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The Presentation Rank of a Direct Product of Finite Groups

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1. INTRODUCTION

Let $1 \to R \to F \to G \to 1$ be a minimal free presentation of the finite group G. This means that the sequence is exact and F is a free group of rank equal to the minimum number of generators $d(G)$ of G. Suppose the corresponding relation module $\bar{R} = R/R'$ is decomposed as

$$\bar{R} = A \oplus P,$$

where P is $\mathbb{Z}G$-projective and A has no projective direct summand. By a theorem of Swan [7], $P \otimes \mathbb{Q} \simeq (\mathbb{Q}G)^s$ (the direct sum of s copies of $\mathbb{Q}G$) for some non-negative integer s. It is known that this integer is independent of the particular minimal free presentation and the particular decomposition of $\bar{R}$ (cf. [3, pp. 263–264]). It is therefore an invariant of G that we call the *presentation rank* of G and write $\mathrm{pr}(G) = s$.

It has been shown elsewhere that all 2-generator groups (and therefore all known simple groups) as well as all soluble groups have zero presentation rank ([3, p. 267] and [4], respectively). This might suggest that there do not exist any groups with non-zero presentation rank. Our aim here is to show that this is far from being the case by establishing

THEOREM 1. *Let* $G^{(k)}$ *denote the direct product of* k *copies of* G. *Then* $\lim_{k\to\infty} \mathrm{pr}(G^{(k)})$ *is* ∞ *or* 0 *according as* G *is perfect or not perfect.*

The presentation rank is intimately connected with both $d(G)$ and the minimal number of module generators $d_G(\mathfrak{g})$ of the augmentation ideal $\mathfrak{g}$ of $\mathbb{Z}G$. We have the following result due to Roggenkamp [6].

PROPOSITION 1. $d(G) = d_G(\mathfrak{g}) + \mathrm{pr}(G)$.

We take the opportunity later (Section 2) of giving an alternative proof of this result. Theorem 1 follows from it together with

THEOREM 2. *Let* $D = G_1 \times \cdots \times G_k$. *Then*

$$d_D(\mathfrak{d}) = \max\{d_{G_i}(\mathfrak{g}_i),\ i = 1,\dots, k;\ d(D/D')\}.$$

In particular, if each G_i equals G and G is perfect, then Theorem 2 tells us that $d_D(\mathfrak{d}) = d_G(\mathfrak{g})$, whatever the value of k. However, since a free group of finite rank has only finitely many distinct homomorphisms onto G, it is clear that $\lim_{k\to\infty} d(G^{(k)}) = \infty$. Proposition 1 now shows that $\lim_{k\to\infty} \mathrm{pr}(G^{(k)}) = \infty$ and thus half of Theorem 1 is established.

Suppose G is a non-abelian simple group. Many years ago Philip Hall [5] developed a method for calculating the discontinuities of the function $d(G^{(k)})$. These are then also the discontinuities of $\mathrm{pr}(G^{(k)})$. For example, if G is the alternating group of degree 5, Hall found that $d(G^{(k)}) = 2$ for $k \leqslant 19$ but $d(G^{(20)}) = 3$. Hence $G^{(20)}$ is the smallest direct power of G with non-zero presentation rank.

We complete now the proof of Theorem 1. Assume G is non-perfect and let $d(G/G') = c$. Again let $G_i = G$ in Theorem 2. Clearly $d(D/D') = kc$ and hence if k is large enough $d_D(\mathfrak{d}) = kc$. But Wiegold [9] has recently proved that $d(D) = kc$ for $k > 2d(G)$. Hence $d(D) = d_D(\mathfrak{d})$ for all sufficiently large k. In view of Proposition 1 this establishes the second half of Theorem 1.

Theorem 2 depends, in its turn, on the following result which allows us to work in a suitable modular group algebra.

PROPOSITION 2. *There exists a prime* p *dividing* $|G|$ *such that* $d_G(\mathfrak{g}) = d_G(\mathfrak{g}/p\mathfrak{g})$.

In the terminology of [4], the proposition asserts that $\mathfrak{g}$ *is a Swan module.*

The proof of Theorem 2 involves an argument that leads quickly to a formula expressing $d_G(\mathfrak{g})$ as a function of certain representation theoretic data. To express these we introduce some notation.

For each p dividing $|G|$ and each irreducible $\mathbb{F}_pG$-module M, let $E = \mathrm{Hom}_G(M, M)$. Then

$$\mathrm{Hom}_G(\mathbb{F}_pG, M) \cong E^{r_M},$$

$$H^1(G, M) \cong E^{s_M},$$

for certain non-negative integers r_M, s_M. Further, let $\zeta_M = 0$ if $M = \mathbb{F}_p$, $\zeta_M = 1$ if $M \neq \mathbb{F}_p$; and if a is a rational number, write $[a]$ for the *smallest* integer $\geqslant a$.

THEOREM 3. $d_G(\mathfrak{g}) = \max\{[s_M/r_M + \zeta_M]\}$, *where the maximum is calculated over all irreducible* $\mathbb{F}_pG$*-modules* M *for all* p *dividing* $|G|$.

If G is soluble, then as we have already remarked, $\mathrm{pr}(G) = 0$ and so $d_G(\mathfrak{g}) = d(G)$. In this case, the formula in Theorem 3 is due to Gaschütz [2]. However, the methods of [2] do not seem applicable in the general case.

2. PROOFS

Proof of Proposition 2. Let $m = \max\{d_G(\mathfrak{g}/p\mathfrak{g});\ \text{all } p/|G|\}$. If $m \geqslant 2$ then, by Swan's criterion [8, Lemma 4.4], $\mathfrak{g}$ is a Swan module. There remains the case $m = 1$. We show that then G is cyclic. The proposition is obviously true in that case.

Let $\mathbb{Z}_{(G)} = \{a/b \in \mathbb{Q};\ (b, |G|) = 1\}$. Then $\mathbb{Z}_{(G)}\mathfrak{g}$ can be generated by a single element as $\mathbb{Z}_{(G)}G$-module [8, Lemma 4.3] and so we have an exact sequence

$$0 \to A \to \mathbb{Z}_{(G)}G \to \mathbb{Z}_{(G)}\mathfrak{g} \to 0. \tag{1}$$

Tensoring with $\mathbb{Q}$ over $\mathbb{Z}_{(G)}$ shows that $A \otimes \mathbb{Q} \cong \mathbb{Q}$ and hence A is the trivial $\mathbb{Z}_{(G)}G$-module $\mathbb{Z}_{(G)}$.

We could now proceed exactly as on p. 268 of [3] (Case 2). However, here is an alternative argument not involving cohomology. The image of A in $\mathbb{Z}_{(G)}G$ must be the $\mathbb{Z}_{(G)}$-submodule with generator $\sum_{x \in G} x$. Tensoring (1) with $\mathbb{Z}_{(G)}$ over $\mathbb{Z}_{(G)}G$, we obtain

$$A/A\mathfrak{g} \to \mathbb{Z}_{(G)} \to \mathfrak{g}/\mathfrak{g}^2 \to 0.$$

Since $\mathfrak{g}/\mathfrak{g}^2 \cong G/G'$, this implies

$$\mathbb{Z}/|G|\mathbb{Z} \cong G/G'.$$

It follows that $G' = 1$ and G is cyclic.

Proposition 2 is extremely useful in many contexts. For example it yields a new proof of Roggenkamp's formula (Proposition 1). To see this we argue as follows.

Let d be the minimum number of generators of $\mathbb{Z}_{(G)}\mathfrak{g}$. By Proposition 2, $d = d_G(\mathfrak{g})$. There exists an exact sequence

$$0 \to K \to (\mathbb{Z}_{(G)}G)^d \to \mathbb{Z}_{(G)}\mathfrak{g} \to 0 \tag{2}$$

with K containing no free direct summand. (Recall that all $\mathbb{Z}_{(G)}G$-projectives are free.) We also have the relation sequence

$$0 \to \bar{R} \rightarrowtail (\mathbb{Z}G)^{d(G)} \to \mathfrak{g} \to 0 \tag{3}$$

(e.g. [3, pp. 34, 37]).

Applying Schanuel's lemma to (2) and to (3) tensored with $\mathbb{Z}_{(G)}$ over $\mathbb{Z}$, we obtain

$$(\bar{R} \otimes \mathbb{Z}_{(G)}) \oplus (\mathbb{Z}_{(G)}G)^d \cong K \oplus (\mathbb{Z}_{(G)}G)^{d(G)}. \tag{4}$$

Let $\bar{R} = A \oplus P$, where $P \otimes \mathbb{Q}$ is $\mathbb{Q}G$-free of rank $\operatorname{pr}(G)$. By cancellation, (4) yields $A \otimes \mathbb{Z}_{(G)} \cong K$ and $\operatorname{pr}(G) + d = d(G)$, as required.

We turn now to the *proofs of Theorems* 2 *and* 3. In view of Proposition 2, we have only to establish the mod-p version of both theorems. Moreover, Theorem 2 has clearly only to be proved when $k = 2$, and for Theorem 3 we recall that $r_{\mathbb{F}} = 1$ and $H^1(G, \mathbb{F}_p) \cong G/G'G^p$.

Explicitly, therefore, we must prove the following results.

THEOREM 2p. *If $D = G \times H$, then*

$$d_D(\mathfrak{d}/p\mathfrak{d}) = \max\{d_G(\mathfrak{g}/p\mathfrak{g}), d_H(\mathfrak{h}/p\mathfrak{h}), d(D/D'D^p)\}.$$

THEOREM 3p. $d_G(\mathfrak{g}/p\mathfrak{g}) = \max\{d(G/G'G^p), [s_M/r_M + 1]\}$, *where M varies over all non-trivial irreducible $\mathbb{F}_pG$-modules.*

We begin with the simple

LEMMA 1. *Let Λ be a semi-simple algebra and*

$$W = M_1^{k_1} \oplus \cdots \oplus M_r^{k_r},$$

where $M_1, \ldots, M_r$ are distinct irreducible Λ-modules. If M_i occurs r_i times in Λ, then

$$d_\Lambda(W) = \max\{[k_i/r_i], i = 1, \ldots, r\}.$$

Proof. We assert first that

$$d_\Lambda(W) = \max\{d_\Lambda(M_i^{k_i}), i = 1,..., r\}.$$

For if the right side is n and the left side m, then obviously $m \geqslant n$. But from given projections $\Lambda^n \to M_i^{k_i}$ we obtain, because of the semi-simplicity, a projection $\Lambda^n \to W$. Hence $m \leqslant n$.

Now write $M = M_i$, $k = k_i$, $r = r_i$ and $k = qr + s$, with $0 \leqslant s < r$. If $d_\Lambda(M^k) = n$, then clearly $n \leqslant q$ or $q + 1$ according as $s = 0$ or $s \neq 0$. On the other hand, an epimorphism $\Lambda^n \to M^k$ yields, by semi-simplicity, that $rn \geqslant k$. Hence $n = [k/r]$.

Let J be the Jacobson radical of $\mathbb{F}_pG$ and U the indecomposable projective $\mathbb{F}_pG$-module with $U/UJ \cong \mathbb{F}_p$. If $\mathbb{F}_pG = U \oplus W$, then $\mathbb{F}_p\mathfrak{g}$ $(=\mathfrak{g}/p\mathfrak{g}) = UJ \oplus W$ and $W\mathfrak{g} = W$. Therefore

$$UJ/UJ\mathfrak{g} \cong \mathbb{F}_p\mathfrak{g}/\mathbb{F}_p\mathfrak{g}^2 \cong G/G'G^p$$

and the semi-simple module $\mathbb{F}_p\mathfrak{g}/\mathbb{F}_p\mathfrak{g}J$ breaks up as a direct sum

$$\mathbb{F}_p\mathfrak{g}/\mathbb{F}_p\mathfrak{g}J \cong G/G'G^p \oplus U(G) \oplus W/WJ,$$

where we have written

$$U(G) = UJ\mathfrak{g}/UJ^2.$$

Clearly, $d_G(\mathbb{F}_p\mathfrak{g}) = d_G(\mathbb{F}_p\mathfrak{g}/\mathbb{F}_p\mathfrak{g}J)$. Lemma 1 now yields

$$d_G(\mathbb{F}_p\mathfrak{g}) = \max\{d(G/G'G^p), 1 + d_G(U(G))\}. \tag{5}$$

At this point the proofs of Theorems 2p and 3p diverge. The former is an immediate consequence of (5) and

LEMMA 2. *$U(D) \cong U(G) \oplus U(H)$, where H acts trivially on $U(G)$ and G acts trivially on $U(H)$.*

For then, using also Lemma 1, $d_D(U(D))$ is the maximum of $d_G(U(G))$ and $d_H(U(H))$ and Theorem 2p is finished. We postpone the proof of Lemma 2 in order first to conclude Theorem 3p.

In view of (5), we need to show that

$$d_G(U(G)) = \max\{[s_M/r_M]\};$$

and in view of Lemma 1, this amounts to verifying that s_M is precisely the number of occurrences of M in UJ/UJ^2. But this is immediate from the definition of $H^1(G, -)$ and the use of the resolution $0 \to UJ \to U \to \mathbb{F}_p \to 0$.

Proof of Lemma 2. We begin with some facts about graded modules.[1] Let A be a finite dimensional algebra over a field K and U an A-module with filtration $U = U_0 \geqslant U_1 \geqslant \cdots$. We write $\bar{U}_i = U_i/U_{i+1}$ so that the corresponding graded module is $\mathrm{gr}(U) = \bigoplus_{i=0}^{\infty} \bar{U}_i$. Suppose B is a second finite dimensional algebra over K and V is a B-module with filtration $V = V_0 \geqslant V_1 \geqslant \cdots$. Then $W = U \otimes_K V$ is a module over $A \otimes B$ and has the "product filtration" $W = W_0 \geqslant W_1 \geqslant \cdots$, where $W_n = \sum_{i+j=n} U_i \otimes V_j$. The obvious module homomorphism

$$\mathrm{gr}(U) \otimes \mathrm{gr}(V) \to \mathrm{gr}(U \otimes V) \tag{6}$$

is here an isomorphism of graded $A \otimes B$-modules (e.g. [1, Chapter 3, p. 106]).

Assume next that K is a finite field and that A, B are augmented K-algebras. Let J_A, J_B denote the Jacobson radicals of A, B, respectively. Then A/J_A, B/J_B are separable and so $A/J_A \otimes B/J_B$ is semi-simple. Consequently, in the exact sequence

$$(J_A \otimes B) \oplus (A \otimes J_B) \xrightarrow{\theta} A \otimes B \longrightarrow A/J_A \otimes B/J_B \longrightarrow 0,$$

we have Image $(\theta) \geqslant J_{A\otimes B}$, the Jacobson radical of $A \otimes B$. But Image (θ) is nilpotent and hence

$$J_{A\otimes B} = \text{Image}\ (\theta). \tag{7}$$

Suppose U is the indecomposable projective A-module with $U/UJ_A \cong K$ and similarly V for B. The kernel of φ: $U \otimes V \to K \otimes K = K$ is precisely the image of $(UJ_A \otimes V) \oplus (U \otimes VJ_B)$ and by (7)

$$\mathrm{Ker}(\varphi) = (U \otimes V)\, J_{A\otimes B}. \tag{8}$$

Hence $U \otimes V$ is the indecomposable projective $A \otimes B$-module with image K.

We consider the filtrations $(UJ_A{}^n)$, $(VJ_B{}^n)$ in U, V, respectively. Then (8) shows that the filtration $((U \otimes V)\, J_{A\otimes B}{}^n)$ is exactly the product filtration. The isomorphism (6) in this context and applied only in degree 1 yields

$$(U \otimes V)\, J_{A\otimes B}/(U \otimes V)\, J_{A\otimes B}{}^2 \cong UJ_A/UJ_A{}^2 \oplus VJ_B/VJ_B{}^2, \tag{9}$$

where B operates via K on the first summand on the right hand side and similarly A on the second.

Now let $K = \mathbb{F}_p$, $A = \mathbb{F}_pG$, $B = \mathbb{F}_pH$ and apply $\mathfrak{d}$ to the formula (9). The result is precisely the formula of Lemma 2.

[1] We are grateful to John Moore for drawing these to our attention. Their use has simplified our original argument.

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