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CLASSIFICATION THEOREMS FOR TORSIONFREE GROUPS

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I shall report on some little-known results, which are neither particularly new nor "mainstream" infinite group theory, but which can at least be transposed to a key not too far removed from that of this meeting.

As classifying infinite groups up to isomorphism is a hopeless goal (even in the abelian case), let us consider a coarser equivalence relation: call two groups "similar" if each is isomorphic to some section of an unrestricted direct product of suitably many copies of the other. It is not hard to see that two finite simple groups are similar if and only if they are isomorphic, but all nontrivial torsionfree abelian groups are in a single similarity class: so this seems a reasonable classification principle for the present occasion. To keep a long story short, until the concluding remarks we shall consider only torsionfree groups.

The easiest result to state extends our observation concerning abelian groups: the torsionfree metabelian groups which are nilpotent of a given nilpotency class c form a single similarity class, say, M_c. It may make more sense to write E for M_0 (the class of all groups of order 1) and A for M_1 (the class of all other torsionfree abelian groups). Other similarity classes of immediate interest are: the similarity class N_c of $F_r/\gamma_{c+1}F_r$ where F_r is a free group of rank r (finite or infinite, but not smaller than c) and $\gamma_{c+1}F_r$ is the term of its lower central series which yields a quotient of class precisely c. (It is, of course, nontrivial that N_c is independent of r.) Note $N_0 = E$, $N_1 = A$, $N_2 = M_2$, $N_3 = M_3$, but $N_4 \neq M_4$. Before more can be said, we need to define a partial order on the set of all similarity classes of torsionfree groups: put $X \leqslant Y$ whenever $G \in X$ and $H \in Y$ imply that G is isomorphic to some section of an unrestricted direct product of suitably many copies of H. Then clearly

$$E < A < N_1 < N_2 < N_3 < M_4 \begin{matrix} < N_4 \\ < M_5 \end{matrix} < N_5 < \dots .$$

This poset is in fact a lattice (with $2^{\aleph_0}$ elements). It is an open and, as far as I know, unexplored question whether this lattice is modular; the sublattice consisting of the classes of torsionfree nilpotent-by-abelian groups certainly is, and we

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shall not step beyond that today. We shall see from the next result that the lattice is not distributive.

Let $(12\ldots c)$ be a cyclic permutation, C the subgroup it generates in the symmetric group S of degree c; take any faithful irreducible complex character γ of C, induce to S and decompose:

$$\gamma^S = \sum_\lambda d_\lambda \chi_\lambda$$

where the χ_λ are the irreducible characters of S indexed as usual by the partitions λ of c. For each λ, let Q_λ denote the lattice of all subspaces of a rational vector space of dimension d_λ, and form the direct product lattice $\Pi_\lambda Q_\lambda$. (If $d_\lambda = 0$, interpret Q_λ as a singleton, redundant in this direct product.)

(*) *The sublattice* $\{X \mid N_{c-1} \leqslant X \leqslant N_c\}$ *is isomorphic to* $\Pi_\lambda Q_\lambda$.

For each c in $\{1, 2, 3\}$, one d_λ is 1 and all others are 0: thus there are no similarity classes strictly between N_{c-1} and N_c.

For $c = 4$, two d_λ are 1, and all others 0, so in that case we have Figure 1 where the unnamed point can be identified as the class of torsionfree nilpotent groups of class precisely 4 with all 2-generator subgroups of class at most 3.

For $c = 5$, five of the d_λ are 1 and the rest 0, so then we have a 5-dimensional cube pictured so that one diameter is vertical. Indeed, one can show that by now we have seen all (39) members of the sublattice $\{X \mid X \leqslant N_5\}$, the only one not covered by (*) being M_5: this contains M_4, and its join with N_4 is a vertex of the 5-cube adjacent to N_4. "Recognition theorems" exist for each of these 39 classes.

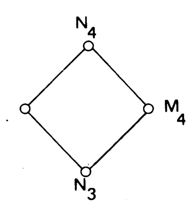

FIGURE 1

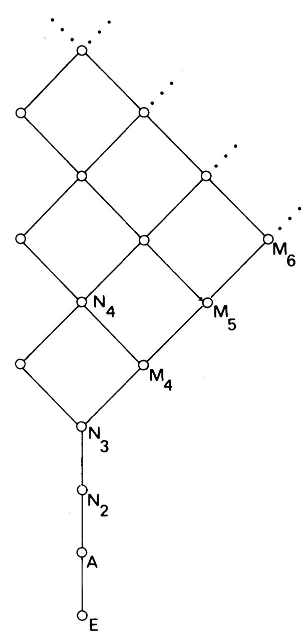

FIGURE 2

For each c from 6 on, at least one d_λ is greater than 1, so the corresponding Q_λ is infinite and nondistributive. Very little is known about similarity classes of torsionfree nilpotent groups not covered by (*): for instance, we do not know whether we are missing finitely or infinitely many in $\{X \mid X \leqslant N_6\}$.

The only area where we have further conclusive information in the nilpotent case is that of the centre-by-metabelian torsionfree nilpotent groups; the lattice of their similarity classes is shown in Figure 2.

Of the nonnilpotent story, we only know the metabelian case. Let M be the class of the largest metabelian quotient of any noncyclic free group. If G is a torsionfree metabelian group not in M, then for some positive integer n the subgroup of G generated by the commutators and the nth powers must be nilpotent. This is a relatively elementary observation, but good enough to start us on the long road to the following conclusions. The lattice of similarity classes of torsionfree metabelian groups is distributive and satisfies the descending chain condition, so each element of it has a unique expression as a finite irredundant join of join-irreducibles. The join-irreducibles other than E or M are parametrized by ordered pairs (c, n) of positive integers: $M_{c,n}$ is the largest member of the lattice which consists of groups in which the subgroup generated by all commutators and nth powers is nilpotent of class c. The lattice can now be fully described by adding that $M_{b,m} \leqslant M_{c,n}$ if and only if $b \leqslant c$ and $m|n$. (Note our previous M_c is now called $M_{c,1}$.)

Does this mean anything for finite groups? The nilpotent classification described here did, in fact, start in the context of groups of prime exponent and is valid there, *mutatis mutandis*, provided the prime is larger than the class. Extension to the prime-power exponent case presents no serious difficulties. It could be feasible to relax the small class restriction, say, to $c < 2p$, by exploiting the fact that the symmetric group of degree c has cyclic Sylow p-subgroups under this assumption. One would need to use not only the appropriate modular representation theory of the symmetric group, but also the corresponding part of the representation theory of $\mathrm{GL}(n, p)$ over $\mathrm{GF}(p)$, extended to representations of the semigroup of all $n \times n$ matrices over $\mathrm{GF}(p)$. There is certainly scope in this area for using some of the expertise present at this conference.

I must own up, though: there seems to be no information on similarity of infinite simple groups–for all we know, all simple groups of infinite exponent could be similar to the noncyclic free groups. Also, similarity is an *ad hoc* term appropriated for the occasion; to contact with the literature, one has to translate "are similar" as "generate the same variety". The rest of our notation was also *ad hoc*.

For a detailed exposition and references, see a recent paper of mine **[5]**. The basic ideas for the nilpotent case owe much to Magnus, but it was perhaps Thrall who first glimpsed the whole picture [7]. G. Higman **[2]** and A. A. Kljačko [3] appear to have rediscovered it independently. Most of the torsionfree nilpotent version was elaborated in 1967 by M. F. Newman and me, but remained unpublished until last year **[4]**. The nilpotent centre-by-metabelian case is due essentially to A. G. R. Stewart **[6]**. The torsionfree metabelian results were found by M. F. Newman and me; they rely heavily on, and were eventually incorporated into, the work of R. A. Bryce **[1]**.

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