TENSOR PRODUCTS OF REPRESENTATIONS OF FINITE GROUPS

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1. Throughout this note K will be a field and G a finite group. By a KG-module we shall mean a finite-dimensional right module for the group algebra KG. A well-known theorem states that if V is a KG-module which is faithful for G then each irreducible KG-module is a composition factor of some tensor power $V^{(r)}$ of V. (See Burnside [**2**; §226], Fong and Gaschütz [**3**], Steinberg [**5**] and Brauer [**1**; Theorem 1* and Remarks 4, 5].) Here we shall give an extremely simple proof of the somewhat stronger fact: *each principal indecomposable KG-module is isomorphic to a direct summand of some tensor power of V.* Our main result is

THEOREM 1. *If, for each $g \in G \setminus 1$, V_g is a KG-module on which g has non-scalar action, then the regular KG-module is isomorphic to a direct summand of*

$$\bigotimes_{g \in G \setminus 1} V_g.$$

Here, and below, we have the convention that any empty tensor product of KG-modules is equal to the trivial one-dimensional KG-module T. By the regular KG-module we mean KG itself, regarded as a KG-module; and an element g of G is said to have scalar action on a KG-module V if $V(k-g) = 0$ for some $k \in K$.

If V is a KG-module, the elements of G which have scalar action on $T \oplus V$ are precisely those which act trivially on V. Thus an immediate consequence of Theorem 1 is

COROLLARY. *If, for each $g \in G \setminus 1$, V_g is a KG-module on which g acts non-trivially, then the regular KG-module is isomorphic to a direct summand of*

$$\bigotimes_{g \in G \setminus 1} (T \oplus V_g) \cong \bigoplus_{S \subseteq G \setminus 1} \left(\bigotimes_{g \in S} V_g \right).$$

If G has order n and V is faithful for G then it follows from this that the regular KG-module is isomorphic to a direct summand of $(T \oplus V)^{(n-1)}$, and so, by the Krull–Schmidt Theorem, each principal indecomposable KG-module is isomorphic to a direct summand of some $V^{(r)}$. In fact we shall prove a refinement of the special case of the corollary.

THEOREM 2. *Let V be a KG-module which is faithful for G and let M be the cyclic subgroup of G consisting of those elements which have scalar action on V. If G has order n, M has order m, and $l \geqslant (n/m)-1$, then the regular KG-module is isomorphic to a direct summand of*

$$V^{(l)} \oplus V^{(l+1)} \oplus \dots \oplus V^{(l+m-1)}.$$

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Under the hypotheses of Theorem 2, if $V^{(s)}, \ldots, V^{(s+m-1)}$ is any sequence of m consecutive powers of V then the restrictions of the $V^{(s+i)}$ to M are homogereous modules associated with the m distinct irreducible KM-modules. Hence the sets of composition factors of the $V^{(s+i)}$ are pairwise disjoint, and a principal indecomposable KG-module can be a direct summand of at most one of the $V^{(s+i)}$. It follows also that no direct sum of fewer than m consecutive powers of V can contain the regular KG-module, or even involve every irreducible KG-module. Brauer **[1]** showed that every irreducible KG-module is a composition factor of

$$W = T \oplus V \oplus \ldots \oplus V^{(c-1)}$$

where c is the cardinality of the set of values of the (Brauer) character of V. However, although $c \geqslant m$, the regular KG-module is not necessarily a direct summand of W: for example, if G is the quaternion group of order 8 and V its 2-dimensional faithful irreducible module over the complex field, then $c = 3$ and so W has dimension only 7.

It is perhaps of interest to note that Theorem 1 may be put in a rather different form. Let G have order n and, for $i = 1, 2, \ldots, n-1$, suppose that V_i is a KG-module and M_i the subgroup of G consisting of those elements which have scalar action on V_i. By the well-known theorem of Hall **[4]**, the elements of $G \setminus 1$ form a system of distinct representatives for the sets $G \setminus M_i$ if and only if, for each r, $1 \leqslant r \leqslant n-1$, the union of any r of the $G \setminus M_i$ has cardinality at least r. Thus the regular KG-module is isomorphic to a direct summand of

$$\bigotimes_{i=1}^{n-1} V_i$$

if, for each r, $1 \leqslant r \leqslant n-1$, the intersection of any r of the M_i has order at most $n-r$.

2. If W is a KG-module and A is a subset of W we shall write $\langle A \rangle$ for the K-subspace of W spanned by A. We need some simple facts which are summarized in the following lemma.

LEMMA. (i) *The element g of G has non-scalar action on the KG-module W if and only if $w \notin \langle wg \rangle$ for some $w \in W$.* (ii) *The regular KG-module is isomorphic to a direct summand of the KG-module W if and only if $w \notin \langle w(G \setminus 1) \rangle$ for some $w \in W$.* (iii) *If U and V are KG-modules, R and S are subsets of $G \setminus 1$, $u \in U$ is such that $u \notin \langle uR \rangle$ and $v \in V$ is such that $v \notin \langle vS \rangle$, then $u \otimes v \in U \otimes V$ satisfies*

$$u \otimes v \notin \langle (u \otimes v)(R \cup S) \rangle.$$

Proof. (i) is easy. To prove (ii) we use the fact that the regular KG-module is injective. Thus it is isomorphic to a direct summand of W if and only if it is isomorphic to a submodule of W. This holds if and only if for some $w \in W$ the set wG is linearly independent over K. But $w \notin \langle w(G \setminus 1) \rangle$ if and only if

$$wg \notin \langle w(G \setminus 1)g \rangle = \langle w(G \setminus g) \rangle$$

for all $g \in G$.

Finally we prove (iii). Since $u \notin \langle uR \rangle$, U has a K-space endomorphism α satisfying $u\alpha = u$ and $uR\alpha = 0$. Similarly V has a K-endomorphism β satisfying $v\beta = v$ and $vS\beta = 0$. If $g \in R \cup S$ then

$$(u \otimes v)\, g(\alpha \otimes \beta) = ug\alpha \otimes vg\beta = 0,$$

since $ug\alpha = 0$ or $vg\beta = 0$. Thus

$$\langle (u \otimes v)(R \cup S) \rangle (\alpha \otimes \beta) = 0.$$

But

$$(u \otimes v)(\alpha \otimes \beta) = u \otimes v \neq 0,$$

and (iii) follows.

To prove Theorem 1, we take, using (i), for each $g \in G \setminus 1$ an element v_g of V_g such that $v_g \notin \langle v_g\, g \rangle$. Thus, by repeated application of (iii),

$$w = \bigotimes_{g \in G \setminus 1} v_g \in \bigotimes_{g \in G \setminus 1} V_g$$

satisfies $w \notin \langle w(G \setminus 1) \rangle$; and the theorem follows by (ii).

Under the hypotheses of Theorem 2, we have already observed that for any s the restriction of

$$U = V^{(s)} \oplus V^{(s+1)} \oplus \ldots \oplus V^{(s+m-1)}$$

to M contains every irreducible KM-module, and hence a regular KM-module. Thus, by (ii), there is $u \in U$ such that $u \notin \langle u(M \setminus 1) \rangle$. Let D be a set of representatives for the non-trivial cosets of M in G. Then, by (i), for each $d \in D$ there is $v_d \in V$ such that $v_d \notin \langle v_d\, d \rangle$; whence also $v_d \notin \langle v_d\, dM \rangle$. Since

$$G \setminus 1 = (M \setminus 1) \cup \Big(\bigcup_{d \in D} dM \Big),$$

repeated application of (iii) shows that

$$w = u \otimes \Big(\bigotimes_{d \in D} v_d \Big) \in U \otimes V^{((n/m)-1)}$$

satisfies $w \notin \langle w(G \setminus 1) \rangle$; and the theorem follows by (ii).

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