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On Finite Soluble Groups

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The purpose of this note is to point out two ways in which restrictions on the structure of Sylow subgroups imply restrictions on the structure of a finite group. The first is yet another p-length theorem (cf. HALL and HIGMAN [3]):

Theorem 1. *If a Sylow p-subgroup of a finite p-soluble group can be generated by d elements, then the p-length of G is at most d.*

The second is an easy, but apparently unnoticed, consequence of Satz 4 of GASCHÜTZ [2]:

Theorem 2. *If each Sylow subgroup of a finite soluble group G can be generated by d elements, then G can be generated by $d+1$ elements.*

This is best possible in the sense that to each positive value of d one can easily construct such groups G (in fact, abelian-by-cyclic groups) which cannot be generated by d elements. A suitable rearrangement of the steps of GASCHÜTZ [2] can give further results such as the following: if every abelian-by-cyclic factor group of G can be generated by k elements and $k \geqq \frac{1}{2}(d+1)$, then G can be generated by k elements.

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It is convenient to put the first steps of the proof of Theorem 1 separately:

Lemma 1. *Let G be any finite group, H a normal subgroup of G contained in the Frattini subgroup $\Phi(G)$ of G, A/H a normal p'-subgroup of G/H, and H_p the Sylow p-subgroup of H. Then A is the direct product $H_p \times B$ of H_p and a normal p'-subgroup B of G.*

Proof. Since $\Phi(G)$ is nilpotent ([4], IV.2.g), H_p is normal in G; as H_p has p'-index in A, the Schur-Zassenhaus Theorem (VII.2.c in [4]) guarantees that it has a complement B in A, and that all such complements are conjugate in A. Thus if $\mathcal{N}_G(B)$ denotes the normalizer of B in C, then

$$G = A\,\mathcal{N}_G(B) = H_p B\,\mathcal{N}_G(B) = H_p\,\mathcal{N}_G(B) = \mathcal{N}_G(B)$$

on account of the Frattini argument (IV.2.e in [4]) and the fact that H_p consists of non-generators of G ([4], p. 31). Hence B is normal in G; it is obviously a p'-group; and $A = H_p \times B$.

Corollary 1. *If N is the largest normal p'-subgroup of G, then NH/H is the largest normal p'-subgroup of G/H.*

For, if A/H is the largest normal p'-subgroup of G/H, then $A \geqq NH$ and $N \geqq B$, so $A = NH$.

Corollary 2. *If $R/N = \Phi(G/N)$, then G/R has no non-trivial normal p'-subgroup. If G/R is p-soluble of p-length l, so is G.*

The first statement is obtained by applying Corollary 1 to G/N in place of G and R/N in place of H; the place of N is taken by N/N. For the second statement, note that R/N is nilpotent and has no normal p'-subgroup, so that it is a p-group. (Corollary 2 is a slightly stronger version of the Corollary to Lemma 1.2.5 of Hall and Higman [3].)

Lemma 2. *Let G be a finite group with a Sylow p-subgroup P which can be generated by d elements. If M is a non-trivial, complemented, normal p-subgroup of G, then a Sylow p-subgroup of G/M can be generated by $d-1$ elements.*

Proof. Let K be a complement of M in G; then $L = P \cap K$ is a complement of M in P. The mutual commutator subgroup $[P, M]$ is properly contained in M and is normal in P, so

$$P/[P, M] = M/[P, M] \times L[P, M]/[P, M]$$

where $P/[P, M]$ can be generated by d elements, $M/[P, M]$ is non-trivial, and $L[P, M]/[P, M] \cong L \cong P/M$. This implies that P/M, which is a Sylow p-subgroup of G/M, can be generated by $d-1$ elements.

Proof of Theorem 1. (The notation is maintained; the assumptions of Theorem 1 are now taken up.) If $d=0$, then G is a p'-group, that is, of p-length 0. Suppose $d>0$ and proceed by induction on d. Since a Sylow p-subgroup of G/R is a homomorphic image of one of G, it can be generated by d elements; on the other hand, Corollary 2 shows that the p-lengths of G and G/R are equal. Thus it can and will be assumed, without loss of generality, that $R=1$: that is, G has no non-trivial normal p'-subgroup and $\Phi(G)=1$. The Fitting subgroup F of G is then just the greatest normal p-subgroup of G, and it is complemented in G (cf. Gaschütz [1]). Thus Lemma 2 (applied with $M=F$) and the induction hypothesis give that G/F has p-length at most $d-1$; hence the p-length of G is at most d.

Proof of Theorem 2. It is immediate from Satz 4 of Gaschütz [2] that if G could not be generated by $d+1$ elements then, for at least one prime p, a chief series of G would have at least $d+1$ complemented p-factors. Therefore it is sufficient to prove that no chief series of G can have more than d complemented p-factors. This is obvious if $d=0$; assume $d>0$ and proceed by induction. Let $1 = G_0 < \cdots < G_n = G$ be a chief series of G; if this has any complemented p-factors, let G_{i+1}/G_i be the one with smallest i. By Lemma 2 (applied to G/G_i with $M=G_{i+1}/G_i$) and the induction hypothesis, the chief series $1<G_{i+1}/G_i<\cdots<G_n/G_i$ of G/G_i has at most $d-1$ complemented p-factors; so $G_0<\cdots<G_n$ has at most d.

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