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On the existence of Baur-soluble groups of arbitrary height

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A hierarchy of generalized soluble groups has been defined by HERIBERT BAUR in [1]: abelian groups are to be termed Baur-soluble of height 0; for an ordinal h other than 0, a group is to be called Baur-soluble of height h if it is not Baur-soluble of height less than h and has an invariant system (in the sense of KUROSH [3]) whose factors are all Baur-souble of height less than h. Thus Baur-soluble groups of height 1 are precisely the non-abelian SI-groups. It is easy to show, by induction on h, that

(1) subgroups and factor groups of Baur-soluble groups of height h are Baur-soluble of height at most h.

A further simple observation is that

(2) every restricted direct product of Baur-soluble groups is Baur-soluble, and its height is the least upper bound of the heights of the factors.

The purpose of this note is to demontstrate the existence of Baur-soluble groups of height h, for every ordinal h. In fact, the following result will be proved, by induction on h:

Theorem. *There exist perfect Baur-soluble groups of height h, for every ordinal h.* (A group is called perfect if it is its own commutator subgroup.)

Proof. The initial step: for $h=0$, the unit group is the only example. Observation (2) and the fact that restricted direct products of perfect groups are perfect take care of the limit step in the induction.

It remains *to construct, from any given perfect Baur-soluble group* $\cancel{G}$ *of height* H/ *h, a perfect Baur-suloble group of height* $h+1$. The construction is based on wreath powers, which were introduced and studied by HALL in [2]; his notation is adopted throughout. Important use is made of a torsion free, characteristically simple group A constructed by MCLAIN in [5]. This group has the additional property that it is the product of its abelian normal subgroups. Thus, firstly, A is a perfect SI-group, and secondly, A is locally nilpotent. Therefore, firstly, A is a perfect Baur-soluble group of height 1, and so there is no need to consider the case $h=0$. Secondly, as a torsion free locally nilpotent group, A can be (fully) ordered (cf. MAL'CEV [4], or Corollary 6. 2 of B. H. NEUMANN [6]). Let Λ be the ordered set of elements of A, and let W be the wreath power Wr H^{Λ}. For each element λ of Λ, HALL defines (in section 2. 4 of [2]) a subgroup D_λ of W and shows that

(3) each D_λ is a direct product of isomorphic copies of H;

(4) the D_λ generate W; and

(5) D_λ normalizes D_μ whenever $\mu \in \Lambda$, $\mu \leqq \lambda$. Define D_Γ, for each lower section Γ of Λ, to be the subgroup of W generated by the D_λ with $\lambda \in \Gamma$. Contrary to HALL's convention, allow the empty set $\varnothing$ as a lower section of Λ, and put $D_\varnothing$ equal the unit subgroup. It follows from (4) and (5) that the D_Γ are all normal subgroups of W. As unions of ascending chains of lower sections, as well as intersections of any number of lower sections, are again lower sections, it follows that the D_Γ form an invariant system of W. Let D_Δ/D_Γ be a factor of this system; then no member of the system can lie properly between D_Γ and D_Δ, and so no lower section properly contained in Δ can properly contain Γ. From this it is easy to deduce that Δ contains just one element λ which is not contained in Γ. Hence $D_\Delta = D_\Gamma D_\lambda$; so that D_Δ/D_Γ is isomorphic to a factor group of D_λ. Now (1), (2) and (3) give that each factor of this invariant system of W is Baur-soluble of height at most h, and therefore W is Baur-soluble of height at most $h+1$. On the other hand, as W is generated by the perfect Baur-soluble groups D_λ of height h, W is perfect, and its height is at least h.

If the height of W is $h+1$, the construction is complete. Assume that W is of height h. Let the elements of A act on Λ as right translations; then A becomes a transitive and hence irreducible group of order-preserving permutations of Λ. Take G to be the natural split extension of W by A. (In [2], HALL takes A to be the group of *all* order-preserving permutations of Λ, but the omission of "all" does not effect his subsequent results.) Since $G/W \cong A$ and A is Baur-soluble of height 1, G is Baur-soluble of height at most $h+1$; also, as W and A are perfect, so is G. It follows from Theorem D of HALL [2] that W is a minimal normal subgroup of G; therefore every invariant system of G has a factor which contains an isomorphic copy of W and therefore cannot be Baur-soluble of height less than $\cancel{H}$. Consequently, the height of G cannot be less than $h+1$. This completes the construction in case W is of height h. h/

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