p-1-j}{j-1}\right] y_i^{p-2j}$$

where y_n is to be read as y_o (thus in fact this ring is generated by y_o alone). The monomials whose degree in each indeterminate is at most $p-1$ from another additive basis; we have

$$V(a,\ i) = \sum (-1)^k \binom{a-k}{k} y_i^{a-2k} \qquad \text{whenever} \quad 0 \le a \le p-1\ ,$$

and $V_\alpha = \Pi V(\alpha_i,\ i)$ for each thin V_α, to give us the change of basis in one direction; and in the other,

$$y_i^a = \sum \left[\binom{a}{j} - \binom{a}{j-1}\right] V(a-2j,\ i) \qquad \text{for} \quad 0 \le a \le p-1\ ,$$

with summation over all j such that $0 \leq 2j \leq a$. Clearly, $x_i \mapsto y_i$ is a ring homomorphism; the matrix describing it in terms of the original additive bases yields the conversion from Krull-Schmidt multiplicities to Jordan-Hölder multiplicities, in the same sense as J did in L .

5.

The results concerning G , from which those for H are derived, naturally have their inspiration in the theory of algebraic groups. A key contribution from that context is Cline's theorem [5] on $\mathrm{Ext}^1_G(U_\alpha, U_\beta)$ for thin U_α , U_β . This is re-proved in the Andersen, Jørgensen, Landrock preprint [3]: by contrast, their work is essentially bare-handed, except for two lemmas which still rely on the algebraic nature of G (in using infinitesimal arguments). I believe I can replace these by a bare-handed proof of Cline's result, which is perhaps best expressed in the following terms.

Call a module *short uniserial* if it has precisely one proper nonzero submodule: that is, if it is a non-split extension of one irreducible by another. The result gives all short uniserial G-modules with thin composition factors, as twisted tensor products. To this end, we need some further building blocks. Let $\mathbb{F}[x, y]$ be the polynomial ring over $\mathbb{F}$ in two (commuting) indeterminates, and let G act on $\mathbb{F}[x, y]$ by linear homogeneous substitutions:

$$(x^i y^j)g = (g_{11}x+g_{12}y)^i(g_{21}x+g_{22}y)^j .$$

The space W_d of all homogeneous polynomials of degree d is then a G-module (of dimension $d + 1$), for each nonnegative integer d. These are known as (the duals of) the Weyl modules of G; their submodule structure has been completely described by Carter and Cline [4] (see also Cline [6], Deriziotis [7], Schooneveldt [8]; this can also be done with bare hands). The W_d with $0 \leq d \leq p - 1$ are irreducible: in this case, $W_d \cong U_d$. For the sequel, let $d = d_0 + d_1 p$ with $0 \leq d_0 \leq p - 2$ and $1 \leq d_1 \leq p - 1$. Then W_d is a short uniserial, with submodule $U(d_0, 0) \otimes U(d_1, 1)$ and factormodule $U(p-2-d_0, 0) \otimes U(d_1-1, 1)$. Of course, the contragradient module W_d^* has the same composition factors in opposite order. (If $p \leq d \leq 2p - 2$, then the unique maximal submodule of U_d is isomorphic to W_d^*, and the quotient of U_d over its unique minimal submodule is isomorphic to W_d.) Set $W(d, k) = W_d^{(p^k)}$. The theorem says that *the short uniserials with thin composition factors are precisely the* $W(d, k) \otimes U_\alpha$ *and the* $W(d, k)^* \otimes U_\alpha$ *with* U_α *thin and* $\alpha_k = \alpha_{k+1} = 0$ *(and* d *restricted as stated).* It is pleasing to have this result, and hence all of [3], accessible asscesible without any reference to the theory (or even the language) of algebraic groups.

6.

In conclusion, a little more about Schooneveldt's conjecture confirmed above. Various deep questions concerning finite p-groups have been handled successfully over the decades by Lie ring methods which go back to Magnus, Witt and Lazard: one famous example is Graham Higman's

work on Suzuki 2-groups. These start by translating the problem to the following setting. Let L be a free Lie algebra of finite rank over the field of order p, with L_c the homogeneous component of degree c. We have a finite group H acting on L by Lie algebra automorphisms which respect the grading of L, and need to analyze the action of H on L_c in terms of its action on L_1. We extend the field to $\mathbb{F}$: the resulting $\mathbb{F}H$-modules are the same as if we had started with the free Lie algebra over $\mathbb{F}$. Now we can choose convenient free generators for L: say, eigenvectors for some p'-element h of H; the usual bases of the L_c will then also consist of eigenvectors of h. This enables us, for instance, to calculate the Brauer character of H afforded by L_c in terms of that afforded by L_1.

Schooneveldt was investigating (among other things) the case of $H = SL(2, p)$ acting naturally on the free Lie algebra L of rank 2. The irreducible and (all) the indecomposable $\mathbb{F}H$-modules were, of course, well known, and he could obtain the Jordan-Hölder multiplicities of the L_c: the outstanding problem was to calculate the Krull-Schmidt multiplicities. His idea was to extend the field to $\mathbb{F}$ and exploit not only the freedom to choose better bases but also the fact that the action of H comes by restriction from the natural action of G. When c is not a multiple of p, it is easy to see that L_c is a (G-module) direct summand of $V^{\otimes c}$, so L_c lies in the class $\mathcal{L}$ and therefore, as we have seen in section 2, the Krull-Schmidt multiplicities of L_c (as G-module) can be recovered from the known Jordan-Hölder multiplicities. The comment at the end of the

discussion of the first ring in section 4 indicates how one obtains then the Krull-Schmidt multiplicities of L_c as $SL(2, p)$-module. So in this case Schooneveldt's program works. This idea may well find further applications, extending the role of Lie ring methods for p-groups. (The problem of the L_c with c a multiple of p is still open.)

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