

A REMARK ON CRITICAL GROUPS

L. G. KOVÁCS

(Received 9 October 1967)

Problem 24 of Hanna Neumann's book [3] reads: Does there exist, for a given integer $n > 0$, a Cross variety that is generated by its k -generator groups and contains $(k+n)$ -generator critical groups? In such a variety, is every critical group that needs more than k generators a factor of a k -generator critical group, or at least of the free group of rank k ? In a recent paper [1], R. G. Burns pointed out that the answer to the first question is an easy affirmative, and asked instead the question which presumably was intended: Given two positive integers k, l , does there exist a variety \mathfrak{B} generated by k -generator groups and also by a set S of critical groups such that S contains a group G minimally generated by $k+l$ elements and $S \setminus \{G\}$ does not generate \mathfrak{B} ? The purpose of this note is to record a simple example which shows that the answer to the question of Burns is affirmative at least for $k = 2, l = 1$, and also that the answer to the second question of Hanna Neumann's Problem 24 is negative.

Let \mathfrak{B} be the variety defined by the law $x^9[x, y, u]^3[x, y, u, v]$. It can be read off from Bjarni Jónsson's description [2] of the lattice of nilpotent varieties of class at most 3 that \mathfrak{B} has precisely two maximal subvarieties: the subvariety \mathfrak{U} defined by the additional law $[x, y, y]$, and the subvariety \mathfrak{B} defined by the additional law $[x, y]^3$; moreover, \mathfrak{U} is certainly not of class 2. Since \mathfrak{B} is defined (within \mathfrak{B}) by a two-variable law, it cannot contain the \mathfrak{B} -free group F of rank 2; nor can F be contained in \mathfrak{U} , for the two-generator groups of \mathfrak{U} are all of class at most 2 while \mathfrak{B} contains the wreath product of two cyclic groups of order 3, a two-generator group of class 3. Thus F generates \mathfrak{B} . A direct calculation shows that the proper subgroups of F are all of class at most 2: this, and a similar calculation below, is somewhat simplified by the observation that the Frattini subgroup of any group in \mathfrak{B} is contained in the second term of the upper central series of the group.

Next, consider the \mathfrak{U} -free group H of rank 3, on the free generators a, b, c . As the only relators of these generators are the laws of \mathfrak{U} , neither $[a, b]^3$ nor $[a, b, c]$ can be 1. On the other hand, as \mathfrak{U} has class 3 and $[x, y]^3, u$ is a law even in \mathfrak{B} , the element $[a, b]^3[a, b, c]$ is central in H . Let N be maximal among the normal subgroups of H which contain $[a, b]^3[a, b, c]$ but not $[a, b, c]$, and put $H/N = G$. By construction, G is

monolithic, has class precisely 3, and does not belong to \mathfrak{B} . As $G \in \mathfrak{U}$ and the two-generator groups of \mathfrak{U} are all of class at most 2, it follows that G cannot be generated by two elements. It is now easy to see that G cannot be isomorphic to any factor of F , and that no proper factor of G can have class greater than 2: in particular, G is critical. If S is the set consisting of G and of the critical groups of \mathfrak{B} , then S generates \mathfrak{B} but $S \setminus \{G\}$ does not.

References

- [1] R. G. Burns, 'Verbal wreath products and certain product varieties of groups', *J. Austral. Math. Soc.* 7 (1967), 356–374.
- [2] Bjarni Jónsson, 'Varieties of groups of nilpotency 3', *Notices Amer. Math. Soc.* 13 (1966), 488.
- [3] Hanna Neumann, *Varieties of groups* (Springer, Berlin etc., 1967).

Australian National University
Canberra