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HANNA NEUMANN'S PROBLEMS ON VARIETIES OF GROUPS

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This is an informal report on the present status of the displayed problems in Hanna Neumann's book *Varieties of groups* [50]. The reader should have the book at hand, not only for notation and terminology but also because we do not re-state the problems nor repeat the comments available there. Our aim is to be up to date, not to present a complete historical survey; superseded references will be mostly ignored regardless of their significance at the time. Details of solutions will not be quoted from papers already published, unless needed to motivate further questions. The discussion and the problems highlighted in it reflect our personal interests rather than any considered value-judgement.

The preparation of this report was made much easier by access to the notes Hanna Neumann had kept on these problems. We are indebted to several colleagues who took part in a seminar on this topic and especially to Elizabeth Ormerod for keeping a record of these conversations. The report has also gained a lot from the response of conference participants; in particular, Professor Kostrikin supplied much useful information. Of course, all errors and omissions are our own responsibility: we shall be very grateful for information leading to corrections or additions.

PROBLEM 1 (page 6 in [50]).

As Hanna Neumann wrote, "this is of no great consequence". The answer is negative; see Kovács and Vaughan-Lee [47].

PROBLEMS 2, 3, 11 (pages 22, 92).

The celebrated "finite basis problem" asked whether every variety can be defined by a finite set of laws. Ol'šanskiĭ proved (in about 1968, unpublished) that this is equivalent to the problem: is the set of varieties countable? (See also Kovács [42].) A positive answer would have created a simple situation to report on; however, in general, the answer is negative. A comprehensive survey of the positive partial results is beyond the scope of this report, and the listing of open questions provoked by the complexity of the situation is also without any claim to completeness.

The negative answer was first obtained by Ol'šanskiĭ [55] in September 1969: he proved that there are continuously many locally finite varieties of soluble length at most 5 and exponent dividing $8pq$ whenever p, q are distinct, odd primes. This settled Problems 2 and 3. By December 1969, Vaughan-Lee [62] constructed (by entirely different means) continuously many varieties within $(\underline{\underline{B}}_4 \wedge \underline{\underline{N}}_2)^2$; and, early in 1970, Adyan ([1], see also [3]) gave an infinite independent set of very simple two-variable laws.

Given the negative solution, Zorn's Lemma yields the existence of at least one just non-finitely-based variety (a variety minimal with respect to not being definable by a finite set of laws). One may then ask:

QUESTION 1. *What is the cardinality of the set of just non-finitely-based varieties?*

Simplifying and extending Vaughan-Lee's construction, Newman [53] proved also that to each odd prime p there is at least one just non-finitely-based variety in $\left(\underline{\underline{A}}_p^2 \wedge \underline{\underline{N}}_p\right)(\underline{\underline{B}}_p \wedge \underline{\underline{N}}_2)$: so the answer to Question 1 is certainly 'infinite'.

Vaughan-Lee [64] was the first to show that the product of two finitely based varieties need not be finitely based.

Problem 11 was solved simultaneously and independently by Kleĭman [41] and Bryant [16]: $\underline{\underline{B}}_4\underline{\underline{B}}_2$ is not finitely based. This is still the easiest example to name. Further results of theirs (see also a forthcoming paper of Kleĭman), with little extra overlap between them, extend the scope of the work beyond what can be fully reported here. All we mention is that Bryant [16], starting from the method of Vaughan-Lee and Newman, produced varieties $\underline{\underline{U}}$ and $\underline{\underline{V}}$ such that there are continuously many varieties $\underline{\underline{W}}$ with $\underline{\underline{U}} \leq \underline{\underline{W}} \leq \underline{\underline{V}}$ and no such $\underline{\underline{W}}$ can have a finite basis. This killed all hope that the set of finitely based varieties might in some sense be dense in the set of all varieties.

One of the questions provoked by the nature of these examples is the following.

QUESTION 2. *If all nilpotent groups in a (locally finite) variety are abelian, must the variety be finitely based?*

Another is prompted by the observation that elements of finite order appear to occur in the free groups of all these examples. Now it is easy to see that $\underline{\underline{AV}}$ is always torsion-free (in the sense that its free groups are torsion-free) and that if $\underline{\underline{V}}$ is not finitely based then neither is $\underline{\underline{AV}}$: so, torsion-free non-finitely-based varieties can also be made. However, here this was done at the cost of increasing soluble length and losing properties such as local nilpotence. Thus one may ask:

QUESTION 3. *Are all torsion-free metanilpotent varieties finitely based? Are* *all torsion-free subvarieties of* $\underline{\underline{A}}^3$ *(or even those of* $\underline{\underline{A}}^4$ *) finitely based? Are all torsion-free locally nilpotent varieties finitely based?*

The last of these questions is closely related, at least in one direction, to another problem mentioned by Professor Kostrikin in these Proceedings:

QUESTION 4. *Is every torsion-free locally nilpotent variety soluble (equivalently, nilpotent)?*

(The equivalence follows, for instance, from Lemma 4 of Groves [27].) While in general the meet of two torsion-free varieties need not be torsion-free, the intersection of a descending chain of torsion-free varieties is always torsion-free. Hence if the answer were negative, by Zorn's Lemma there would also exist minimal examples: torsion-free, insoluble, locally nilpotent varieties whose torsion-free proper subvarieties are all nilpotent.

We now turn to some of the positive results achieved since the publication of the book [50]. Many of these take the form that "all subvarieties of $\underline{\underline{V}}$ are finitely based": we shall paraphrase this as "$\underline{\underline{V}}$ is *hereditarily* finitely based". Perhaps the deepest result in this area, superseding many earlier ones and developing the technique initiated by Cohen [22] to its present limits, is due to Bryant and Newman [17]: $\underline{\underline{N}}_c\underline{\underline{A}} \wedge \underline{\underline{N}}_2\underline{\underline{N}}_c$ is hereditarily finitely based for every positive integer c. Others assert that the following varieties are hereditarily finitely based: $\underline{\underline{A}}_r\underline{\underline{A}}_s\underline{\underline{A}}_t$ provided $(s, rt) = 1$ and t is prime, Bryce and Cossey [19]; $\underline{\underline{A}}_m(\underline{\underline{B}}_n \wedge \underline{\underline{N}}_2)$ provided $(m, n) = 1$, Brady, Bryce, and Cossey [8]; and $\underline{\underline{B}}_6$, Atkinson [4].

QUESTION 5. *Which, if any, of the following varieties are hereditarily finitely based:* $\underline{\underline{A}}^3$, $\underline{\underline{A}}\underline{\underline{N}}_2$, $\underline{\underline{A}}_2(\underline{\underline{B}}_4 \wedge \underline{\underline{N}}_2)$, $\underline{\underline{N}}_c\underline{\underline{A}}$, $(\underline{\underline{B}}_4 \wedge \underline{\underline{N}}_3)\underline{\underline{A}}_2$, $(\underline{\underline{B}}_m \wedge \underline{\underline{N}}_2)(\underline{\underline{B}}_n \wedge \underline{\underline{N}}_2)$ *when* $(m, n) = 1$, $\underline{\underline{B}}_4$ *?*

Other positive results give ways of making new finitely based varieties from old. Abstracting the essence of an argument of Higman (34.23 in [50]), Brooks, Kovács, and Newman [10] introduced the concept of *strongly finitely based* variety and showed that if $\underline{\underline{U}}$ is strongly finitely based and $\underline{\underline{V}}$ is finitely based then $\underline{\underline{U}}\underline{\underline{V}}$ is finitely based. Indeed, if $\underline{\underline{V}}$ is also *strongly* finitely based, the same holds for $\underline{\underline{U}}\underline{\underline{V}}$. Bryant [14] proved that if $\underline{\underline{U}} \leq \underline{\underline{A}}\underline{\underline{N}}_c \wedge \underline{\underline{N}}_c\underline{\underline{A}}$ for some c and $\underline{\underline{V}}$ is (strongly) finitely based, then $\underline{\underline{U}} \vee \underline{\underline{V}}$ is (strongly) finitely based. On the other hand, a result of Vaughan-Lee [64] quoted above implies that not all finitely based varieties are strongly finitely based; in fact, Bryant pointed out in [16] that even $\underline{\underline{B}}_4 \wedge [\underline{\underline{A}}^2, \underline{\underline{E}}]$ fails to be strongly finitely based. Of the host of questions one might ask in this context, let us highlight just one.

QUESTION 6. *Are all Cross varieties strongly finitely based?*

One of the most intriguing general questions remains:

QUESTION 7. *Is the join of two finitely based varieties always finitely based?*

Deep positive partial results may be found in Bryant [14], [15]. We quote just one more: if $\underline{\underline{U}}$ is Cross while $\underline{\underline{V}}$ is locally finite and (hereditarily) finitely based, then $\underline{\underline{U}} \vee \underline{\underline{V}}$ is also (hereditarily) finitely based. This is derived in [15] by extending the method of proof of the Oates-Powell Theorem to its present limits. Finally, consider the following two propositions. If $\underline{\underline{U}} \leq \underline{\underline{W}}$ and $\underline{\underline{V}}$ is finitely based, then $\underline{\underline{UV}}$ is also finitely based. If $\underline{\underline{U}} \leq \underline{\underline{W}}$ and $\underline{\underline{V}}$ is finitely based, then $\underline{\underline{U}} \vee \underline{\underline{V}}$ is also finitely based. Both propositions are known to be valid provided $\underline{\underline{W}} \leq \underline{\underline{AN}}_c \wedge \underline{\underline{N}}_c\underline{\underline{A}}$ for some c, and neither is *known* to be valid otherwise. While this may be pure coincidence, the expectation is that the answer to Question 7 will be negative. This is further encouraged by the fact that Jónsson [37], [38] has shown the join of two finitely based varieties of lattices need not be finitely based.

PROBLEM 4 (page 23; insert "finitely generated" before "group").

No progress. One way towards a positive answer has been closed by the result that $\underline{\underline{K}}_p$ is not nilpotent (and hence not even soluble) if $p > 3$. This is due to Razmyslov [58]; for the case $p = 5$ it was found first by Bachmuth, Mochizuki, and Walkup [5]. In addition, Razmyslov identified [59] a just-non-Cross subvariety, satisfying the $(p-2)$th Engel condition, in each $\underline{\underline{K}}_p$ with $p > 3$.

PROBLEM 5 (page 42).

No progress: this is perhaps the most tantalizing problem of all. Many people feel there is a connection with the question (usually attributed to Tarski) concerning the existence of infinite groups in which all proper nontrivial subgroups are of prime order, but nobody seems to be able to prove even a one-way implication.

To facilitate discussion, call a variety *pseudo-abelian* if it is nonabelian but all its finite groups (equivalently, all its soluble groups) are abelian. In particular, a pseudo-abelian variety would not be generated by its finite groups. So far, the only way known for showing the existence of varieties not generated by their finite groups is still to point to $\underline{\underline{B}}_p$ with a large prime p and quote both Kostrikin's positive solution of the restricted Burnside problem and the negative solution by Novikov and Adyan of the unrestricted Burnside problem. This is one indication of the difficulties which would have to be overcome here.

At one time it was thought proved that all groups in a pseudo-abelian variety would have to be T-groups (groups in which normality is transitive; that is, all subnormal subgroups are normal). This claim survives as a conjecture supported by an unpublished partial result obtained independently by Kovács and Peter M. Neumann: if an element of squarefree order fails to normalize a subnormal subgroup of a group

G , then G has a metabelian, nonabelian factor and so cannot belong to any pseudo-abelian variety.

Any pseudo-abelian variety would have to contain a minimal pseudo-abelian variety (whose proper subvarieties are all abelian). This enables one to show that if a product contains a pseudo-abelian variety then at least one of the factors must also contain some (possibly different) pseudo-abelian variety. It follows that the class of those varieties which have no pseudo-abelian subvarieties, is closed with respect to all usual operations (meet, join, product, commutator) except (possibly) infinite joins, and contains, of course, all locally finite and all locally soluble varieties. One might therefore try to work within this class and prove theorems which would become generally valid if the existence of pseudo-abelian varieties were disproved. Unfortunately, this approach has failed in every case tried so far. As an example, let us note that if the existence of pseudo-abelian varieties were disproved one would wish to move on to questions like this:

QUESTION 8. *If all finite groups of a variety* $\underline{\underline{U}}$ *lie in a given (nonabelian) Cross variety* $\underline{\underline{V}}$ *, does* $\underline{\underline{U}} \leq \underline{\underline{V}}$ *follow?*

No counter-examples are known. However, assuming that $\underline{\underline{U}}$ has no pseudo-abelian subvarieties (or even that no pseudo-abelian varieties exist) seems to be no help at all, even if $\underline{\underline{V}}$ is taken as $\underline{\underline{B}}_p \wedge \underline{\underline{N}}_2$ for some large prime p . Apparently a positive answer to Question 8 cannot be derived from a dogma to the effect that there are no ghosts: what is needed is a surefire method for exorcising the pseudo-abelian ghost, and then an appropriate modification of the method might also yield the answer to Question 8.

The case of Question 8 with the special choice of $\underline{\underline{V}}$ mentioned above is vaguely related to another question: would the join of two pseudo-abelian varieties have to be pseudo-abelian? This could be answered positively if one knew that neither $\underline{\underline{B}}_4 \wedge \underline{\underline{N}}_2$ nor any $\underline{\underline{B}}_p \wedge \underline{\underline{N}}_2$ (with p an odd prime) can be contained in a join of two pseudo-abelian varieties (and used that $\underline{\underline{A}}_p\underline{\underline{A}}_q$, with p and q distinct primes, certainly cannot: see Kovács [45]).

Finally, we recall that Problem 5 arose in a discussion on what 'small' subvarieties must a variety possess. Theorem 21.4 of the book [50] could have been stated as follows: if $\underline{\underline{V}}$ is neither abelian nor pseudo-abelian, then it contains either an $\underline{\underline{A}}_p\underline{\underline{A}}_q$ (with p and q distinct primes) or $\underline{\underline{B}}_4 \wedge \underline{\underline{N}}_2$ or a $\underline{\underline{B}}_p \wedge \underline{\underline{N}}_2$ (with p some odd prime). Further, very much deeper, results of this kind are to be found in the contexts of just-non-Cross varieties (principal references: Brady [7], Ol'šanskiĭ [56]) and dichotomies (Groves [28], [29], [30], [31], Kargapolov and Čurkin [39]).

PROBLEM 6 (page 42).

No real progress. Peter M. Neumann (unpublished) improved the comment of the book [50]: a nonabelian variety other than $\mathbf{O}$ in which verbal products with one normal amalgamation exist, would have to be pseudo-abelian. Meskin [49] noted that if $\mathbf{V}$ is a variety of exponent 0 in which verbal products with one amalgamation exist, then the set of non-laws of $\mathbf{V}$ is closed under those endomorphisms of the word group which map each variable to a nontrivial power of itself.

PROBLEM 7 (page 60).

The problem seems to have been asked, at least partly, in the hope that a positive answer would help make further examples of indecomposable varieties. This direction has been explored with some success through positive partial solutions of the second half of the problem by Brumberg [11] and Cossey [24]. However, the solution to both halves of the problem is, in general, negative: see Cossey [25].

The first half of the problem is easily seen to be equivalent to the following. Can a product of two nontrivial varieties ever be written as a join of two incomparable varieties, other than by writing the first factor as such a join and using the distributivity of right multiplication over joins? From this formulation the negative answer is almost evident; the example given by Cossey is a very simple one indeed.

The second part may be re-formulated similarly, with 'commutator' in place of 'join' (and one may as well omit 'incomparable'), but here the negative solution is far from obvious. Perhaps the most interesting positive result is due to Dunwoody [26] and Brumberg [11]; we describe it as a basis for an analogy to be drawn below. If $\mathbf{U}$, $\mathbf{V}$, $\mathbf{X}$, $\mathbf{Y}$ are varieties such that $\mathbf{X} \neq \mathbf{E}$ and $\mathbf{XY} = [\mathbf{U}, \mathbf{V}] \neq \mathbf{O}$, then there exist varieties $\mathbf{U}'$, $\mathbf{V}'$ such that $\mathbf{U}'\mathbf{Y} \leq \mathbf{U}$, $\mathbf{V}'\mathbf{Y} \leq \mathbf{V}$, and $\mathbf{X} = [\mathbf{U}', \mathbf{V}']$ (so $\mathbf{XY} = [\mathbf{U}'\mathbf{Y}, \mathbf{V}'\mathbf{Y}] = [\mathbf{U}, \mathbf{V}]$). Thus if a product $\mathbf{XY}$ (with $\mathbf{X} \neq \mathbf{E}$, $\mathbf{XY} \neq \mathbf{O}$) admits a commutator-decomposition, so does the first factor $\mathbf{X}$; and in fact a commutator-decomposition of $\mathbf{X}$ may be chosen so that the commutator-decomposition of the product obtained from it (by the appropriate distributive law) is 'smaller than or equal to' the original. In particular, if a commutator-decomposition $[\mathbf{U}, \mathbf{V}]$ of the product $\mathbf{XY}$ is minimal (in the sense that $\mathbf{U}_1 < \mathbf{U}$, $\mathbf{V}_1 < \mathbf{V}$ imply $[\mathbf{U}_1, \mathbf{V}] < [\mathbf{U}, \mathbf{V}] > [\mathbf{U}, \mathbf{V}_1]$) then it comes from a minimal commutator-decomposition of the first factor $\mathbf{X}$, and *vice versa*. Thus at least the minimal commutator-decompositions of $\mathbf{X}$ and $\mathbf{XY}$ are in one-to-one correspondence. It is easy to see that each commutator-decomposition $[\mathbf{U}, \mathbf{V}]$ of a variety is comparable to a minimal one: if U, V are the corresponding verbal subgroups of an absolutely free group F of infinite rank, let $C/[U, V]$ be the centralizer of $V/[U, V]$ in $F/[U, V]$, and $D/[U, V]$ the centralizer of $C/[U, V]$; then $C \geq U$, $D \geq V$, $[C, D] = [U, V]$, while C and D are verbal in F; so $[\mathrm{var}F/C, \mathrm{var}F/D]$ is a minimal commutator-decomposition of $[\underline{\underline{U}}, \underline{\underline{V}}]$, comparable to the original. Consequently, the negative solution demonstrates and exploits the existence of distinct, comparable, commutator-decompositions of certain varieties, while the positive partial solutions are obtained in situations where such ambiguities can be ruled out.

As this account shows, the context of the second half of Problem 7 is, by now, fairly well understood. By contrast, the situation surrounding the first half of the problem has remained largely unexplored. The first question suggested by analogy is whether a product can ever be a (proper, finite) join without the first factor being (trivial or such) a join. In other words:

QUESTION 9. *If* $\underline{\underline{X}}$ *is a nontrivial join-irreducible and* $\underline{\underline{Y}}$ *an arbitrary variety, is the product* $\underline{\underline{XY}}$ *necessarily also join-irreducible?*

Kovács [45] shows that the answer is positive if either $\underline{\underline{X}}$ is abelian or the infinite-rank free groups of $\underline{\underline{X}}$ have no nontrivial abelian verbal subgroups. However, it is not known whether $(\underline{\underline{B}}_4 \wedge \underline{\underline{N}}_2)\underline{\underline{Y}}$ or $(\underline{\underline{B}}_p \wedge \underline{\underline{N}}_2)\underline{\underline{Y}}$ (for odd primes p) is join-irreducible for every $\underline{\underline{Y}}$. (For $\underline{\underline{Y}} = \underline{\underline{A}}_m$ with m a divisor of $p - 1$, this problem has been settled positively by Woeppel [66]; see also [67].) If the answer were positive in general, one would proceed to ask whether each proper join-decomposition of a product is comparable to one obtained from one for the first factor:

QUESTION 10. *Does* $\underline{\underline{Y}} \neq \underline{\underline{XY}} = \underline{\underline{U}} \vee \underline{\underline{V}} \neq \underline{\underline{U}} \nleq \underline{\underline{V}}$ *imply that* $\underline{\underline{X}} = \underline{\underline{U}}' \vee \underline{\underline{V}}'$ *for some* $\underline{\underline{U}}'$, $\underline{\underline{V}}'$ *with* $\underline{\underline{U}}'\underline{\underline{Y}} \leq \underline{\underline{U}}$ *and* $\underline{\underline{V}}'\underline{\underline{Y}} \leq \underline{\underline{V}}$ *? Equivalently, does every minimal proper join-decomposition* $\underline{\underline{U}} \vee \underline{\underline{V}}$ *of a product* $\underline{\underline{XY}}$ *(with* $\underline{\underline{X}} \neq \underline{\underline{E}}$ *) come from a (necessarily minimal) proper join-decomposition* $\underline{\underline{X}} = \underline{\underline{U}}' \vee \underline{\underline{V}}'$ *(in the sense that* $\underline{\underline{U}} = \underline{\underline{U}}'\underline{\underline{Y}}$ *,* $\underline{\underline{V}} = \underline{\underline{V}}'\underline{\underline{Y}}$ *)?*

Here $\underline{\underline{U}} \vee \underline{\underline{V}}$ is a minimal join-decomposition if $\underline{\underline{U}}_1 < \underline{\underline{U}}$, $\underline{\underline{V}}_1 < \underline{\underline{V}}$ imply $\underline{\underline{U}}_1 \vee \underline{\underline{V}} < \underline{\underline{U}} \vee \underline{\underline{V}} > \underline{\underline{U}} \vee \underline{\underline{V}}_1$. The equivalence claimed depends on the fact that every join-decomposition is comparable to a minimal one. (Prove this as the corresponding fact for commutator-decompositions, but instead of using centralizers choose $C/(U \cap V)$ maximal among the verbal subgroups of $F/(U \cap V)$ which avoid $V/(U \cap V)$, and so on: Zorn's Lemma makes this possible.) The 'converse' to Question 10 may be a separate question:

QUESTION 11. *If* $\underline{\underline{U}} \vee \underline{\underline{V}}$ *is a minimal join-decomposition and* $\underline{\underline{Y}} \neq \underline{\underline{O}}$ *, is* $\underline{\underline{UY}} \vee \underline{\underline{VY}}$ *necessarily also minimal?*

Of course, a positive answer to Question 10 would imply one for Question 11, but it is not known whether the implication goes the other way as well.

PROBLEMS 8, 9 (page 69).

The only directly relevant work we know of is Baumslag's paper [6], which solved

Problem 8 partly and Problem 9 fully, and was already reported in the footnote on this page of the book [50].

One may well ask the question more generally (that is, not only for product varieties):

QUESTION 12. *Suppose* $F_k(\underline{\underline{V}})$ *generates* $\underline{\underline{V}}$, *and* $n > k$. *Is* $F_n(\underline{\underline{V}})$ *residually k-generator? Is* $F_n(\underline{\underline{V}})$ *residually* $F_k(\underline{\underline{V}})$ *? Is* $F_n(\underline{\underline{V}})$ *residually* $F_{n-1}(\underline{\underline{V}})$ *?*

Of course, these three questions are related, but it does not seem to be known whether any two of them are actually equivalent. Some answers are available for certain varieties defined via commutators: see the forthcoming paper [33] of Gupta and Levin. In particular, they show that $F_n([\underline{\underline{A}}^2, \underline{\underline{E}}])$ is residually $F_{n-1}([\underline{\underline{A}}^2, \underline{\underline{E}}])$ if and only if $n \neq 2$ and $n \neq 4$ (see also the comments after Question 13 below).

This leads on to residual properties of relatively free groups in general. The most interesting question seems to be:

QUESTION 13. *Are all soluble relatively free groups residually finite?*

The difficulties of progress beyond the results reported on in the book (26.31 in [50]) are best illustrated by the fact that even the residual finiteness of the $F_n([\underline{\underline{A}}^2, \underline{\underline{E}}])$ had not been conclusively established until recently, C.K. Gupta [32]. She showed also that for $n \leq 3$ these groups are torsion-free but for $n > 3$ they are not: indeed, $[\underline{\underline{A}}^2, \underline{\underline{E}}]$ is generated by its free group of rank 4 , and is the proper join of a nilpotent variety of 2-groups with the torsion-free variety generated by $F_2([\underline{\underline{A}}^2, \underline{\underline{E}}])$.

PROBLEM 10 (page 72).

No progress to report.

PROBLEM 11 (page 92).

The negative solution has been discussed with Problems 2 and 3.

PROBLEM 12 (page 101).

Unsolved. The very deep work of Ward mentioned in the remark preceding the problem was published in [65].

PROBLEM 13 (page 101).

The solution is in the negative. The classification of all subvarieties of $\underline{\underline{B}}_p \wedge [\underline{\underline{A}}^2, \underline{\underline{E}}] \wedge \underline{\underline{N}}_{p-1}$ by Stewart [61] yields this (see Stewart [60] for an explicit derivation), for instance, with $k = 3$, $c = 5$, $\underline{\underline{U}}_6 = \underline{\underline{B}}_7 \wedge [\underline{\underline{A}}^2, \underline{\underline{E}}] \wedge \underline{\underline{N}}_6$: the smallest free group to generate $\mathfrak{U}_6$ is $F_3(\mathfrak{U}_6)$, the centre of $F_6(\mathfrak{U}_6)$ contains the second derived group which is not in the last term of the lower central series, but $\mathfrak{U}_5$ $(= \mathfrak{U}_6 \wedge \mathfrak{N}_5)$ is also generated by its free group of rank 3 - indeed, even by its free group of rank 2. On the other hand, the condition cannot be omitted altogether; Cossey had shown [23] that it cannot even be replaced by insisting that the varieties in question be torsion-free, or that they have prime-power exponent.

Hanna Neumann's lead-up to Problem 13 started with the comment: "As one might expect, the minimal rank of a generating group of a nilpotent variety is in general a non-decreasing function of the class". Her 35.21 is a specific instance of this general and intuitive statement. The nature of the negative solution of Problem 13 prompts one to look for other formulations. For example:

QUESTION 14. *If* $[\mathfrak{U}, \mathfrak{E}]$ *is generated by its free group of rank* k, *must the same be true of* $\mathfrak{U}$ *(at least if* $\mathfrak{U}$ *is nilpotent)? Does* $[\mathfrak{U}, \mathfrak{E}] = [\operatorname{var}F_k(\mathfrak{U}), \mathfrak{E}]$ *imply* $\mathfrak{U} = \operatorname{var}F_k(\mathfrak{U})$ *(at least if* $\mathfrak{U}$ *is nilpotent)?*

The first hypothesis implies the second. Note that Cossey has shown [25] that $[\mathfrak{U}, \mathfrak{E}] = [\mathfrak{U}_0, \mathfrak{E}]$ and $\mathfrak{U} \geq \mathfrak{U}_0$ need not imply $\mathfrak{U} = \mathfrak{U}_0$: however, in his example $\mathfrak{U}$ was not nilpotent, and $\mathfrak{U}_0$ was not generated by a free group of $\mathfrak{U}$. Note that a positive answer for the nilpotent case of Question 14 would, like 35.21, yield Corollary 35.22 of [50]. A positive answer to the following question would also be good enough for 35.22:

QUESTION 15. *If* $\mathfrak{U} \leq \mathfrak{N}_c$ *and* $[\mathfrak{U}, \mathfrak{E}]$ *is generated by its free group of rank* k, *must the same be true of* $\mathfrak{N}_c \wedge [\mathfrak{U}, \mathfrak{E}]$ *?*

PROBLEM 14 (page 102).

The suggestion that $d(c)$ might be $[c/2] + 1$ (and part of 35.35 of [50]) has to be replaced by $d(c) = c - 1$, established (for $c > 2$) by Kovács, Newman, Pentony [46] and Levin [48]. The latter paper contains also some further information, as does Vaughan-Lee [63]. Much work has been done on the varieties generated by groups of the form $F_k(\mathfrak{N}_c)$ by Chau [20], [21] and especially Pentony [57].

PROBLEM 15 (page 113).

No new information is available.

PROBLEM 16 (page 114).

Unpublished work of Peter M. Neumann together with the results of Groves [31] provide a positive solution for the case of metanilpotent varieties. In fact, they show that if in a metanilpotent variety all finitely generated groups are Hopf, then these groups are also residually finite and satisfy the maximum condition for normal subgroups.

PROBLEM 17 (page 125).

The negative solution is due to Adyan [2]; see also his paper [3] in these Proceedings.

PROBLEM 18 (page 128).

The positive solution was given by Bronšteĭn [9].

PROBLEMS 19, 20 (page 133).

No new information has come to our attention.

PROBLEMS 21, 22 (pages 141, 142).

For the exponent zero case, the solution to both problems is negative: it was obtained independently by Peter M. Neumann [51], [52] and Ol'šanskiĭ [54]. The counterexamples are made by joining $\underline{\underline{A}}$ to a variety of finite exponent which is known to be 'bad' and showing that the join remains 'bad'. One should replace 'exponent zero' by 'torsion-free' to revive these parts of the problem; the prime-power-exponent parts are open. Neumann and Ol'šanskiĭ give a lot of detailed information. Houghton's results on direct decompositions, partly reported in the book [50], have been published in [35] and [35a]. Bryant [13] and Bryce [18] have done much to explore splitting groups.

PROBLEM 23 (page 166).

Unsolved. Heineken and Peter M. Neumann claimed [34] and Jones eventually proved [36] that no variety other than $\underline{\underline{O}}$ contains infinitely many isomorphism types of the nonabelian finite simple groups which are now known. Related questions are whether any variety other than $\underline{\underline{O}}$ can contain an infinite simple locally finite group (Question IV.7 in the book [40] by Kegel and Wehrfritz), and whether any locally finite variety can contain infinite simple groups (Question IV.6 in [40]); see also Kovács [44].

PROBLEM 24 (page 171).

The paper of Burns quoted by Hanna Neumann in the lead-up to this problem had already answered positively the first half, and suggested the alternative which she presumably intended to put here. That, and the second half, have also been answered (at least for some small values of the parameters) in Kovács [43]: the first positively, the second negatively.

PROBLEM 25 (page 174).

The positive solution, and a lot more, was given by Bryant in [12].

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