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Just-non-Cross varieties

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It is convenient here to define a *Cross variety* as a variety generated by a single finite group: we shall not use the rather deep result (Sheila Oates and M. B. Powell [9]) that this definition is equivalent to the usual one. It is easy to see (e.g., from Lemma 4.3 of Graham Higman [3]) that the nilpotent groups of a Cross variety are of bounded class and the chief factors of the finite groups of a Cross variety are of bounded order. Let $\mathfrak{V}$ be a variety whose laws are finitely based; then the laws of every subvariety of $\mathfrak{V}$ are finitely based if and only if every set of subvarieties of $\mathfrak{V}$ has a minimal element (with respect to partial order by inclusion). The question to be considered here is a very special case of the suggested equivalent of the finite basis problem: For what varieties $\mathfrak{V}$ is it true that among the non-Cross subvarieties of $\mathfrak{V}$ there are minimal ones? Call a non-Cross variety *just-non-Cross* if every proper subvariety of it is Cross, and rephrase the question: What non-Cross varieties have just-non-Cross subvarieties? Also, find as many just-non-Cross varieties as possible.

Notation. For a nonnegative integer e, $\mathfrak{A}_e$ denotes the variety of abelian groups of exponent dividing e. If q is an odd prime, $\mathfrak{Q}_q$ denotes the variety of nilpotent groups of class (at most) 2 and exponent dividing q; $\mathfrak{Q}_2$ is the variety of groups of exponent dividing 4 with central derived groups of exponent dividing 2. The letters p, q, r will denote arbitrary but distinct primes.

The just-non-Cross varieties we know are $\mathfrak{A}_0$, $\mathfrak{A}_p\mathfrak{A}_p$, $\mathfrak{A}_p\mathfrak{Q}_q$, $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$. The claim that $\mathfrak{A}_0$ is one is trivial. To see that $\mathfrak{A}_p\mathfrak{A}_p$ is non-Cross, note that it contains nilpotent wreath products of arbitrary large class. Using results of N. D. Gupta and the second author [2], we prove in [6] that every proper subvariety of $\mathfrak{A}_p\mathfrak{A}_p$ is Cross. Higman's paper [3] shows that $\mathfrak{A}_p\mathfrak{Q}_q$ is just-non-Cross and $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$ is not Cross. The fact

that all proper subvarieties of $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$ are Cross has been proved by P. J. Cossey [1].

It may be conjectured that each soluble non-Cross variety must contain one of these just-non-Cross varieties. It has been shown [6] that if the nilpotent groups of a soluble variety are not of bounded class, then it must contain an $\mathfrak{A}_p\mathfrak{A}_p$. Cossey has proved [1] that if the nilpotent groups of a soluble variety are all abelian, then it contains either $\mathfrak{A}_0$ or an $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$. However, the gap is still wide.

In both of the results just quoted, the assumption of solubility is essentially used. It is hard to see what, if any, extension of the first may hold in the locally soluble case: the variety $\mathfrak{K}_p$ of locally soluble groups of exponent dividing p (where p is a prime; cf. [5]) presents a case we dare not hope to handle. On the other hand, the prospects of generalizing Cossey's result to cover all insoluble cases seem good; he is giving attention to this.

Our original reason for suspecting that the product varieties mentioned above are just-non-Cross was Theorem 6.3 of A. L. Šmel'kin [10]: The product of two nontrivial varieties $\mathfrak{U}$, $\mathfrak{V}$ is Cross if and only if (i) the exponents of $\mathfrak{U}$ and $\mathfrak{V}$ are coprime, (ii) $\mathfrak{U}$ is nilpotent, and (iii) $\mathfrak{V}$ is abelian. The three types of just-non-Cross product varieties can be obtained by minimally violating one of these conditions and at the same time maximally satisfying the others. The "if" part of the theorem is proved with an argument of D. C. Cross (cf. Higman [4]). Šmel'kin proved the "only if" part with the help of verbal wreath products. Peter M. Neumann has a proof (only partly published in [8]) which he obtained from estimates of orders of finite relatively free groups. We indicate yet another proof for the "only if" part and at the same time show that there are no further just-non-Cross product varieties.

Let $\mathfrak{U}$, $\mathfrak{V}$ be as above. If (i) is false, $\mathfrak{U}\mathfrak{V}$ obviously contains an $\mathfrak{A}_p\mathfrak{A}_p$. If $\mathfrak{U}$ contains at least one non-nilpotent finite group, then a subvariety generated by a finite non-nilpotent group of least order in $\mathfrak{U}$ is an $\mathfrak{A}_p\mathfrak{A}_q$; so if at the same time (i) holds, then $\mathfrak{U}\mathfrak{V}$ contains an $(\mathfrak{A}_p\mathfrak{A}_q)\mathfrak{A}_r$. If $\mathfrak{V}$ contains at least one non-abelian finite group, then a subvariety generated by a finite non-abelian group of least order in $\mathfrak{V}$ is either an $\mathfrak{A}_q\mathfrak{A}_r$ or a $\mathfrak{Q}_q$; so if at the same time (i) holds, then $\mathfrak{U}\mathfrak{V}$ contains an $\mathfrak{A}_p(\mathfrak{A}_q\mathfrak{A}_r)$ or an $\mathfrak{A}_p\mathfrak{Q}_q$. Suppose that $\mathfrak{U}\mathfrak{V}$ is Cross. Then $\mathfrak{U}\mathfrak{V}$ cannot contain any $\mathfrak{A}_p\mathfrak{A}_p$ or $\mathfrak{A}_p\mathfrak{Q}_q$ or $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$, and so (i) must hold; moreover, as every

Cross variety is locally finite (cf. B. H. Neumann [7]), the above argument implies that (ii) and (iii) must also hold. Suppose that $\mathfrak{U}\mathfrak{V}$ is just-non-Cross. Then $\mathfrak{U}$ and $\mathfrak{V}$ are Cross and by the "if" part of the theorem at least one of (i)–(iii) must fail: so the above argument gives that either $\mathfrak{U}=\mathfrak{A}_p$ and $\mathfrak{V}=\mathfrak{A}_q\mathfrak{A}_r$ or $\mathfrak{Q}_q$, or $\mathfrak{U}=\mathfrak{A}_p\mathfrak{A}_q$ and $\mathfrak{V}=\mathfrak{A}_r$.

We started from the finite basis problem. Although the results reported on are very incomplete, in one direction some hope is already showing that these investigations may lead back to the positive solution of some special cases of that problem.

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