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Minimal verbal subgroups

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1. *Introduction.* In this note we obtain a description of the structure of the minimal verbal subgroups of a finite group which has, among others, the following consequences:

THEOREM 1. *A finite group which belongs to the variety generated by its proper subgroups and proper factor-groups belongs either to the variety generated by its proper subgroups or to the variety generated by its proper factor-groups.*

THEOREM 2. *If G is a finite monolithic group with monolith M (in other words, if G is a finite group with a unique minimal normal subgroup M), then M is a verbal subgroup of G.*

These generalize the corresponding results for finite p-groups established by Weichsel ((4), Theorems 2·1, 2·3). The second answers affirmatively a (privately communicated) question of Sheila Oates.

In order to give our description we need some notation. Verbal subgroups can be defined in a number of ways. For our purposes the following procedure seems the most efficient. A *variety* of groups is a class of groups closed under forming subgroups, homomorphic images, and unrestricted direct products. Given a class $\mathbf{C}$ of groups there is a smallest variety containing it; this will be denoted var $\mathbf{C}$ or var G if $\mathbf{C}$ consists of the one group G. Let $\mathbf{V}$ be a variety, then in every group G there is a unique normal subgroup N minimal with respect to the property that the factor group G/N lies in $\mathbf{V}$. This normal subgroup is the *verbal subgroup* of G corresponding to the variety $\mathbf{V}$ and is denoted $\mathbf{V}(G)$. Clearly $\mathbf{V}(G) = E$ (the identity subgroup) if and only if $G \in \mathbf{V}$. A *minimal verbal subgroup* of a group G is a non-trivial verbal subgroup of G which does not properly contain a non-trivial verbal subgroup of G.

Let S be a subgroup of a group G. The *centralizer* of S in G, that is,

$$\{x: x \in G \quad \text{and} \quad xs = sx \quad \text{all} \quad s \in S\}$$

is denoted $C(S, G)$. If $x \in S$ and $y, z \in G$, and if $y^{-1}z \in C(S, G)$, then $x^y = y^{-1}xy = x^z$; this common value is denoted $x^{yC(S,G)}$. Let M, N be normal subgroups of groups G, H respectively. If there is an isomorphism θ from M to N and an isomorphism ψ from $G/C(M, G)$ to $H/C(N, H)$ such that

$$\{x^{yC(M,G)}\}\theta = \{x\theta\}^{yC(M,G)\psi}$$

for all x in M and y in G, then M is *similar* qua normal subgroup of G to N qua normal subgroup of H; this will be denoted $(M \trianglelefteq G) \sim (N \trianglelefteq H)$; where context allows we shall say simply that M is similar to N and write $M \sim N$.

THEOREM 3. *A minimal verbal subgroup of a finite group G is the direct product of similar minimal normal subgroups of G.*

Theorem 2 is an immediate consequence of this, Theorem 1 can also be deduced but it follows more simply from the proof of Theorem 3 given in the next section. Our attempts to obtain a more precise description of minimal verbal subgroups have foundered on the groups obtained by directly multiplying the binary icosahedral group (alias SL (2, 5)) with the cyclic group of order 4, the alternating group on 4 symbols, and the alternating group on 5 symbols (PSL (2, 5)) in turn.

Another consequence of the proof of Theorem 3 is that, if a finitely generated monolithic group with non-abelian monolith belongs to the variety generated by a finite group G, then it is isomorphic to a factor of G. The restriction 'finitely generated' can be dispensed with. For in the variety generated by a finite group of order n, the centralizer of every chief factor has index at most n (combining 4·3 and 4·4 of (3)), and so every monolithic group with non-abelian monolith in the variety generated by a finite group is finite. Thus we have:

Theorem 4. *If a monolithic group with non-abelian monolith belongs to the variety generated by a finite group G, then it is isomorphic to a factor of G.*

A group is *verbally simple* if it has no proper non-trivial verbal subgroups. If G is a finite verbally simple group, then, by Theorem 3, G is the direct product of finitely many similar minimal normal subgroups. Each of these is simple because a normal subgroup of a direct factor of a group is normal in the whole group. Thus we have the following result.

Theorem 5. *A finite verbally simple group is a direct product of isomorphic simple groups.*

This generalizes the result that every finite fully-invariantly simple group is the direct product of isomorphic simple groups (Baer ((1)), p. 25, Proposition 1).

2. *Proofs.* We start with some technical results on similarity. In each case the routine verification that the given mappings have the required properties is omitted.

(2·1). If A, B are normal subgroups of a group G which intersect trivially, then $(A \trianglelefteq G) \sim (AB/B \trianglelefteq G/B)$.

Clearly $C(A,G)/B = C(AB/B, G/B)$. The mappings θ, ψ defined by $a\theta = aB$ for all a in A, $xC(A,G)\psi = (xB)C(AB/B, G/B)$ for all x in G give the result.

(2·2). Let S be a subgroup of the direct product of two groups A and B which has full projection onto B (i.e. for all b in B there is an s in S such that $bs^{-1} \in A$). If N is a normal subgroup of S contained in $B \cap S$, then $(N \trianglelefteq B) \sim (N \trianglelefteq S)$.

Clearly $N \leqslant B$ and $C(N,B)A \cap S = C(N,S)$. The mappings θ, ψ defined by $n\theta = n$ for all n in N, $sC(N,S)\psi = bC(N,B)$ for all s in S with bs^{-1} in A give the result.

(2·3). If $(M \trianglelefteq G) \sim (N \trianglelefteq H)$ under the mappings θ, ψ, and if L is a normal subgroup of G in M, then $(L \trianglelefteq G) \sim (L\theta \trianglelefteq H)$. It is easy to check that ψ maps $C(L,G)/C(M,G)$ onto $C(L\theta,H)/C(N,H)$ and so induces an isomorphism ψ' from $G/C(L,G)$ onto $H/C(L\theta,H)$. If θ' is the restriction of θ to L, then the mappings θ', ψ' give the result.

(2·4). If $N_1, \ldots, N_s$ are similar minimal normal subgroups of a group G, and if M is a normal subgroup of G in $N_1 \ldots N_s$, then M is the product of minimal normal subgroups of G all similar to N_1.

Proof (by induction on s). If $s = 1$ or $M = N = N_1 \dots N_s$, there is nothing to prove. If $s > 1$ and $M < N$, then one of $N_1, \dots, N_s$ intersects M trivially; suppose, without loss of generality, $N_s \cap M = E$. By 2·1 $(M \trianglelefteq G) \sim (MN_s/N_s \trianglelefteq G/N_s)$. By the inductive hypothesis MN_s/N_s is a product of minimal normal subgroups of G/N_s each similar to $N_1 N_s/N_s$. Hence it follows by repeated application of 2·3 and 2·1 that M is a product of minimal normal subgroups of G each similar to N_1.

After one more preliminary result and a reminder of some definitions we can state our main lemma.

(2·5). If N is a product of minimal normal subgroups of a group G, and if M is a normal subgroup of G contained in N, then there is a normal subgroup L of G which complements M in N. Take for L a normal subgroup of G in N which is maximal with respect to avoiding M.

If $LM < N$, there would be a minimal normal subgroup K of G in N such that $LM \cap K = E$ and it would follow that $LK \cap M = E$ contradicting the maximality of L.

A *factor* of a group G is a factor group of a subgroup of G; *a proper factor* of G is a factor other than G. A group is *critical* if it does not belong to the variety generated by its proper factors.

(2·6). Let $\mathbf{U}$ be a variety of groups, C a finite group not in $\mathbf{U}$ all of whose proper factors lie in $\mathbf{U}$, and let $\mathbf{V} = \operatorname{var}(\mathbf{U} \cup \{C\})$. For every finitely generated group G in $\mathbf{V}$, the verbal subgroup $\mathbf{U}(G)$ is the direct product of minimal normal subgroups each similar qua normal subgroup of G to $\mathbf{U}(C)$ qua normal subgroup of C.

Proof. By the conditions on $\mathbf{U}$ and C, $\mathbf{U}(C)$ is the only minimal normal subgroup of C. Since G is finitely generated and C is finite, G is isomorphic to a factor S/T of the direct product of a group A in $\mathbf{U}$ and a finite number of copies $C_1, \dots, C_r$ of C. [This can be seen by an argument similar to the proof of Lemma 4·3 in Higman ((2)), or by Lemma 4·3 itself under the additional assumption that $\mathbf{V}$ is contained in a variety generated by a finite group: this assumption is always satisfied when we apply (2·6).] Suppose S/T is such that r is minimal. If $r = 0$, there is nothing to prove. If r is positive, then S projects fully onto each C_i, and intersects each C_i non-trivially and, therefore, contains $\mathbf{U}(C_i)$ for all $i \in \{1, \dots, r\}$. Hence, by 2·2, $(\mathbf{U}(C_i) \trianglelefteq C_i) \sim (\mathbf{U}(C_i) \trianglelefteq S)$. Since $T \cap \mathbf{U}(S) < \mathbf{U}(S)$, there is, by (2·5), a normal subgroup L of S in $\mathbf{U}(S)$ such that $L(T \cap \mathbf{U}(S)) = \mathbf{U}(S)$ and $L \cap (T \cap \mathbf{U}(S)) = E$. Since $L \leqslant \mathbf{U}(S) \leqslant \mathbf{U}(C_1) \dots \mathbf{U}(C_r)$, it follows from (2·4) that L is a product of minimal normal subgroups of S each similar to $\mathbf{U}(C) \trianglelefteq C$. Since $L \cap T = E$, it follows from (2·1) and (2·3) that $\mathbf{U}(S/T) = LT/T$ is a product of minimal normal subgroups of S/T each similar to $\mathbf{U}(C) \trianglelefteq C$. The directness of the product follows by a standard argument.

A final lemma completes the preparations for the proofs of the theorems.

(2·7). Let G be a finite group and $\mathbf{V}$ a subvariety of $\operatorname{var} G$. If H belongs to $\operatorname{var} G$ but not to $\mathbf{V}$, then there is (i) a variety $\mathbf{U}$ containing $\mathbf{V}$, (ii) a factor C of G all of whose proper factors lie in $\mathbf{U}$ such that H belongs to $\operatorname{var}(\mathbf{U} \cup \{C\})$ but not to $\mathbf{U}$.

Proof. Let $C_1, \dots, C_s$ be the critical factors of G not in $\mathbf{V}$ ordered by order (i.e. $|C_i| \leqslant |C_{i+1}|$). There is an i in $\{1, \dots, s\}$ such that $H \in \operatorname{var}(\mathbf{V} \cup \{C_1, \dots, C_i\})$ but

$H \notin \mathbf{U} = \operatorname{var}(\mathbf{V} \cup \{C_1, \ldots, C_{i-1}\})$. The result follows taking C_i for C because all the proper critical factors of C_i have smaller order than C_i and so belong to $\mathbf{U}$.

Proof of Theorem 3. Let $\mathbf{W}(G)$ be a minimal verbal subgroup of G; then G does not belong to $\mathbf{V} = \mathbf{W} \cap \operatorname{var} G$ and $\mathbf{V}(G) = \mathbf{W}(G)$. Hence, by (2·7), there is a subvariety $\mathbf{U}$ of $\operatorname{var} G$ containing $\mathbf{V}$ but not G and a critical factor C of G all of whose proper factors lie in $\mathbf{U}$ and such that $G \in \operatorname{var}(\mathbf{U} \cup \{C\})$. It follows from 2·6 that $\mathbf{U}(G)$ is the direct product of similar minimal normal subgroups of G. On the other hand $\mathbf{U}(G) = \mathbf{W}(G)$ because $\mathbf{U}$ contains $\mathbf{V}$ but not G, and $\mathbf{W}(G) = \mathbf{V}(G)$: the result follows.

Proof of Theorem 1. Let G be a finite group which does not belong to the variety generated by its proper factor-groups, then G is monolithic with monolith M say. If, moreover, G does not belong to the variety $\mathbf{S}$ generated by its proper subgroups, then, by 2·7, there is a variety $\mathbf{U}$ containing $\mathbf{S}$ but not G and a factor C of G such that G belongs to $\operatorname{var}(\mathbf{U} \cup \{C\})$. Hence, by 2·6, $\mathbf{U}(G) = M$. Therefore G is critical because $\mathbf{U}$ contains all the proper factors of G. The result follows.

Proof of Theorem 4. As remarked in the introduction, it suffices to consider finitely generated groups. Let H be a finitely generated group in $\operatorname{var} G$ with non-abelian monolith M. As usual 2·7 gives a subvariety $\mathbf{U}$ of var G which does not contain H and a factor C of G such that $H \in \operatorname{var}(\mathbf{U} \cup \{C\})$. It follows from 2·6 that $\mathbf{U}(H) = M$ and $(M \trianglelefteq H) \sim (\mathbf{U}(C) \trianglelefteq C)$. Hence $\mathbf{U}(C)$, being isomorphic to M, is non-abelian, and then H is isomorphic to C because $C(M, H) = E$ and $C(\mathbf{U}(C), C) = E$.

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