

ON NON-CROSS VARIETIES OF GROUPS

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To Bernhard Hermann Neumann on his 60th birthday

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1. Introduction

Our title has become something of a misnomer, however we retain it since drafts of this note have been quoted with it.

Unless otherwise stated our terminology and notation follow that in Hanna Neumann's book [12].

The Oates-Powell Theorem ([12] p. 151) allows us to say that a variety is *Cross* if and only if it can be generated by a finite group, and to assert that the laws of a Cross variety are finitely based. A variety is *just-non-Cross* if it is not Cross but every proper subvariety of it is Cross.

We asked in [9]: what non-Cross varieties have just-non-Cross subvarieties? The answer is: all of them.

THEOREM 1. *Every non-Cross variety has a just-non-Cross subvariety.*

The proof is an easy application of Zorn's Lemma. If $\{\mathfrak{B}_\lambda : \lambda \in A\}$ is a descending chain of non-Cross subvarieties of a non-Cross variety such that the intersection $\mathfrak{D} = \bigwedge \{\mathfrak{B}_\lambda : \lambda \in A\}$ is properly contained in each \mathfrak{B}_λ , then the union of the corresponding chain $\{B_\lambda : \lambda \in A\}$ of fully invariant subgroups of the word group X_∞ ([12] p. 4) is not finitely generated, hence \mathfrak{D} is not finitely based, and a fortiori \mathfrak{D} is still non-Cross.

In [9] we claimed that for every prime p the product variety $\mathfrak{A}_p \mathfrak{A}_p$ is just-non-Cross (\mathfrak{A}_p is the variety of abelian groups of exponent dividing p). Here we substantiate this as a consequence of a detailed description, in section 2, of the lattice of subvarieties of $\mathfrak{A}_{p^\infty} \mathfrak{A}_p$.

The variety \mathfrak{A} of all abelian groups and the varieties $\mathfrak{A}_p \mathfrak{A}_p$ are just-non-Cross and nilpotent-by-abelian. The converse is also true.

THEOREM 2. *The only nilpotent-by-abelian just-non-Cross varieties are \mathfrak{A} and the $\mathfrak{A}_p \mathfrak{A}_p$.*

This theorem is related to the so-called external result we state in section 3, and is proved with it in section 5.

2. The subvariety lattice of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$

In this section we give a description of the lattice of subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$. Proofs are deferred to section 4.

Lattice terminology follows Birkhoff [1].

We begin with some notation. The set of positive integers is denoted by P . As usual \mathfrak{A}_n , \mathfrak{B}_n , \mathfrak{N}_n denote, respectively, the variety of abelian groups of exponent dividing n , the variety of groups of exponent dividing n , and the variety of groups of nilpotency class at most n . The variety of all groups will, for convenience, be denoted \mathfrak{N}_ω . Our description of the subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ will be in terms of these varieties and one more family whose members will be denoted \mathfrak{N}_{n*} . The variety \mathfrak{N}_{n*} is the subvariety of \mathfrak{N}_n defined by the additional law $\prod_{s=2}^n [x_s, x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_n]$. Note that $\mathfrak{N}_{n*} \supseteq \mathfrak{N}_{n-1}$. For any particular prime p only certain of these additional varieties are needed, namely those for which n is at least 3 and is divisible by p . We therefore introduce for each prime p an ordered extension $P(p)$ of P defined by:

$$\begin{aligned} P(p) &= \{1, \dots, p-1, p*, p, \dots, pr-1, pr*, pr, \dots, \omega\} \quad \text{for } p \text{ odd,} \\ P(2) &= \{1, 2, 3, 4*, 4, \dots, 2r-1, 2r*, 2r, \dots, \omega\} \end{aligned}$$

with the order as indicated. The $P(p)$ and $\{0, 1, \dots, \alpha+1\}$ taken in this order may be considered as lattices – we do this. For each p the varieties \mathfrak{B}_{p^β} and $\mathfrak{A}_{p^\tau}\mathfrak{N}_v$ for v in $P(p)$ play a distinguished role. We denote them $\mathfrak{B}(\beta)$ and $\mathfrak{N}(\tau, v)$ respectively.

With each subvariety \mathfrak{B} of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ we associate an element $\beta(\mathfrak{B})$ of $\{0, \dots, \alpha+1\}$ and elements $v(0, \mathfrak{B}), \dots, v(\alpha-1, \mathfrak{B})$ of $P(p)$ as follows:

$$\begin{aligned} \beta(\mathfrak{B}) &= \min \{\beta : \mathfrak{B} \subseteq \mathfrak{B}(\beta)\}; \\ \text{for } \tau \in \{0, \dots, \alpha-1\}, \\ v(\tau, \mathfrak{B}) &= \min \{v : \mathfrak{B} \subseteq \mathfrak{N}(\tau, v)\}. \end{aligned}$$

The subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$ are characterized by the above invariants:

2.1 *If \mathfrak{B} is a subvariety of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$, then*

$$\mathfrak{B} = \mathfrak{A}_{p^\alpha}\mathfrak{A}_p \wedge \mathfrak{B}(\beta(\mathfrak{B})) \wedge \bigwedge_{\tau=0}^{\alpha-1} \mathfrak{N}(\tau, v(\tau, \mathfrak{B})).$$

If $\alpha = 1$, it follows that every proper subvariety of $\mathfrak{A}_p\mathfrak{A}_p$ is nilpotent, and hence Cross because it has finite exponent. As $\mathfrak{A}_p\mathfrak{A}_p$ is obviously not Cross, this yields the following.

THEOREM 3. *For every prime p the variety $\mathfrak{A}_p\mathfrak{A}_p$ is just-non-Cross.*

This discharges a debt incurred in [9]. The proof here – due primarily to one of us (MFN) – supersedes an earlier one which motivated the papers [4], [5] (and in which the result was also announced).

It is clear that for all subvarieties $\mathfrak{U}, \mathfrak{B}$ of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$

$$\beta(\mathbb{U} \vee \mathbb{B}) = \max \{\beta(\mathbb{U}), \beta(\mathbb{B})\}$$

and

$$v(\tau, \mathbb{U} \vee \mathbb{B}) = \max \{v(\tau, \mathbb{U}), v(\tau, \mathbb{B})\}$$

for all τ in $\{0, \dots, \alpha-1\}$. The next point to prove is that the corresponding result for meets also holds.

2.2 For all subvarieties \mathbb{U}, \mathbb{B} of $\mathfrak{A}_{p^*}\mathfrak{A}_p$

$$\beta(\mathbb{U} \wedge \mathbb{B}) = \min \{\beta(\mathbb{U}), \beta(\mathbb{B})\}$$

and

$$v(\tau, \mathbb{U} \wedge \mathbb{B}) = \min \{v(\tau, \mathbb{U}), v(\tau, \mathbb{B})\}$$

for all τ in $\{0, \dots, \alpha-1\}$.

Now it follows from 2.1 that the mapping $\chi: \mathbb{B} \mapsto (\beta(\mathbb{B}), v(0, \mathbb{B}), \dots, v(\alpha-1, \mathbb{B}))$ is an embedding of the lattice of subvarieties of $\mathfrak{A}_{p^*}\mathfrak{A}_p$ into the direct product \mathcal{A} of the lattice $\{0, \dots, \alpha+1\}$ with α copies of $P(p)$. A sublattice of a direct product of distributive lattices with descending chain condition is a distributive lattice with descending chain condition.

THEOREM 4. *The lattice of subvarieties of $\mathfrak{A}_{p^*}\mathfrak{A}_p$ is distributive with descending chain condition.*

The description of the lattice of subvarieties of $\mathfrak{A}_{p^*}\mathfrak{A}_p$ is now completed by giving its image under χ . Let Σ be the subset of the direct product lattice \mathcal{A} defined by:

$(\beta, v_0, \dots, v_{\alpha-1}) \in \Sigma$ if and only if

$$\begin{aligned} v_\beta &= \dots = v_{\alpha-1} = 1 && \text{for } \beta < \alpha, \\ v_{\beta-1} &< p && \text{for } 1 \leq \beta \leq \alpha; \\ v_{\tau+1} &\leq \begin{cases} v_\tau & \text{for } v_\tau \in \{1, \omega\}, \\ v_\tau - p + 1 & \text{for } v_\tau \in P \text{ and } v_\tau > p, \\ pr & \text{for } v_\tau = p(r+1)* \text{ with } r \in P, \\ 2 & \text{for } 2 \leq v_\tau \leq p; \end{cases} \\ v_{\tau+2} &= 1 && \text{for } v_\tau \leq 2p-1. \end{aligned}$$

2.3 The image of χ is Σ .

While the description of the lattice of subvarieties of $\mathfrak{A}_{p^*}\mathfrak{A}_p$ afforded by all this is adequate, it is somewhat *ad hoc*. Because the lattice is distributive with descending chain condition, it follows (cf. section 2 of Chapter VIII of [1] – suitably corrected) that every element of the lattice can be uniquely written as an irredundant finite join of (finitely) join-irreducible elements. Moreover, a finite set of join-irreducibles gives its join irredundantly if and only if no two distinct elements of the set are comparable. Hence such a lattice can easily be reconstructed from the partially ordered set of its join-irreducible elements. The reconstruction

can be carried out so as to yield a faithful representation of the lattice in the lattice of all subsets of the set of its join-irreducible elements. These facts suggest that a canonical way of describing such lattices is to give the partially ordered sets of their join-irreducible elements. We do this for the lattice of subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$. An advantage of this approach is that our results are then more readily comparable with related results of Brooks [2] and Bryce [3], and better suited for the extension of the present results to a description of the subvarieties of $\mathfrak{A}\mathfrak{A}_n$ with square-free n (to be given in [10]).

Given 2.3 and the explicit description of the sublattice Σ of \mathcal{A} , it is an elementary exercise to derive the desired information. We simply give the result after a hint to the derivation we used.

If an element $(\beta, v_0, \dots, v_{\alpha-1})$ of Σ is join-irreducible, then $(\beta', v_0, \dots, v_{\alpha-1}) \notin \Sigma$ for $\beta' < \beta$ because

$$(\beta, v_0, \dots, v_{\alpha-1}) = (\beta, 1, \dots, 1) \vee (\beta', v_0, \dots, v_{\alpha-1}).$$

Similarly $(\beta, v_0, \dots, v_{\tau-1}, \mu, v_{\tau+1}, \dots, v_{\alpha-1}) \notin \Sigma$ for $\mu < v_\tau$ and $\tau \in \{0, \dots, \alpha-2\}$. Hence if $v_{\alpha-1} = v \neq 1$, the conditions defining Σ determine $\beta, v_0, \dots, v_{\alpha-2}$:

$$\begin{aligned} \beta &= \begin{cases} \alpha & \text{for } v < p, \\ \alpha+1 & \text{for } v \geq p; \end{cases} \\ v_\tau &= \begin{cases} \omega & \text{for } v = \omega, \\ \langle v \rangle + (p-1)(\alpha-1-\tau) & \text{for } v \neq \omega \quad \text{except } v = 2, \tau \in \{\alpha-3, \alpha-2\}; \end{cases} \\ v_{\alpha-2} = 2, v_{\alpha-3} = 2p^* & \quad \text{for } v = 2: \end{aligned}$$

here, and in the sequel, $v \mapsto \langle v \rangle$ denotes the mapping of $P(p) \setminus \{\omega\}$ to P which is the identity on P and for which $\langle pr^* \rangle = pr$ whenever $pr^* \in P(p) \setminus P$. Finally, if $v_{\alpha-1} = 1$, then $\beta = \alpha+1$ or the corresponding variety lies in $\mathfrak{A}_{p^{\alpha-1}}\mathfrak{A}_p$; if $\beta = \alpha+1$, then $v_0 = \dots = v_{\alpha-2} = 1$. It is straightforward to check that the resulting elements of Σ are join-irreducible.

We can now describe the partially ordered set $J(p^\alpha)$ of the join-irreducible subvarieties of $\mathfrak{A}_{p^\alpha}\mathfrak{A}_p$. Clearly $J(p^0)$ consists of \mathfrak{E} and \mathfrak{A}_p with $\mathfrak{E} \subset \mathfrak{A}_p$. For α in P the set $J(p^\alpha)$ consists of $J(p^{\alpha-1})$ and for each v in $P(p)$ a variety $\mathfrak{J}(p^\alpha, v)$ defined as follows:

$$\begin{aligned} \mathfrak{J}(p^\alpha, 1) &= \mathfrak{A}_{p^{\alpha+1}}; \\ \mathfrak{J}(p^\alpha, 2) &= \mathfrak{A}_{p^\alpha}\mathfrak{A}_p \wedge \mathfrak{B}_{p^\alpha} \wedge \mathfrak{N}_{2+(p-1)(\alpha-1)} \wedge \mathfrak{A}_{p^{\alpha-3}}\mathfrak{N}_{2p^*} \wedge \mathfrak{A}_{p^{\alpha-2}}\mathfrak{N}_2 \end{aligned}$$

here the second term must be omitted when $p = 2$, and the fourth and fifth when they are not meaningful (also, the third term is redundant when α is 2 or 3); for $v \in P(p) \setminus \{1, 2, \omega\}$,

$$\mathfrak{J}(p^\alpha, v) = \mathfrak{A}_{p^\alpha}\mathfrak{A}_p \wedge \mathfrak{B}_{p^\alpha} \wedge \mathfrak{N}_{\langle v \rangle + (p-1)(\alpha-1)} \wedge \mathfrak{A}_{p^{\alpha-1}}\mathfrak{N}_v$$

here the second term must be omitted when $v \geq p$ and the last term is redundant when $v \in P$; and

$$\mathfrak{F}(p^\alpha, \omega) = \mathfrak{A}_{p^\alpha} \mathfrak{A}_p.$$

Note that the only non-nilpotent join-irreducible varieties in $\mathfrak{A}_{p^\alpha} \mathfrak{A}_p$ are the $\mathfrak{A}_{p^\tau} \mathfrak{A}_p$ with $\tau \in \{1, \dots, \alpha\}$. In contrast to this Brooks [2] has shown that there is an infinite number of non-nilpotent join-irreducible subvarieties in $\mathfrak{A}_p \mathfrak{A}_{p^2}$.

It is a routine matter to check that the partial order on $J(p^\alpha)$ is generated by that on $J(p^{\alpha-1})$ and the inclusions:

$$\mathfrak{A}_p \subset \mathfrak{F}(p^\alpha, 1) \subset \mathfrak{F}(p^\alpha, p),$$

$$\mathfrak{A}_p \subset \mathfrak{F}(p^\alpha, 2),$$

$$\mathfrak{F}(p^\alpha, \mu) \subset \mathfrak{F}(p^\alpha, \nu) \quad \text{whenever } \mu, \nu \in P(p) \text{ and } 2 \leq \mu < \nu;$$

if $\alpha > 1$ then also

$$\mathfrak{F}(p^{\alpha-1}, 1) \subset \mathfrak{F}(p^\alpha, 1),$$

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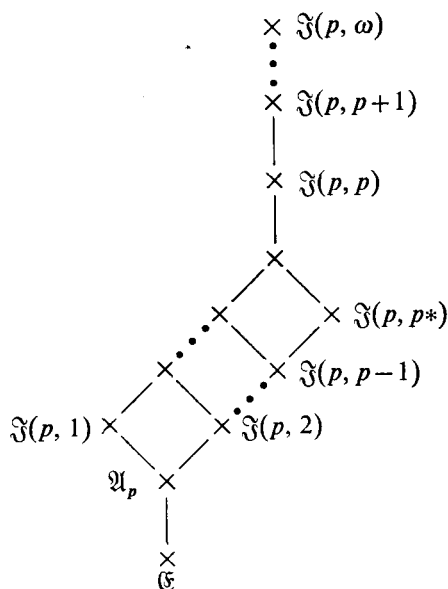
$$\mathfrak{F}(p^{\alpha-1}, \langle v \rangle + p - 1) \subset \mathfrak{F}(p^\alpha, \nu) \quad \text{for all } \nu \text{ in } P(p) \setminus \{1, 2, \omega\},$$

$$\mathfrak{F}(p^{\alpha-1}, \omega) \subset \mathfrak{F}(p^\alpha, \omega);$$

and if $\alpha > 2$ then

$$\mathfrak{F}(p^{\alpha-2}, 2p^*) \subset \mathfrak{F}(p^\alpha, 2).$$

It is easy to indicate diagrammatically the lattice in the case $\alpha = 1$ and, say, $p \neq 2$:



3. External result on $\mathfrak{A}_p\mathfrak{A}_p$

By an external result on a variety \mathfrak{B} we mean a result of the form: A variety which does not contain $\mathfrak{B} \cdots$. For example, a variety which does not contain \mathfrak{A} has finite exponent. For $\mathfrak{A}_p\mathfrak{A}_p$ we can prove the following.

THEOREM 5. *A soluble variety which does not contain $\mathfrak{A}_p\mathfrak{A}_p$ cannot contain any non-nilpotent p -group and therefore has a bound on the nilpotency class of its p -groups.*

The proof is given in section 5.

One might hope for a stronger result which we state as a problem.

In a variety which does not contain $\mathfrak{A}_p\mathfrak{A}_p$ is every locally finite p -group nilpotent?

The local finiteness is needed in view of the result of Novikov-Adyan [13] which implies that for all large enough primes p there are infinite finitely generated groups of exponent p . Note also that this result implies the existence of just-non-Cross varieties of exponent p .

A special case of the above problem is the well-known question: is there a bound on the nilpotency class of finite groups of exponent p ?

It is perhaps worth recording some consequences of Theorem 5.

COROLLARY 1. *A soluble variety which does not contain $\mathfrak{A}_p\mathfrak{A}_p$ has a bound on the nilpotency class of nilpotent torsion free groups in it.*

COROLLARY 2. *A soluble variety in which the nilpotent groups do not form a subvariety contains $\mathfrak{A}_p\mathfrak{A}_p$ for some prime p .*

This discharges another debt incurred in [9].

COROLLARY 3. *The variety generated by the two-generator free metabelian-of-exponent- q groups for an infinite set of primes q contains $\mathfrak{A}_p\mathfrak{A}_p$ for some prime p .*

The last statement is in fact valid for all p but this requires additional argument which is not given in this note.

4. Proofs for section 2

Most of the discussion is set in a free group H of $\mathfrak{A}_p\mathfrak{A}_p$ of countably infinite rank freely generated by $\{a_i : i \in P\}$. Much of the argument will involve the verbal subgroups $\mathfrak{A}_p(H)$, $\mathfrak{B}(\beta)(H)$ and $\mathfrak{N}(\tau, \nu)(H)$; we denote them A_p , $B(\beta)$, $N(\tau, \nu)$ respectively.

We first write down relationships between the subgroups $N(\tau, \nu)$. The first two are obvious:

4.01 *For all τ in $\{0\} \cup P$ and all $\mu \leq \nu$ in $P(p)$,*

$$N(\tau, \mu) \geq N(\tau, \nu) \text{ and } \mathfrak{A}_p(N(\tau, \nu)) = N(\tau + 1, \nu).$$

Further relations are easy consequences of some well-known results about

commutators. We record here all such results which are used frequently in what follows. Notation: $[u, v] = u^{-1}v^{-1}uv$, $[u, v, w] = [[u, v], w]$, $[u, 0v] = u$ and $[u, nv] = [u, (n-1)v, v]$ for all n in P . The identity is denoted e .

4.02 For h, h_1, h_2, \dots in H and d in A_p ,

$$\begin{aligned} [h_1, h_2 h_3] &= [h_1, h_2][h_1, h_3][h_1, h_2, h_3]; \\ [h_1 h_2, h_3] &= [h_1, h_3][h_2, h_3][h_1, h_3, h_2]; \\ [h_1, h_2, h_3] &= [h_1, h_3, h_2][h_3, h_2, h_1]; \\ [d, h_1, \dots, h_m] &= [d, h_{1\pi}, \dots, h_{m\pi}] \end{aligned}$$

for all m in P and all permutations π of $\{1, \dots, m\}$;

$$[h_1, h_2^m] = \prod_{i=1}^m [h_1, ih_2]^{m!/i!(m-i)!} \quad \text{for all } m \text{ in } P;$$

in particular, since $[d, h^p] = e$ and $[h_1^p, h_2^p] = e$,

$$\prod_{i=1}^p [d, ih]^{p!/i!(p-i)!} = e$$

and

$$\prod_{i=1}^p \prod_{j=1}^p [h_2, ih_1, (j-1)h_2]^{(p!)^2/i!j!(p-i)!(p-j)!} = e.$$

The last two equations have the following immediate consequences.

4.03 For all τ and all n in P ,

$$N(\tau+1, n+1) \leq N(\tau, n+p) \text{ and } N(\tau+2, 1) \leq N(\tau, 2p-1).$$

In fact a little more is true.

4.04 For all τ and all r in P ,

$$N(\tau+1, pr) \leq N(\tau, p(r+1)*).$$

Before proving this, we introduce some further notation. For s, n in P with $2 \leq s < n$, let

$$b(s, s) = [a_s, a_1, a_2, \dots, a_{s-1}]$$

and

$$b(s, n) = [b(s, n-1), a_n].$$

We denote by ι the identity endomorphism of H , and by $\pi_{i,j}$ with i, j in P the endomorphism which fixes all the generators except a_i, a_j which it interchanges.

PROOF OF 4.04. Let ψ be the endomorphism of H which maps a_j to a_{pr+1} if $pr+1 \leq j \leq p(r+1)$ and to a_j otherwise. From

$$\prod_{s=2}^{p(r+1)} (b(s, p(r+1))\psi)^{p^r} \in N(\tau, p(r+1)*)$$

it is easy to derive, using 4.02 and the inclusions

$$N(\tau+1, pr+1) \leq N(\tau, p(r+1)) \leq N(\tau, p(r+1)*)$$

(which hold on account of 4.03 and 4.01), that

$$h = \prod_{s=2}^{pr} b(s, pr+1)^{p^{r+1}} \in N(\tau, p(r+1)*).$$

Then, applying $\iota - \pi_{2, pr+1}$ to h and using 4.02, one gets

$$[a_2, a_{pr+1}, a_1, a_3, a_4, \dots, a_{pr}]^{p^{r+1}} \in N(\tau, p(r+1)*)$$

and the result follows.

The story is completed by obtaining suitable generating sets for the $N(\tau, v)$.

Let \mathcal{B} be the subset of H defined by: $b \in \mathcal{B}$ if and only if $b = [a_i, m_j a_j, m_{j+1} a_{j+1}, \dots, m_s a_s]$ where $i > j \leq s$; $m_j - 1, m_{j+1}, \dots, m_s \in \{0, \dots, p-1\}$, $m_s \neq 0$; and if $m_j = p$ then firstly $i \leq s$ implies $m_i < p-1$ and secondly $m_k = 0$ whenever $j < k < i$ and $k \leq s$.

4.05 The set $\mathcal{B} \cup \{a_i^p : i \in P\}$ is a free generating set for A_p as free \mathfrak{A}_{p^*} -group.

PROOF. It follows easily from 4.02 that A_p is generated by $\mathcal{B}^* = \mathcal{B} \cup \{a_i^p : i \in P\}$. If there were a non-trivial relation between the elements of \mathcal{B}^* this would involve only finitely many of $\{a_i : i \in P\}$. It therefore suffices to consider for each k in P the subgroup H_k of H generated by $\{a_1, \dots, a_k\}$ and to show that $\mathfrak{A}_p(H_k) \cap \mathcal{B}^*$ is independent in $\mathfrak{A}_p(H_k)$. By the Schreier formula for the rank of subgroups of absolutely free groups, $\mathfrak{A}_p(H_k)$ has rank $(k-1)p^k + 1$. On the other hand the number of elements in $\mathfrak{A}_p(H_k) \cap \mathcal{B}^*$ is

$$k + \sum_{j=1}^{k-1} \{(k-j)(p-1)p^{k-j} + \sum_{i=j+1}^k (p-1)p^{k-i}\}$$

where the first term in $\{\dots\}$ comes from counting the commutators with $m_j \neq p$ and the second term from the rest. The sum comes to $(k-1)p^k + 1$ and the result follows.

Note that this proof implies that every element of \mathcal{B} can be uniquely written in the way it is defined.

It follows that the commutator subgroup $N(0, 1)$ of H is a free \mathfrak{A}_{p^*} -group freely generated by \mathcal{B} . The other terms $N(0, n)$ of the lower central series of H are a little more complicated to describe. This we do next after first defining weights for elements of \mathcal{B} .

The weight $\text{wt}(b)$ of an element $b = [a_i, m_j a_j, \dots, m_s a_s]$ of \mathcal{B} is $1 + \sum_{k=j}^s m_k$. The weight $\text{wt}(b, a_k)$ of b in the generator a_k is 'the number of occurrences of a_k in b ', that is,

$$\text{wt}(b, a_k) = \begin{cases} 0 & \text{if } k \notin \{j, \dots, s\} \cup \{i\}, \\ m_k & \text{if } k \in \{j, \dots, s\} \setminus \{i\}, \\ m_k + 1 & \text{if } k = i \in \{j, \dots, s\}, \\ 1 & \text{if } k = i \notin \{j, \dots, s\}. \end{cases}$$

4.06 For n in P the subgroup $N(0, n)$ is generated by the set \mathcal{B}_n of elements of the form b^{p^k} where $b \in \mathcal{B}$, $k \in \{0, \dots, \alpha-1\}$, $\text{wt}(b) + k(p-1) > n$ and $k \text{ wt}(b) \neq 2$ unless $n = 1$.

PROOF. A routine argument from 4.02 shows that $N(\tau, n)$ is generated by the $[a_{i_1}, \dots, a_{i_m}]^{p^k}$ with $m > n$, $k \in \{\tau, \dots, \alpha-1\}$ and $i_1 > i_2 \leq i_3 \leq \dots \leq i_m$. From this and 4.03 one gets immediately that $N(0, n)$ is generated by the set \mathcal{S} of the elements $[a_{i_1}, \dots, a_{i_m}]^{p^k}$ with $m \geq 2$, $k \in \{0, \dots, \alpha-1\}$, $km \neq 2$ unless $n = 1$, $m + (p-1)k > n$ and $i_1 > i_2 \leq \dots \leq i_m$. Clearly \mathcal{B}_n is a subset of \mathcal{S} . An induction on m using 4.02 shows that each element of \mathcal{S} lies in the subgroup generated by \mathcal{B}_n . The result follows.

Note that if $b^{p^k} \in \mathcal{B}_n$ and $k < \alpha-1$, then $b^{p^{k+1}} \in \mathcal{B}_n$. Thus for all n every element of $N(0, n)$ can be uniquely written (up to order) in the form $\prod_{i=1}^t b_i^{\beta(i)}$ where the b_i are distinct elements of \mathcal{B}_n and the $\beta(i) \in \{1, \dots, p-1\}$.

Similar generating sets can be given for the $N(0, pr^*)$.

4.07 For pr^* in $P(p)$ the subgroup $N(0, pr^*)$ is generated by the union \mathcal{B}_{pr^*} of

- (i) \mathcal{B}_{pr^*} ,
- (ii) the set of elements of the form $b^{p^{k+1}}$ where $b \in \mathcal{B}$, $k \in \{0, \dots, \alpha-2\}$, $\text{wt}(b) + (k+1)(p-1) = pr$, and $k \text{ wt}(b) \neq 2$, and
- (iii) the set of elements of the form $\prod_{s=2}^{pr} b(s, pr)\psi$ where ψ is an endomorphism of H such that $a_j\psi = a_{i_j}$ where $i_1 \leq i_2 \leq \dots \leq i_{pr}$ and no p of i_2, \dots, i_{pr} are equal.

PROOF. The argument is essentially the same as that in the proof of 4.06. It is routine to derive from 4.02 that $N(\tau, pr^*)$ is generated by the $[a_{i_1}, \dots, a_{i_m}]^{p^k}$ with $m \geq pr+1$, $k \in \{\tau, \dots, \alpha-1\}$ and $i_1 > i_2 \leq \dots \leq i_m$, and the

$$\prod_{s=2}^{pr} [a_{i_s}, a_{i_1}, a_{i_2}, \dots, a_{i_{s-1}}, a_{i_{s+1}}, \dots, a_{i_{pr}}]^{p^k} \quad \text{with } i_1 \leq i_2 \leq \dots \leq i_{pr}.$$

Hence, by 4.03 and 4.04, $N(0, pr^*)$ is generated by the set of elements $[a_{i_1}, \dots, a_{i_m}]^{p^k}$ with $m \geq 2$, $k \in \{0, \dots, \alpha-1\}$, $km \neq 2$, $(k-1)m\delta_{2r} \neq 2$, $m + (p-1)k \geq pr + \delta_{k0}$ (where $\delta_{uv} = 1$ if $u = v$ and 0 if $u \neq v$), and $i_1 > i_2 \leq i_3 \leq \dots \leq i_m$, and the elements

$$\prod_{s=2}^{pr} [a_{i_s}, a_{i_1}, \dots, a_{i_{s-1}}, a_{i_{s+1}}, \dots, a_{i_{pr}}] \quad \text{with } i_1 \leq i_2 \leq \dots \leq i_{pr}.$$

An induction on m (using some ideas from the proof of 4.04) then yields the result.

Observe that an element of \mathcal{B} occurs in at most one of the products in (iii) above. It follows that *every element of $N(0, pr^*)$ can be uniquely written (up to order) as $\prod_{i=1}^t b_i^{\beta(i)}$ where the b_i are distinct elements of \mathcal{B}_{pr^*} and each $\beta(i) \in \{1, \dots, p-1\}$.*

It is a straight-forward consequence of 4.01 and the remarks after the proofs of 4.06 and 4.07 that *the $N(\tau, v)$ for τ in $\{0, \dots, \alpha-1\}$ and v in $P(p) \setminus \{\omega\}$ are non-trivial and distinct* and that the following relations between them hold:

4.08 For all τ ,

$$\begin{aligned} N(\tau, n) \cap N(\tau+1, 1) &= \begin{cases} N(\tau+1, n-p+1) & \text{for } 2p \leq n \in P, \\ N(\tau+1, n-p+1)N(\tau+2, 1) & \text{for } p < n < 2p, \\ N(\tau+1, 2)N(\tau+2, 1) & \text{for } 2 \leq n \leq p; \end{cases} \\ N(\tau, pr^*) \cap N(\tau+1, 1) &= \begin{cases} N(\tau+1, p(r-1)) & \text{for } 1 \neq r \in P, \\ N(\tau+1, 2)N(\tau+2, 1) & \text{for } r = 1; \end{cases} \\ \bigcap_{n \in P} N(\tau, n)N(\tau+1, 1) &= N(\tau+1, 1). \end{aligned}$$

The next step is to prove that every fully invariant subgroup of H can be expressed in terms of the $B(\beta)$ and the $N(\tau, v)$.

4.09 If V is a fully invariant subgroup of H , then there is a unique element, call it $\beta(V)$, in $\{0, \dots, \alpha+1\}$ such that $V = B(\beta(V))(V \cap N(0, 1))$.

Observe that if \mathfrak{B} is a subvariety of $\mathfrak{A}_{p^*}\mathfrak{A}_p$, then $\beta(\mathfrak{B})$ (see section 2) is the same as $\beta(\mathfrak{B}(H))$.

PROOF OF 4.09. Recall that $N(0, 1)$ is the commutator subgroup of H . Clearly there is precisely one β in $\{0, \dots, \alpha+1\}$ such that $VN(0, 1) = B(\beta)N(0, 1)$. Then $a_1^{p^\beta} = vd$ where $v \in V$, $d \in N(0, 1)$. Applying to this the endomorphism of H which maps a_1 to a_1 and all the other generators to the identity yields $a_1^{p^\beta} \in V$. Thus $B(\beta) \leq V$ and the result follows.

4.10 For τ in $\{0, \dots, \alpha-1\}$, if V is a fully invariant subgroup of H contained in $N(\tau, 1)$, then there is just one element $v(\tau, V)$ of $P(p)$ such that $V = N(\tau, v(\tau, V))(V \cap N(\tau+1, 1))$.

Observe that if \mathfrak{B} is a subvariety of $\mathfrak{A}_{p^*}\mathfrak{A}_p$ then

$$v(\tau, \mathfrak{B}) = v(\tau, \mathfrak{B} \vee (\mathfrak{N}(\tau, 1) \wedge \mathfrak{A}_{p^*}\mathfrak{A}_p))$$

so that

$$v(\tau, \mathfrak{B}) = v(\tau, \mathfrak{B}(H) \cap N(\tau, 1)).$$

4.10 is proved in two stages. The first will be stated as a separate result. The endomorphism of H which maps a_j to e and fixes the other generators will be denoted δ_j .

4.11 Let $n-1$ be in P and τ in $\{0, \dots, \alpha-1\}$. If p does not divide n or if $p = n = 2$,

there is no fully invariant subgroup of H strictly between $N(\tau, n)N(\tau+1, 1)$ and $N(\tau, n-1)N(\tau+1, 1)$. Otherwise $n = pr$ and $N(\tau, pr*)N(\tau+1, 1)$ is the only fully invariant subgroup of H strictly between them.

PROOF. Let V be a fully invariant subgroup of H such that $N(\tau, n)N(\tau+1, 1) < V \leq N(\tau, n-1)N(\tau+1, 1)$. There are two cases.

(a) If V contains $w = \prod_{i=1}^t b_i^{\beta(i)}$ where the b_i are distinct elements of \mathcal{B} (of 4.05) of weight n , each $\beta(i) \in \{p^\tau, 2p^\tau, \dots, (p-1)p^\tau\}$ and $\text{wt}(b_i, a_j) = p$ for some i and some j , then $V = N(\tau, n-1)N(\tau+1, 1)$.

Clearly it suffices to consider the case $\text{wt}(b_1, a_j) = p$. Put $f(0) = 0$ and $f(k) = f(k-1) + \text{wt}(b_1, a_k)$ and let θ be the endomorphism of H which maps a_k to $a_{f(k-1)+1} \cdots a_{f(k)}$ [to the identity if $f(k-1) = f(k)$]. Using 4.02 gives

$$w\theta \prod_{m=1}^n (1 - \delta_m) = \prod_{s=1}^p b(f(j-1) + s, n)^r w'$$

where

$$r = \pm(p-1)! \prod_{k \neq j} \text{wt}(b_1, a_k)! \beta(1),$$

$w' \in N(\tau, n)N(\tau+1, 1)$ and $b(1, n)$ is interpreted to be the identity. Hence V contains $\prod_{s=1}^p b(f(j-1) + s, n)^{p^\tau}$ because $p^{\tau+1}$ does not divide r . Applying $\iota - \pi_{1, f(j-1)+2}$ to this and using 4.02 yields that $b(f(j-1) + 2, n)^{p^\tau}$ is in V and the result follows.

(b) The only products of the form $\prod_{i=1}^t b_i^{\beta(i)}$ where the b_i are distinct elements of \mathcal{B} of weight n and the $\beta(i) \in \{p^\tau, \dots, (p-1)p^\tau\}$ are such that $\text{wt}(b_i, a_j) < p$ for all i, j .

For k in $P \cup \{0\}$ and m in $\{1, \dots, p-1\}$ let $\Pi_{k,m}$ be the set of products of the above form in V in which $\text{wt}(b_i, a_k) \leq m$ for all i [take $\text{wt}(b_i, a_0) = 0$], and for all j exceeding k the $\text{wt}(b_i, a_j)$ are independent of i and equal to 0 or 1. Let $V_{k,m}$ be the fully invariant closure in H of $\Pi_{k,m}$ and $N(\tau, n)N(\tau+1, 1)$. Clearly $V_{k,p-1} \leq V_{k+1,1}$ and $V_{k,m} \leq V_{k,m+1}$ for all k and m in $\{1, \dots, p-2\}$. If $w \in \Pi_{k+1,1}$, then both $w(\iota - \delta_{k+1})$ and $w\delta_{k+1}$ are in $\Pi_{k,p-1}$; hence w is in $V_{k,p-1}$ and $V_{k+1,1} = V_{k,p-1}$. The argument which follows establishes $V_{k,m} = V_{k,m+1}$. Let θ, ψ be the endomorphisms of H defined by:

$$a_j \theta = \begin{cases} a_j & \text{for } j < k, \\ a_k \cdots a_{k+m} & \text{for } j = k, \\ a_{j+m} & \text{for } j > k; \end{cases}$$

$$a_j \psi = \begin{cases} a_j & \text{for } j < k, \\ a_k & \text{for } j \in \{k, \dots, k+m\}, \\ a_{j-m} & \text{for } j > k+m. \end{cases}$$

It is easy to verify, using 4.02, that if $w \in \Pi_{k,m+1}$, then $w_1 = w\theta(\iota - \delta_k) \cdots (\iota - \delta_{k+m})$ is in $V_{k,1}$ and $w^{(m+1)!}(w_1\psi)^{-1}$ is in $V_{k,m}$, and hence that w is in $V_{k,m}$. From these equalities it follows that $V = V_{0,p-1}$; that is, V is the fully invariant closure of $N(\tau, n)N(\tau+1, 1)$ and the products of the form $\prod_{s=2}^n b(s, n)^{\beta(s)}$ (with $\beta(s) \in \{0, p^r, \dots, (p-1)p^r\}$) which lie in it. If $w = \prod_{s=2}^n b(s, n)^{\beta(s)} \in V$, then $w(\iota - \pi_{s,t}) \in V$ for all s, t in $\{2, \dots, n\}$. But $w(\iota - \pi_{s,t}) = [a_s, a_t, a_1, \dots]^{\beta(s) - \beta(t)}$ by 4.02, so $V = N(\tau, n-1)N(\tau+1, 1)$ or $\beta(s) = \beta(t)$ for all s, t and all relevant w . In the latter case V is the fully invariant closure of $N(\tau, n)N(\tau+1, 1)$ and $x = \prod_{s=2}^n b(s, n)^{p^r}$. If $n = 2$, then $V = N(\tau, n-1)N(\tau+1, 1)$. If $n \neq 2$ and p divides n , then $V = N(\tau, n*)N(\tau+1, 1)$. If p does not divide n , then $x(\iota - \pi_{1,2}) = b(2, n)^{p^r} \in V$ and so $V = N(\tau, n-1)N(\tau+1, 1)$.

PROOF OF 4.10. It follows from 4.06 and 4.07 that if $\mu \neq v$ in $P(p)$, then $N(\tau, v)N(\tau+1, 1) \neq N(\tau, \mu)N(\tau+1, 1)$. Thus there is at most one v in $P(p)$ such that $V = N(\tau, v)(V \cap N(\tau+1, 1))$. If $V \leq N(\tau+1, 1)$, put $v(\tau, V) = \omega$. If $V \not\leq N(\tau+1, 1)$, then by 4.08 there is an n in P such that $V \leq N(\tau, n-1)N(\tau+1, 1)$ but $V \not\leq N(\tau, n)N(\tau+1, 1)$, and it follows from 4.11 that $VN(\tau, n)N(\tau+1, 1)$ is either (a) $N(\tau, n-1)N(\tau+1, 1)$ or (b) $N(\tau, n*)N(\tau+1, 1)$.

Case (a): This implies $N(\tau, n-1) \leq VN(\tau, n)N(\tau+1, 1)$. It follows that $N(\tau, m-1) \leq VN(\tau, m)N(\tau+1, 1)$ for all m in P with $m \geq n$. Hence $N(\tau, n-1) \leq VN(\tau, pn)N(\tau+1, 1)$. Therefore $b(2, n)^{p^r} = v \prod_{i=1}^t b_i^{\beta(i)}$ where $v \in V$, the b_i are distinct elements of \mathcal{B} , p^r divides each $\beta(i)$, and for each i either $\text{wt}(b_i) > pn$ or p^{r+1} divides $\beta(i)$. By a standard argument (applying in turn the mappings $\iota - \delta_1, \iota - \delta_2, \dots$) it can be assumed that, for all i , $\text{wt}(b_i, a_j) \geq 1$ for $j \leq n$ and $\text{wt}(b_i, a_j) = 0$ for $j > n$. Hence $\text{wt}(b_i) \geq n$ and p^{r+1} divides $\beta(i)$ for all i , because no element b of \mathcal{B} satisfies $\text{wt}(b) > pn$ and $\text{wt}(b, a_j) = 0$ for all $j > n$. Thus $b(2, n)^{p^r} \in VN(\tau+1, n-1)$ and so $N(\tau, n-1) \leq VN(\tau+1, n-1)$. It follows that $N(\rho, n-1) \leq VN(\rho+1, n-1)$ for all $\rho \geq \tau$. Therefore $N(\tau, n-1) \leq V$. But $V \leq N(\tau, n-1)N(\tau+1, 1)$ and so the result follows with $v(\tau, V) = n-1$.

Case (b): Now $n = pr \geq 3$ and $N(\tau, pr*) \leq VN(\tau, pr)N(\tau+1, 1)$. By 4.02, $[\prod_{s=2}^{pr} b(s, pr), a_{pr+1}](\iota - \pi_{2, pr+1}) = [a_2, a_{pr+1}, a_1, \dots, a_{pr}]$. Hence $N(\tau, pr) \leq VN(\tau, pr+1)N(\tau+1, 1)$ and so $N(\tau, pr*) \leq VN(\tau, p^2r)N(\tau+1, 1)$. Therefore arguing as in (a) we obtain that

$$\prod_{s=2}^{pr} b(s, pr)^{p^r} = v \prod_{s=2}^{pr} b(s, pr)^{\mu(s)} \prod_{i=1}^t b_i^{\beta(i)}$$

where $v \in V$, the b_i are elements of \mathcal{B} of weight at least $pr+1$ and p^{r+1} divides each $\mu(s)$ and $\beta(i)$. Let π denote the automorphism of H which maps a_i to a_{i+1} if $2 \leq i \leq pr$, a_{pr} to a_2 , and fixes all other generators. Apply the mapping $\sum_{m=0}^{pr-2} \pi^m$ to the last displayed relation above: since $pr-1$ is prime to p , it follows that $\prod_{s=2}^{pr} b(s, pr)^{p^r} \in VN(\tau+1, pr*)$. Hence $N(\tau, pr*) \leq VN(\tau+1, pr*)$. Then arguing as in (a) shows that the result holds with $v(\tau, V) = pr*$.

PROOF OF 2.1. Let \mathfrak{B} be a subvariety of $\mathfrak{A}_{p^*}\mathfrak{A}_p$. By 4.09 and repeated applications of 4.10,

$$\mathfrak{B}(H) = B(\beta(\mathfrak{B}(H))) \prod_{\tau=0}^{\alpha-1} N(\tau, \mathfrak{v}(\tau, \mathfrak{B}(H) \cap N(\tau, 1))).$$

It follows from the observations after 4.09 and 4.10 that

$$\mathfrak{B}(H) = B(\beta(\mathfrak{B})) \prod_{\tau=0}^{\alpha-1} N(\tau, \mathfrak{v}(\tau, \mathfrak{B})).$$

Going over to varieties gives the result.

PROOF OF 2.2. It follows from the argument in the proof of 2.1 that

$$(\mathfrak{U} \wedge \mathfrak{B})(H) = \mathfrak{U}(H)\mathfrak{B}(H) = B(\beta(\mathfrak{U}))B(\beta(\mathfrak{B})) \prod_{\tau=0}^{\alpha-1} N(\tau, \mathfrak{v}(\tau, \mathfrak{U}))N(\tau, \mathfrak{v}(\tau, \mathfrak{B})).$$

Since the $B(\)$'s and the $N(\tau, \)$'s are linearly ordered,

$$(\mathfrak{U} \wedge \mathfrak{B})(H) = B(\min \{\beta(\mathfrak{U}), \beta(\mathfrak{B})\}) \prod_{\tau=1}^{\alpha-1} N(\tau, \min \{\mathfrak{v}(\tau, \mathfrak{U}), \mathfrak{v}(\tau, \mathfrak{B})\}).$$

It follows from 4.09 that

$$\beta(\mathfrak{U} \wedge \mathfrak{B}) = \min \{\beta(\mathfrak{U}), \beta(\mathfrak{B})\}$$

and

$$(\mathfrak{U} \wedge \mathfrak{B})(H) \cap N(0, 1) = \prod_{\tau=0}^{\alpha-1} N(\tau, \min \{\mathfrak{v}(\tau, \mathfrak{U}), \mathfrak{v}(\tau, \mathfrak{B})\}).$$

An induction on ρ , using 4.10, yields

$$\mathfrak{v}(\rho, (\mathfrak{U} \wedge \mathfrak{B})(H) \cap N(\rho, 1)) = \min \{\mathfrak{v}(\rho, \mathfrak{U}), \mathfrak{v}(\rho, \mathfrak{B})\}$$

and

$$(\mathfrak{U} \wedge \mathfrak{B})(H) \cap N(\rho+1) = \prod_{\tau=\rho+1}^{\alpha-1} N(\tau, \min \{\mathfrak{v}(\tau, \mathfrak{U}), \mathfrak{v}(\tau, \mathfrak{B})\}).$$

Hence, by the remark after 4.10,

$$\mathfrak{v}(\tau, \mathfrak{U} \wedge \mathfrak{B}) = \min \{\mathfrak{v}(\tau, \mathfrak{U}), \mathfrak{v}(\tau, \mathfrak{B})\}$$

as required.

Before proving 2.3 we need one more result.

4.12 For $\beta \geq 1$

$$B(\beta) \cap N(0, 1) = \begin{cases} N(\beta-1, 1) & \text{for } p = 2, \\ N(\beta-1, p^*)N(\beta, 1) & \text{for } p \text{ odd.} \end{cases}$$

PROOF. The result is an easy consequence of the case $\beta = 1$ so we only prove that. For $p = 2$ this is an immediate consequence of the well-known fact that all

groups of exponent 2 are abelian. Let p be an odd prime. Since $N(0, 1) \geq B(1) \cap N(0, 1) \geq N(1, 1)$, it follows from 4.09 that there is a v in $P(p)$ such that $B(1) \cap N(0, 1) = N(0, v)N(1, 1)$. There are metabelian groups of exponent p and class precisely p (see [11] Satz 3 or [4] Example 3.2), so $v > p-1$. By 18.4.13 of [6], $[a_2, (p-1)a_1] \in (B(1) \cap N(0, 1))N(0, p)$. By 4.06, $[a_2, (p-1)a_1] \notin N(0, p)N(1, 1)$, so $v < p$. Thus $v = p^*$ and the result follows.

PROOF OF 2.3. Clearly the set Σ is a sublattice of the direct product lattice \mathcal{A} . It is a straight-forward matter to calculate using 4.08 and 4.12 that

$$\beta(\mathfrak{B}(\beta) \wedge \mathfrak{U}_{p^*} \mathfrak{U}_p) = \min \{\beta, \alpha+1\},$$

$$v(\tau, \mathfrak{B}(\beta) \wedge \mathfrak{U}_{p^*} \mathfrak{U}_p) = \begin{cases} \omega & \text{for } \tau < \beta-1, \\ p^* & \text{for } \tau = \beta-1 \text{ and } p \text{ odd}, \\ 1 & \text{for } \tau = \beta-1 \text{ and } p = 2, \\ 1 & \text{for } \tau \geq \beta; \end{cases}$$

and $\beta(\mathfrak{N}(\tau, v) \wedge \mathfrak{U}_{p^*} \mathfrak{U}_p) = \alpha+1$,

$$v(\rho, \mathfrak{N}(\tau, v) \wedge \mathfrak{U}_{p^*} \mathfrak{U}_p) = \begin{cases} \omega & \text{for } \rho < \tau, \\ v & \text{for } \rho = \tau, \\ v(\rho, \mathfrak{N}(\tau+1, \bar{v}) \wedge \mathfrak{U}_{p^*} \mathfrak{U}_p) & \text{for } \rho > \tau \text{ and } v > 2p-1, \\ \bar{v} & \text{for } \rho = \tau+1 \text{ and } v \leq 2p-1, \\ 1 & \text{for } \rho > \tau+1 \text{ and } v \leq 2p-1, \end{cases}$$

where

$$\bar{v} = \begin{cases} v & \text{for } v \in \{1, \omega\} \\ v-p+1 & \text{for } v \in P \text{ and } v > p, \\ pr & \text{for } v = p(r+1)^*, \\ 2 & \text{for } 2 \leq v \leq p. \end{cases}$$

Hence $(\mathfrak{B}(\beta) \wedge \mathfrak{U}_{p^*} \mathfrak{U}_p)\chi$ and $(\mathfrak{N}(\tau, v) \wedge \mathfrak{U}_{p^*} \mathfrak{U}_p)\chi$ belong to Σ and so the image of χ lies in Σ . Moreover it follows that if $(\beta, v_0, \dots, v_{\alpha-1}) \in \Sigma$, then

$$(\mathfrak{B}(\beta) \wedge \bigwedge_{\tau=0}^{\alpha-1} \mathfrak{N}(\tau, v_\tau) \wedge \mathfrak{U}_{p^*} \mathfrak{U}_p)\chi = (\beta, v_0, \dots, v_{\alpha-1}).$$

5. Proof of Theorems 2 and 5

PROOF OF THEOREM 5. Since a group G is nilpotent if it has a nilpotent normal subgroup N such that $G/\mathfrak{U}(N)$ is nilpotent (P. Hall [7] Theorem 7), it suffices to prove the theorem for metabelian varieties. Let \mathfrak{B} be a metabelian variety which does not contain $\mathfrak{U}_p \mathfrak{U}_p$; then there is a positive integer c such that $\mathfrak{B} \wedge \mathfrak{U}_p \mathfrak{U}_p \subseteq \mathfrak{N}_{c-1}$. We show that every p -group in \mathfrak{B} lies in \mathfrak{N}_{c-1} . Suppose not; then there

would be a finitely generated, and therefore finite, p -group in $\mathfrak{B} \setminus \mathfrak{N}_{c-1}$. Since all finite p -groups are nilpotent, it would follow that there is a p -group in $(\mathfrak{B} \wedge \mathfrak{N}_c) \setminus \mathfrak{N}_{c-1}$. The result is therefore a consequence of the following more precise lemma.

LEMMA. *If \mathfrak{B} is a metabelian variety such that $\mathfrak{B} \wedge \mathfrak{A}_p \mathfrak{A}_p \subseteq \mathfrak{N}_v$ for some v in $P(p)$, then for each μ in $P(p) \setminus \{\omega\}$ with $\mu > v$ there is a positive integer k not divisible by p such that $\mathfrak{B} \wedge \mathfrak{N}_\mu \subseteq \mathfrak{N}_k \mathfrak{N}_v$.*

PROOF. There is nothing to prove if $v = \omega$. If $v \neq \omega$, then it clearly suffices to prove the result when μ is the first positive integer exceeding v – call it c . Let G be a free group of $\mathfrak{A} \wedge \mathfrak{N}_c$ freely generated by $\{g_1, \dots, g_c\}$, let $V = \mathfrak{B}(G)$ and $K = \mathfrak{N}_v(G)$. We will show there is an element y of K such that $y^p w \in V$ where $w = [g_1, \dots, g_c]$ if $v = c-1$ and $w = \prod_{s=2}^c [g_s, g_1, \dots, g_{s-1}, g_{s+1}, \dots, g_c]$ if $v = c*$. Since K is finitely generated abelian and the fully invariant closure of w , it will follow that KV/V is a finitely generated abelian group in which every element has a p -th root; hence that KV/V is a finite abelian group of order k not divisible by p ; and therefore $\mathfrak{B} \wedge \mathfrak{N}_c \subseteq \mathfrak{N}_k \mathfrak{N}_v$ as required. Since $\mathfrak{B} \wedge \mathfrak{A}_p \mathfrak{A}_p \subseteq \mathfrak{N}_v$ it follows that $K \leq VD$ where $D = \mathfrak{A}_p \mathfrak{A}_p(G)$ and hence $w = v_0 d_0$ with $v_0 \in V$, $d_0 \in D$. For each i in $\{1, \dots, c\}$ let ε_i be the endomorphism of G which maps g_j to g_j for $j \neq i$ and g_i to e . We now define $v_1, \dots, v_c \in V$ and $d_1, \dots, d_c \in D$ by $v_i = v_{i-1}(v_{i-1}\varepsilon_i)^{-1}$ and $d_i = (d_{i-1}\varepsilon_i)^{-1}d_{i-1}$. It is easy to check for all i that $(v_{i-1}\varepsilon_i)(d_{i-1}\varepsilon_i) = e$, $w = v_i d_i$ and $d_i \varepsilon_j = e$ for all $j \leq i$. It follows ([12] 36.32) that d_c can be uniquely written in the form $\prod_{s=2}^c [g_s, g_1, \dots, g_{s-1}, g_{s+1}, \dots, g_c]^{\beta(s)}$. Let H be the free group of $\mathfrak{A}_p \mathfrak{A}_p$ defined in section 4. Let θ be the homomorphism of G into $H/N(0, c)$ defined by $g_i \theta = a_i N(0, c)$ for all i in $\{1, \dots, c\}$. Then, as $D\theta = \{N(0, c)\}$,

$$\prod_{s=2}^c b(s, c)^{\beta(s)} N(0, c) = d_c \theta = N(0, c),$$

and so p divides $\beta(s)$ for all s . Therefore d_c has a p -th root b in $\mathfrak{N}_{c-1}(G)$ and $w = v_c b^p$. If $c = 2$ or p does not divide c , then $\mathfrak{N}_{c-1}(G) = K$ and the proof is complete. Let π denote the automorphism of G which maps g_1 to g_1 , g_i to g_{i+1} when $1 < i < c$, and g_c to g_2 ; put $\psi = \sum_{m=0}^{c-2} \pi^m$. If p divides c and $c \geq 3$, then applying ψ (cf. the last paragraph of the proof of 4.10) yields $w^{c-1} = v_c \psi(b\psi)^p$ and $b\psi \in K$. The result follows because p does not divide $c-1$.

PROOF OF THEOREM 2. Let \mathfrak{U} be a nilpotent-by-abelian just-non-Cross variety. If $\mathfrak{A} \subseteq \mathfrak{U}$, then $\mathfrak{U} = \mathfrak{A}$. If $\mathfrak{A} \not\subseteq \mathfrak{U}$, then \mathfrak{U} has finite exponent, t say. Hence \mathfrak{U} is generated by its finite groups. We will show that there is a bound, t' , on the order of chief factors of finite groups in \mathfrak{U} . By the Corollary in [8] applied to the class of finite groups in \mathfrak{U} , there is no bound on the class of finite nilpotent groups in \mathfrak{U} and the result follows from Corollary 2 of Theorem 5. Let H/K be a chief factor of a finite group G in \mathfrak{U} . Clearly H/K is an elementary abelian p -group for some p dividing t . Let C be the centralizer of H/K in G ; then G/C is an abelian group which

has a faithful irreducible representation over the field of p elements. Hence G/C is cyclic and so has order dividing t . Therefore the order of H/K is at most t' as required.

REMARK (added in proof, 9 December, 1970). The problem stated in section 2 has a negative solution on account of the results of Bachmuth, Mochizuki, and Walkup ['A nonsolvable group of exponent 5', *Bull. Amer. Math. Soc.* 76 (1970), 638–640] and O. Yu. Razmuslov [to appear]: for all primes $p \geq 5$, there exist nonnilpotent locally finite varieties of exponent p . Our Theorem 2 has been superseded by results of J. M. Brady ['On the classification of just-non-Cross varieties of groups', *Bull. Austr. Math. Soc.* 3 (1970), 293–311; 'On soluble just-non-Cross varieties of groups', *ibid.* 313–323] and O. Yu. Ol'shanskij [to appear].

References

- [1] G. Birkhoff, *Lattice Theory*, Third (New) Edition (Amer. Math. Soc., Providence, 1967).
- [2] M. S. Brooks, 'On lattices of varieties of metabelian groups', *J. Austral. Math. Soc.* to appear.
- [3] R. A. Bryce, 'Metabelian groups and varieties', *Bull. Austral. Math. Soc.* 1 (1969), 15–25.
- [4] N. D. Gupta and M. F. Newman, 'On metabelian groups', *J. Austral. Math. Soc.* 6 (1966), 362–368.
- [5] N. D. Gupta and M. F. Newman, 'On metabelian groups', *Proc. Internat. Conf. Theory of Groups, Austral. Nat. Univ. Canberra, August 1965*, pp. 111–113 (Gordon and Breach, New York, 1967).
- [6] M. Hall Jr., *The Theory of Groups* (Macmillan, New York, 1959).
- [7] P. Hall, 'Some sufficient conditions for a group to be nilpotent', *Illinois J. Math.* 2 (1958), 787–801.
- [8] L. G. Kovács and M. F. Newman, 'Cross varieties of groups', *Proc. Roy. Soc. London Ser. A.* 292 (1966), 530–536.
- [9] L. G. Kovács and M. F. Newman, 'Just-non-Cross varieties', *Proc. Internat. Conf. Theory of Groups, Austral. Nat. Univ. Canberra, August 1965*, pp. 221–223 (Gordon and Breach, New York, 1967).
- [10] L. G. Kovács and M. F. Newman, 'On varieties of metabelian groups' (in preparation).
- [11] H. Meier-Wunderli, 'Metabelsche Gruppen', *Comment. Math. Helv.* 24 (1951), 1–10.
- [12] Hanna Neumann, *Varieties of Groups* (Springer, Berlin, 1967).
- [13] P. S. Novikov and S. I. Adyan, 'On infinite periodic groups I', *Izv. Akad. Nauk S.S.S.R. Ser. Mat.* 32 (1968), 212–244.

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