BULL. AUSTRAL. MATH. SOC. MOS 20E15, 20F50
VOL. 4 (1971), 389-396. (20E10)

On soluble groups of prime-power exponent

Warren Brisley and L. G. Kovács

Let p be a prime and $\underline{\underline{A}}_p^2$ the variety of elementary abelian by elementary abelian p-groups. A result of Brisley and Macdonald is generalized as follows. If H is a finite group in $\underline{\underline{A}}_p^2$ and G is a soluble group of p-power exponent such that no section of G is isomorphic to H, then G is nilpotent and its class is bounded by a function of three variables: H, the exponent of G, and the soluble length of G. It is a corollary that if the variety generated by a soluble group G of finite exponent contains $\underline{\underline{A}}_p^2$, then each finite group in $\underline{\underline{A}}_p^2$ is isomorphic to some section of G.

In a recent paper [1], Macdonald and the first-named author proved (on the way to their Theorem 3.3) that the class of a metabelian regular p-group of exponent p^m is at most $1 + m(p-1)$. This paper is concerned with extending their result.

The first step (suggested by Dr James Wiegold) is to use a well-known argument based on a theorem of Hall [3] (sharpened by Stewart [6]) to eliminate the assumption "metabelian" at the cost of replacing the class bound $1 + m(p-1)$ by a function which depends also on the soluble length of the group (and which we shall not attempt to optimize). The second is to note that while in [1] regular p-groups were nilpotent by definition, in this context it is sufficient to assume that we are dealing with a soluble group (of finite exponent) whose finite sections are all regular

Received 14 December 1970.

p-groups. Thus if $\underline{\underline{H}}$ is the class of those finite p-groups which are not regular, then $\underline{\underline{H}}$ has the property (p) : if G is a soluble group of p-power exponent such that no section of G belongs to $\underline{\underline{H}}$, then G is nilpotent and its class is bounded in terms of its exponent and soluble length. It is then natural to ask what other classes of groups have (p). (By a class of groups we always mean a union of isomorphism classes, but of course we do not insist that a class should contain the one-element groups.) The answer is that a class $\underline{\underline{H}}$ has (p) if and (obviously) only if it contains at least one finite group from $\underline{\underline{A}}_p^2$ (the variety of elementary abelian by elementary abelian p-groups). Since the standard wreath product of two groups of order p is an irregular group in $\underline{\underline{A}}_p^2$, this answer includes the claims made above. Its nontrivial core is the

THEOREM. *If H is a finite group in $\underline{\underline{A}}_p^2$ and G a soluble group of p-power exponent such that no section of G is isomorphic to H, then G is nilpotent and its class is bounded by a function of three variables: H, the exponent of G, and the soluble length of G.*

The proof splits into two parts. The first, based on the Hall-Stewart Theorem mentioned earlier, results in

LEMMA 1. *Let G be a soluble group of length s and exponent p^m. If the $\underline{\underline{A}}_p^2$-sections of G are all nilpotent, then G is nilpotent, and its class is bounded in terms of s, p^m, and the maximum of the classes of its $\underline{\underline{A}}_p^2$-sections.*

The second part, which exploits properties of wreath products of elementary abelian p-groups and makes use of a result of Gupta and Newman [*2*], yields what is still needed for the Theorem:

LEMMA 2. *Let H be a finite group in $\underline{\underline{A}}_p^2$. If G is a group in $\underline{\underline{A}}_p^2$ such that no section of G is isomorphic to H, then G is nilpotent and its class is bounded in terms of H.*

In fact, the result in [*1*] did not need to use the exponent of the

group: the exponent of the commutator subgroup would have served equally well. Exploring the possibilities suggested by this fact, one may enquire about classes $\underline{\underline{H}}$ with the property (p, c) : if G is a soluble p-group such that the $(c+1)$st term $\underline{\underline{N}}_c(G)$ of the lower central series of G has finite exponent and no section of G belongs to $\underline{\underline{H}}$, then G is nilpotent and its class is bounded in terms of c , the exponent of $\underline{\underline{N}}_c(G)$, and the soluble length of G . The answer (for $c > 0$) is that a class $\underline{\underline{H}}$ has (p, c) if and only if it contains at least one finite elementary abelian by cyclic p-group and at least one finite group from $\underline{\underline{A}}_p^2$. The proof runs along similar lines and we do not present it.

It is an immediate consequence of Lemma 2 that if a group G generates $\underline{\underline{A}}_p^2$ then each finite group in $\underline{\underline{A}}_p^2$ is isomorphic to some section of G . We do not know of any other varieties with this property. For $\underline{\underline{A}}_p^2$, we have the more general

COROLLARY. *If a class* $\underline{\underline{X}}$ *of groups generates a soluble variety of finite exponent which contains* $\underline{\underline{A}}_p^2$ *, then every finite group in* $\underline{\underline{A}}_p^2$ *is isomorphic to some section of some group in* $\underline{\underline{X}}$.

For, suppose H is a finite group in $\underline{\underline{A}}_p^2$ which does not occur as a section of any group in $\underline{\underline{X}}$. By our Theorem, every p-section of every group in $\underline{\underline{X}}$ is nilpotent of class at most c , say. By Lemma 4.3 of Higman [4], each finite group of $\underline{\underline{A}}_p^2$ is isomorphic to some section of some finite direct product $G_1 \times \ldots \times G_n$ of groups from $\underline{\underline{X}}$, and hence to a section of a direct product K of finite subgroups $K_1, \ldots, K_n$ of groups from $\underline{\underline{X}}$. The p-sections of K are sections of a Sylow p-subgroup P of K , and P is isomorphic to a direct product of Sylow p-subgroups P_i of the groups K_i . As the P_i are p-subgroups of groups in $\underline{\underline{X}}$, their classes do not exceed c , so the class of P is also at most c . Thus it would follow that every finite group in $\underline{\underline{A}}_p^2$ has class at most c ,

which is not the case.

Before proceeding to technicalities, we mention that it would be interesting to know more about classes, especially single isomorphism classes, which have the property (p^*) : if G is a soluble group of p-power exponent such that no section of G belongs to $\underline{\underline{H}}$, then G is nilpotent. A slight extension of the proof of Lemma 2, together with Lemma 1, shows that if H is a direct product of a countably infinite elementary abelian p-group and a finite group from $\underline{\underline{A}}_p^2$, then (the isomorphism class of) H has (p^*) . Does every extension of a finite elementary abelian p-group by a countable elementary abelian p-group have (p^*) ? In particular, does a central product of countably many nonabelian groups of order p^3 have (p^*) ? Does the countable, restricted direct power of a cyclic group of order p^2 have (p^*) ? Can any nonnilpotent group have (p^*) ? We have no answers.

The rest of the paper consists of the proofs of the two lemmas. We shall use the notation and terminology of Hanna Neumann's book [5], with three exceptions: we use "section" rather than "factor"; we write the verbal subgroup of a group G corresponding to the variety $\underline{\underline{V}}$ as $\underline{\underline{V}}(G)$ rather than $V(G)$, and we do not reserve the letters G and H for relatively free groups. Some additional notation will be introduced as it becomes necessary.

Our first aim is to prove Lemma 1 under the additional assumption that $G \in \underline{\underline{A}}\,\underline{\underline{A}}_p$. This is done by induction on the exponent, say p^n , of $\underline{\underline{A}}_p(G)$. If $n = 1$ then $G \in \underline{\underline{A}}_p^2$ and there is nothing to prove. For $n > 1$, put $q = p^{n-1}$ and let c denote the class of $G/\underline{\underline{A}}_q\underline{\underline{A}}_p(G)$. If $g_0, g_1, \ldots, g_{2c}$ are arbitrary elements of G , we have that $[g_0, g_1, \ldots, g_c] = g^q$ for some g in $\underline{\underline{A}}_p(G)$ and that $[g, g_{c+1}, \ldots, g_{2c}] \in \underline{\underline{A}}_q\underline{\underline{A}}_p(G)$. Exploiting the facts that G is metabelian and $\underline{\underline{A}}_q\underline{\underline{A}}_p(G)$ has exponent dividing q , standard commutator calculations show that

$$[g_0, g_1, \ldots, g_{2c}] = \left[g^q, g_{c+1}, \ldots, g_{2c}\right] = [g, g_{c+1}, \ldots, g_{2c}]^q = 1$$

so that G is nilpotent of class (at most) $2c$. (Clearly, a little more care would have produced a much better estimate, but we are only concerned here with showing the existence of a bound.)

We are now ready to prove Lemma 1 in general. Note that as G is soluble of length s and has exponent p^m, it lies in $\underline{\underline{A}}_p^{ms}$, so that if k is the least positive integer with $G \in \underline{\underline{A}}_p^k$ then $k \leq ms$. The proof runs by induction on k, starting with trivial initial steps $k = 1, 2$. For the inductive step, put $N = \underline{\underline{A}}_p(G)$ and assume, as $N \in \underline{\underline{A}}_p^{k-1}$, that N is nilpotent and its class d is bounded in terms of the relevant parameters. By the previous paragraph, we know that G/N' is also nilpotent and its class c is similarly bounded. It remains to appeal to Stewart [6] to obtain that G is nilpotent of class at most $cd + (c-1)(d-1)$. This completes the proof of Lemma 1.

In preparation for the proof of Lemma 2, we note that it is sufficient to prove that lemma for the case when H is a (standard) wreath product W_k of a group C of order p and an elementary abelian group C^k of order p^k. Indeed, suppose H is any finite group in $\underline{\underline{A}}_p^2$, say, with $|\underline{\underline{A}}_p(H)| = p^\alpha$ and $|H/\underline{\underline{A}}_p(H)| = p^\beta$. By the embedding theorem (22.21 in [5]), H can be embedded in $C^\alpha \text{ Wr } C^\beta$, which in turn (use 22.14 of [5]) can be embedded in W_k provided $p^k \geq \alpha p^\beta$. Thus if no section of a group G is isomorphic to H, no section of G can be isomorphic to W_k either.

As further preparation, we take the (presumably well-known) fact that W_k is the *unique* extension of the regular C^k-module V_k (over the field of p elements: all modules considered will be over this field) by C^k. Uniqueness turns on the claim that every extension of V_k by C^k splits;

one quick way to see this is by exploiting the details of the embedding theorem (22.21 of [5]) as follows. Let V be an arbitrary extension of V_k by C^k, and identify V_k with the appropriate normal subgroup of V. Embed V and C^k in the wreath product W of V_k by C^k, the latter as a complement of the "base group" B of W, in such a way that the action of C^k on V_k by conjugation in W is the original regular action. Then V_k, as a regular and hence injective submodule of the C^k-module B, has a complement K in B which is also a submodule of B, that is, a normal subgroup of W. Since $VB = W$ and $V \cap B = V_k$ so that $VK = W$ and $V \cap K = 1$,

$$V \cong VK/K = W/K = C^k B/K = C^k K/K . V_k K/K$$

with $C^k K/K \cap V_k K/K = 1$. As in the isomorphism above V_k corresponds to $V_k K/K$, this shows that V splits over V_k as claimed.

We shall need one more fact, a direct consequence of the work of Gupta and Newman [2]: a group G in $\underline{\underline{A}}_p^2$ is nilpotent of class $1 + k(p-1)$ if $[g, (p-1)g_1, \ldots, (p-1)g_k] = 1$ whenever $g \in G'$ and $g_1, \ldots, g_k \in G$.

In order to prove Lemma 2, it is now sufficient to show that if G is a group in $\underline{\underline{A}}_p^2$ and $g \in G'$, $g_1, \ldots, g_k \in G$ such that $[g, (p-1)g_1, \ldots, (p-1)g_k] \neq 1$, then some section of G is isomorphic to W_k. We proceed to select such a section. Let A denote $\underline{\underline{A}}_p(G)$, and S the subgroup of G generated by $g_1, \ldots, g_k$, and A: then S/A is an elementary abelian p-group generated (possibly redundantly, for what we know at this stage) by $g_1A, \ldots, g_kA$, and A is an S/A-module in the obvious sense. Denote by U the normal closure of g in S, that is, the submodule of A generated by g. Write W_k as a semidirect product $C^k V_k$; let $\{c_1, \ldots, c_k\}$ be a generating set of C^k, and v a (free)

generator of V_k qua C^k-module. Consider A also as a C^k-module via the homomorphism α of C^k onto S/A which maps each c_i to g_iA, and let ϕ be the C^k-homomorphism of V_k onto U which maps v to g. Now it is well known that $[v, (p-1)c_1, \ldots, (p-1)c_k]$ generates the unique minimal normal subgroup of W_k, that is, the unique minimal submodule of V_k; since this element is mapped by ϕ to the nontrivial element $[g, (p-1)g_1, \ldots, (p-1)g_k]$, ϕ must be an isomorphism. As V_k is a faithful C^k-module, this implies that so are U and A, and therefore that α is also an isomorphism. Moreover, now we know that U is a regular and hence injective submodule of A, so that U has a complement T in A which is also a submodule, that is, a normal subgroup of S. It follows that S/T is an extension of A/T by S/A, or of V_k by C^k on account of α and the composite of ϕ with the natural isomorphism $U \to UT/T = A/T$. Hence $S/T \cong W_k$, and the proof is complete.

References

[1] Warren Brisley and I.D. Macdonald, "Two classes of metabelian p-groups", *Math. Z.* 112 (1969), 5-12.

[2] N.D. Gupta and M.F. Newman, "On metabelian groups", *J. Austral. Math. Soc.* 6 (1966), 362-368.

[3] P. Hall, "Some sufficient conditions for a group to be nilpotent", *Illinois J. Math.* 2 (1958), 787-801.

[4] Graham Higman, "Some remarks on varieties of groups", *Quart. J. Math. Oxford* (2) 10 (1959), 165-178.

[5] Hanna Neumann, *Varieties of groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37, Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[6] A.G.R. Stewart, "On the class of certain nilpotent groups", *Proc. Roy. Soc. Ser. A* 292 (1966), 374-379.

University of Newcastle,
Newcastle,
New South Wales,

and

Institute of Advanced Studies,
Australian National University,
Canberra, ACT.