Some Sylow subgroups

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Most of the well-known theorems of Sylow for finite groups and of P. Hall for finite soluble groups have been extended to certain restricted classes of infinite groups. To show the limitations of such generalizations, examples are here constructed of infinite groups subject to stringent but natural restrictions, groups in which certain Sylow or Hall theorems fail. All the groups are metabelian and of exponent 6. There are countable such groups in which a Sylow 2-subgroup has a complement but no Sylow 3-complement; or again no complement at all. There are countable such groups with continuously many mutually non-isomorphic Sylow 3-subgroups. There are groups, necessarily of uncountable order, with Sylow 2-subgroups of different orders. The most elaborate example is of an uncountable group in which all Sylow 2-subgroups and 3-subgroups are countable, and none is complemented.

Let Π be a set of prime numbers. A Π-group is a periodic group whose elements have orders divisible by primes in Π only. If G is a group and H a Π-subgroup of G, then H is contained in a maximal, or Sylow, Π-subgroup of G; cf. Kurosh 1956, § 54. If Π consists of a single prime number p, we correspondingly have p-groups, p-subgroups, Sylow p-subgroups.

Eugene V. Schenkman has proved† the following theorem:

If the group G is countable, periodic, and locally soluble, then to every pair Π, Π' of complementary sets of primes there exists a Sylow Π-subgroup S and a Sylow Π'-subgroup T of G such that

$$ST = G \quad \textit{and} \quad S \cap T = \{1\};$$

in other words, S and T are complements in G.

It is clear that the condition of periodicity cannot be omitted; nor can the condition of local solubility, for by a well-known theorem of Hall (1937) a finite group is soluble if to every prime p it has a complemented Sylow p-subgroup. In fact, again by a well-known theorem of Hall (1928), in a finite soluble group every Sylow Π-subgroup is complemented; every Sylow Π'-subgroup complements every Sylow Π-subgroup; moreover, for fixed Π all Sylow Π-subgroups are conjugate and thus isomorphic. These theorems of Hall are used, together with a representation of the given group as a direct limit of a sequence of finite groups, to prove Schenkman's theorem; we omit the details.

In this paper we demonstrate by examples that under the assumptions of Schenkman's theorem: (i) there may be Sylow Π-subgroups that are complemented, but not by Sylow Π'-subgroups; (ii) there may be Sylow Π-subgroups that are not complemented at all; (iii) there may be continuously many mutually non-isomorphic Sylow Π-subgroups for fixed Π. However, it is not difficult to see that in

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Schenkman's groups, and more generally in countable locally finite groups, all Sylow Π-subgroups, for fixed Π, have the same order. We show that in uncountable groups even this weak proposition may fail to hold.

We also present another, rather more elaborate example, to show that the assumption of countability in Schenkman's theorem cannot be dispensed with. This is a group of order $\mathfrak{c}$ (the cardinal of the continuum), whose Sylow p-subgroups are all of countable order, and not complemented.† The same example also allows a different interpretation, as follows.

The theorem that 'the order of a Sylow p-subgroup of a finite group is the highest power of p that divides the group order' allows no immediate translation into the language of infinite groups; but if it is reformulated as 'the product of the orders of the Sylow p-subgroups, over all different primes p, is the group order', then it makes sense in those infinite groups in which the order of a Sylow p-subgroup does not depend, for fixed p, on the particular Sylow p-subgroup chosen. By what has been said, this includes the countable locally finite groups, and for these it is in fact true. In groups with Sylow p-subgroups of different orders (for one and the same p), one would have to weaken the proposition further to 'there are Sylow p-subgroups, one for each prime p, such that the product of their orders is the group order'. However, not even this is true in the uncountable group we construct, though the group is metabelian and of exponent 6. Is it true for countable periodic groups without any solubility assumption? This looks as intractable a problem as the problem (due, we believe, to Tarski) whether there are infinite groups all of whose proper non-trivial subgroups have prime orders.

All our groups will not only be periodic, but boundedly periodic, that is of finite exponent; and the exponent will be as small as it can be, namely 6; moreover, the groups will not only be locally soluble, but soluble and even metabelian; the sets Π will consist of a single prime (usually $p = 2$, occasionally $p = 3$) only; and the Sylow p-subgroups will be elementary abelian or, in one case, nilpotent of class 2. All this may be taken to indicate that 'good' Sylow theorems and Hall theorems for infinite groups are not to be looked for in the direction indicated by Schenkman's theorem, even with some further stringent assumptions on the structure of the group thrown in.

1. Notation and preliminaries

If $\{A_i\}_{i\in I}$ is a family of groups, the *cartesian product*, denoted throughout by P, consists of all functions f on I to $\bigcup_{i\in I} A_i$ that satisfy

$$f(i) \in A_i,$$

with multiplication defined in the obvious way by

$$(fg)(i) = f(i)\,g(i), \quad \text{for all} \quad i \in I.$$

All the laws common to the A_i are laws in P, too; thus if all A_i have finite exponent n, then so has P; and if all A_i are soluble of length l, then so is P.

† [*Added in proof*, 6 *January* 1961.] A note by Kargapolov (1959) has come to our attention. In this he constructs an uncountable metabelian group of exponent 6 in which the Sylow 3-subgroup is not complemented.

The cartesian product contains the *direct product*, denoted throughout by P^*, consisting of all those $f \in P$ whose *support*

$$\sigma(f) = \{i \mid i \in I . f(i) \neq 1\}$$

is finite. We shall here be concerned with groups G contained in the cartesian product and containing the direct product

$$P^* \leqslant G \leqslant P;$$

the name *interdirect product* has been proposed for such groups.

For fixed $i \in I$, those f that satisfy

$$\sigma(f) \subseteq \{i\}$$

form a *co-ordinate subgroup* A_i^0 isomorphic to the *constituent* A_i; the *projection* $\pi_i \colon P \to A_i$ defined by

$$f\pi_i = f(i)$$

is an epimorphism whose restriction to A_i^0 is an isomorphism. The $f\pi_i$ are also called the *components* of f.

Two or more elements of a group will be called *consonant* if they jointly generate a p-subgroup, for some prime p. They must then, of course, be periodic of p-power order. There is a Sylow subgroup containing given elements if, and only if, they are consonant.

Assume that all constituent groups A_i have finite exponent n, so that P also has exponent n; then† elements $f, g, \ldots$ of P of p-power order are consonant if, and only if, their components $f\pi_i, g\pi_i, \ldots$ are consonant, for every $i \in I$.

LEMMA 1·1. *Let the cartesian product P have finite exponent, and let S be a Sylow p-subgroup of P. Then $S^* = S \cap P^*$ is a Sylow p-subgroup of the direct product P^*, and $S\pi_i$ is a Sylow p-subgroup of A_i; moreover, S is the cartesian product of its components $S\pi_i$, and S^* is their direct product. If for fixed p and for every $i \in I$ a Sylow p-subgroup S_i of A_i is given, then there is a unique Sylow p-subgroup S of P, namely the cartesian product of the S_i, and a unique Sylow p-subgroup S^* of P^*, namely their direct product, such that*

$$S\pi_i = S^*\pi_i = S_i.$$

The proof is obvious, and we omit it.

COROLLARY 1·2. *If $P^* \leqslant G \leqslant P$ and if S is a Sylow p-subgroup of G, then $S^* = S \cap P^*$ is a Sylow p-subgroup of P^*. Every Sylow p-subgroup S^* of P^* is contained in one and only one Sylow p-subgroup S of G, and S consists of all those elements of G that are consonant with the elements of S^*.*

One easily extends the definitions and results of this section from p-groups to Π-groups.

For most of our constructions we need the special case of interdirect products that arises when all constituent groups A_i coincide with a single group A. We then use the terms *cartesian power, direct power, interdirect power* of A, and write $P = A^I$

† We are indebted to the referee for reminding us that without the assumption of finite exponent this is not generally true.

for the cartesian power (no special notation will be required for the corresponding direct power P^*, nor for other interdirect powers).

The index set will not always be denoted by I; instead we reserve the letter I for a particular countable index set, namely

$$I = \{1, 2, 3, \ldots\}.$$

2. Sylow subgroups without complements

For our first example we take A to be the symmetric group of degree 3,

$$A = \mathrm{gp}\,(a, b;\ a^2 = b^3 = (ab)^2 = 1);$$

we put $ab = a'$. The group G is to be the subgroup of $P = A^I$ generated by the direct power P^* and one further element g_0, defined by

$$g_0(i) = a' \quad \text{for all} \quad i \in I.$$

Then G is clearly countable, metabelian, and of exponent 6. Denote by S the Sylow 2-subgroup of G defined by

$$S\pi_i = \mathrm{gp}\,(a) \quad \text{for all} \quad i \in I.$$

Then $S \leqslant P^*$; for every element of G outside P^* is of the form

$$g = f^* g_0, \quad f^* \in P^*;$$

and if we choose $i \in I - \sigma(f^*)$, then

$$g(i) = g_0(i) = a',$$

which is not consonant with $S\pi_i$; thus $g \notin S$.

Let T denote the (clearly unique, because normal) Sylow 3-subgroup of G. Then also $T \leqslant P^*$, and

$$ST = P^*.$$

It follows that S cannot be complemented by a Sylow 3-subgroup in G. This does not mean that S is not complemented in G; for let U be the group generated by T and g_0. Then clearly

$$SU = G.$$

Now $U \cap P^* = T$, and as $S \leqslant P^*$, then

$$S \cap U = (S \cap P^*) \cap U = S \cap (P^* \cap U) = S \cap T = \{1\}.$$

Thus S and U are complements in G, and we have proved the following proposition.

Theorem 2·1. *There is a countable metabelian group G of exponent 6 which has a Sylow 2-subgroup S which is not complemented by any Sylow 3-subgroup in G, but which has a complement U containing elements of order 2.*

A slight modification of our construction leads to a group with a Sylow subgroup that is not complemented at all. We replace the symmetric group of degree 3 by its direct product with the tetrahedral group, that is to say, we put

$$A = B \times C,$$

where

$$B = \mathrm{gp}\,(a, b;\ a^2 = b^3 = (ab)^2 = 1),$$
$$C = \mathrm{gp}\,(c, d;\ c^2 = d^3 = (cd)^3 = 1).$$

Then A is again metabelian and of exponent 6. A Sylow 2-subgroup D of A is given by

$$D = \mathrm{gp}\,(a, c, c^d),$$

where (as usual) $c^d = d^{-1}cd$. We note that $\hat{a} = abc$ is not consonant with the elements of D. We again take as our group G the subgroup of $P = A^I$ generated by the direct power P^* and one further element g_0, now defined by

$$g_0(i) = \hat{a} \quad \text{for all} \quad i \in I.$$

Then G is clearly countable, metabelian, and of exponent 6. Denote by S the Sylow 2-subgroup of G defined by

$$S\pi_i = D \quad \text{for all} \quad i \in I.$$

Then $S \leqslant P^*$; for every element of G outside P^* is of the form

$$g = f^* g_0, \quad f^* \in P^*;$$

and if we choose $i \in I - \sigma(f^*)$, then

$$g(i) = g_0(i) = \hat{a},$$

which is not consonant with $S\pi_i$; thus $g \notin S$.

Now let $U \leqslant G$ be such that $SU = G$. Then there are elements $s \in S$ and $u \in U$ such that $su = g_0$. As I is infinite and the support of s is finite, we can choose $i_0 \in I - \sigma(s)$; then

$$u(i_0) = g_0(i_0) = \hat{a}.$$

Next let h_0 denote the element of P^* defined by

$$h_0(i_0) = d,$$
$$h_0(i) = 1 \quad \text{for all} \quad i \neq i_0.$$

Then there are elements $s' \in S$ and $u' \in U$ such that $s'u' = h_0$. Now

$$u'(i_0) = xd,$$
$$u'(i) = y_i \quad \text{for all} \quad i \neq i_0,$$

where $x, y_i \in D$; it follows that $\sigma(u'^2) = \{i_0\}$ and

$$u'^2(i_0) = x'd^2,$$

where $x' \in \mathrm{gp}\,(c, c^d)$. Put $v = [u, u'^2]$. Then $\sigma(v) \subseteq \{i_0\}$, and

$$v(i_0) = [\hat{a}, x'd^2] = [c, x'd^2] = [c, d^2] = c^d \neq 1.$$

It follows that $v \neq 1$ and $v \in S$; but also clearly $v \in U$. Thus the intersection of S and U is non-trivial, and S has no complement in G.

THEOREM 2·2. *There is a countable metabelian group G of exponent* 6 *which has a Sylow* 2*-subgroup S which is not complemented in G.*

Note that by Schenkman's theorem G must also have a Sylow 2-subgroup which is complemented, and even complemented by a Sylow 3-subgroup of G.

3. Non-isomorphic Sylow subgroups

Examples are known of non-periodic groups with non-isomorphic Sylow p-subgroups, and even with Sylow p-subgroups of different finite orders, stringent further assumptions like supersolubility and the maximal condition for subgroups notwithstanding (Zappa 1941). A well-known example of a countable periodic (and even locally finite) group with non-isomorphic Sylow p-subgroups is the ('restricted symmetric') group of all finite permutations of a countably infinite set: this has to every prime p continuously many mutually non-isomorphic Sylow p-subgroups.† By contrast, *if a locally finite group G has a finite Sylow p-subgroup S, then all Sylow p-subgroups of G are conjugate and hence isomorphic to S*; for if T is any other finite p-subgroup of G, then S and T jointly generate a finite subgroup of G, in which S is a Sylow p-subgroup and T is contained in some conjugate of S: thus the order of T is bounded by that of S; if G also contained an infinite p-subgroup, then we could pick from it more elements than there are in S, but finitely many, and they would generate a finite p-subgroup of order strictly greater than that of S: as we have seen, this is impossible. We remark that *in a countable locally finite group all Sylow p-subgroups* (for fixed p) *have the same order*; for if one of them is finite, they are all isomorphic, and if none of them is finite, they are all countably infinite. Moreover, we remind the reader that *all periodic locally soluble groups are locally finite.*

We now proceed to construct a countable locally finite group with continuously many mutually non-isomorphic Sylow 3-subgroups. Unlike the restricted symmetric group referred to above, our group will be soluble, in fact metabelian, and of finite exponent, namely 6. In a countable group of exponent 6 all Sylow 2-subgroups are isomorphic, because they are elementary abelian and have the same order. This is why we consider Sylow 3-subgroups.

We begin by introducing the finite groups from which our group is to be built up. Let n be a positive integer, and put

$$A_n = \mathrm{gp}\,(a_n, b_n, c_{n0}, c_{n1}, \dots, c_{nn}, z_{n0}, z_{n1}, \dots, z_{nn};\ (3\cdot1))$$

with the defining relations (3·1) below. For the present we suppress a suffix n:

$$\left.\begin{array}{r} a^3 = b^2 = (ab)^3 = c_0^3 = c_1^3 = \dots = c_n^3 = z_0^3 = z_1^3 = \dots = z_n^3 = 1; \\ {[a, c_i] = z_i,\ [a, z_i] = [b, c_i] = [b, z_i] = [c_i, c_j] = [c_i, z_j] = [z_i, z_j] = 1} \\ (i, j = 0, 1, \dots, n). \end{array}\right\} \quad (3\cdot1)$$

The generators z_i are redundant, as are also some of the defining relations. The group can be described as the splitting extension of the direct product of a four-group (generated by b and b^a) and an elementary abelian group of order 3^{2n+2} (generated by $c_0, c_1, \dots, c_n, z_0, z_1, \dots, z_n$) by the automorphism of order 3 (induced by a) which maps b on b^a and b^a and bb^a, multiplies c_i by z_i, and fixes z_i ($i = 0, 1, \dots, n$). We put

$$a' = ab;$$

† We do not know where this fact is to be found in the literature.

this is also of order 3, and induces the same automorphism of the direct product as a does. We note that

$$S_n = \mathrm{gp}(a, c_0, c_1, \ldots, c_n)$$

and

$$S'_n = \mathrm{gp}(a', c_0, c_1, \ldots, c_n)$$

are Sylow 3-subgroups of A_n. They are nilpotent of class 2 and have exponent 3; and A_n is metabelian and has exponent 6: we omit the easy verification. We also note that the centre of S_n (and of S'_n) is

$$Z_n = \mathrm{gp}(z_0, z_1, \ldots, z_n);$$

that the elements of

$$C_n = \mathrm{gp}(c_0, c_1, \ldots, c_n, z_0, z_1, \ldots, z_n)$$

have 3 or fewer conjugates each; and that this property characterizes them among the elements of S_n (and of S'_n). Further, the centralizer of C_n in S_n (and in S'_n) is C_n itself; and the index of Z_n in C_n is

$$|C_n : Z_n| = 3^{n+1}.$$

We now consider the different groups A_n simultaneously, and therefore restore the suffix n to the generators. The group we construct is a direct product of countable interdirect powers of A_n, one for each n. It is, however, best constructed in one step; we use a double index set $I^2 = I \times I$, where again $I = \{1, 2, 3, \ldots\}$; the elements of I^2 will be written mn. The cartesian product P is to consist of all functions f on I^2 to $\bigcup A_n$, subject to

$$f(mn) \in A_n.$$

Multiplication is 'componentwise', as always; the projections are π_{mn}. The direct product P^* consists of the functions f^* with finite support $\sigma(f^*)$.

The group G is to be generated by P^* and further elements g_i, one each for $i = 1, 2, 3, \ldots$, defined by

$$g_i(mn) = 1 \quad \text{when} \quad n \neq i,$$
$$g_i(mi) = a_i \quad \text{for all} \quad m \in I.$$

G is then clearly countable, metabelian, and of exponent 6.

Let J be an arbitrary subset of I. We define a Sylow 3-subgroup S_J of G by its components, as follows.

$$S_J \pi_{mn} = S_n \quad \text{for all} \quad m \in I,\ n \in J,$$
$$S_J \pi_{mn} = S'_n \quad \text{for all} \quad m \in I,\ n \in I - J.$$

Our object is to show that if $J \neq J'$, then S_J and $S_{J'}$ are not isomorphic; we do this by showing how J can be uniquely recovered from the *abstract* properties of S_J (as distinct from the representation of S_J as a subgroup of P).

First we note that $g_n \pi_{mn}$ is consonant with S_n but not with S'_n, because $a'_n \in S'_n$ is not consonant with $a_n = g_n \pi_{mn}$. Hence $g_n \in S_J$ if, and only if, $n \in J$. It suffices, therefore, to characterize those numbers n for which $g_n \in S_J$, using the structure of S_J, not, of course, its suffix. To this end we consider elements $s \in S_J$ with precisely

3 conjugates. They are the elements that have just one component with 3 conjugates, and all others central; that is to say, there is a pair $m'n' \in I^2$ such that

$$\left.\begin{aligned} s\pi_{m'n'} &\in C_{n'} - Z_{n'}, \\ s\pi_{mn} &\in Z_n \quad \text{for all} \quad mn \in I^2 - \{m'n'\}. \end{aligned}\right\} \tag{3·2}$$

Such an element s commutes with g_n if $n \neq n'$, but fails to commute with $g_{n'}$.

LEMMA 3·3. *The number n' belongs to J if, and only if, S_J contains an element g with the property*

$\mathscr{P}_{n'}$: g fails to commute with every element s to which there is a number m' such that (3·2) *is satisfied.*

Proof. We have already seen that $g = g_{n'}$ has $\mathscr{P}_{n'}$, and $g_{n'} \in S_J$ if $n' \in J$; this proves the 'only if' part. To prove the converse, assume $g \in S_J$ has property $\mathscr{P}_{n'}$. Write g in the form

$$g = f^* g_{n_1}^{\pm 1} g_{n_2}^{\pm 1} \dots g_{n_r}^{\pm 1}, \quad f^* \in P^*,$$

with $g_{n_1}, g_{n_2}, \dots, g_{n_r} \in S_J$. Now f^* has only a finite number of non-trivial components, and so there can be only a finite number of pairs $m'n'$ such that f^* fails to commute with an s satisfying (3·2). All g_n with $n \neq n'$ commute with all s that satisfy (3·2). Hence n' must occur among $n_1, n_2, \dots, n_r$; and it follows that $n' \in J$. This completes the proof of the lemma.

The lemma does not yet give the structural characterization we look for, as (3·2) is expressed in terms of the pair $m'n'$. However, the property $\mathscr{P}_{n'}$ depends only on the second member of the pair, and we now show how this is determined abstractly from s. To this end we consider the set $K(s)$ of all those elements s' of S_J that are permutable with all elements that are permutable with s; then $K(s)$ is a subgroup, namely the centralizer in S_J of the centralizer in S_J of s. One readily verifies that $K(s)$ consists of the centre Z, say, of S_J together with all elements s' that satisfy

$$\begin{aligned} s'\pi_{m'n'} &\in C_{n'} - Z_{n'}, \\ s'\pi_{mn} &\in Z_n \quad \text{for all} \quad mn \in I^2 - \{m'n'\}, \end{aligned}$$

with the same pair $m'n'$ as for s. It follows that

$$K(s)/Z \cong C_{n'}/Z_{n'},$$

and the index of the centre in $K(s)$ is

$$|K(s):Z| = 3^{n'+1}.$$

Thus n' is abstractly determined by s, and we can paraphrase Lemma 3·3 as follows:

LEMMA 3·4. *The number n' belongs to J if, and only if, S_J contains an element g which fails to commute with every element s with* 3 *conjugates and the property that the index of the centre in the centralizer of the centralizer of s is $3^{n'+1}$.*

COROLLARY 3·5. *If $J \neq J'$, then S_J and $S_{J'}$ are not isomorphic.*

As there are continuously many subsets J of I, we see that G has at least continuously many mutually non-isomorphic Sylow 3-subgroups. But G is countable, and so cannot have more than continuously many subgroups of any kind; and we have proved:

THEOREM 3·6. *There is a countable metabelian group G of exponent 6 with continuously many mutually non-isomorphic Sylow 3-subgroups.*

It may be remarked that here 'metabelian' is best possible, because in a nilpotent group, and even in a locally nilpotent group, the Sylow p-subgroups are unique (cf. Kurosh 1956, p. 229). The exponent 6 is also obviously the smallest possible. Finally, we note that our group G is an extension of an FC-group (namely P^*) by an elementary abelian 3-group, and by contrast in an FC-group all Sylow p-subgroups are isomorphic (cf. Neumann 1958).

We conclude this section by presenting a simple example of a metabelian group of exponent 6 with two Sylow 2-subgroups of different orders. By a remark at the beginning of this section such a group must have uncountable order.

Let A again be the symmetric group of degree 3,

$$A = \mathrm{gp}\,(a, b;\ a^2 = b^3 = (ab)^2 = 1),$$

and put, as before,

$$a' = ab.$$

We form the cartesian power $P = A^I$, and define G as the subgroup of P generated by the direct power P^* and further elements g_J, indexed by the subsets J of I, and defined by

$$g_J(i) = a' \quad \text{for} \quad i \in J,$$
$$g_J(i) = 1 \quad \text{for} \quad i \in I - J.$$

Clearly G is metabelian and of exponent 6, and of order

$$|G| = \mathfrak{c} = 2^{\aleph_0}.$$

We note that the product of two generators g_J, $g_{J'}$ is itself a generator

$$g_J g_{J'} = g_{J''},$$

namely the one that belongs to the symmetric difference

$$J'' = (J - J') \cup (J' - J).$$

Hence every element of G can be written in the form

$$g = f^* g_J,$$

where $f^* \in P^*$ and J is empty or infinite.

Now consider the Sylow 2-subgroups S, S' of G defined by

$$S\pi_i = \mathrm{gp}\,(a) \quad \text{for all} \quad i \in I,$$
$$S'\pi_i = \mathrm{gp}\,(a') \quad \text{for all} \quad i \in I.$$

Then $S \leqslant P^*$; for if $g \in G - P^*$, then

$$g = f^* g_J$$

with $f^* \in P^*$ and J infinite; choosing $i \in J - \sigma(f^*)$, we have

$$g(i) = g_J(i) = a',$$

and this is not consonant with $S\pi_i$; thus $g \notin S$. It follows that S is countable. On the other hand, every g_J is consonant with every $S'\pi_i$, whence it is seen that S' contains every g_J and thus has order $\mathfrak{c}$:

THEOREM 3·7. *There is a metabelian group G of exponent 6 and order $\mathfrak{c}$ which has a countable Sylow 2-subgroup S and a Sylow 2-subgroup S' of order $\mathfrak{c}$.*

4. Uncountable groups with countable Sylow subgroups

For the construction of our next example we require a set of continuously many subsets of a countable set with pairwise finite intersections. The existence of such a set was first proved by Sierpiński (1928). The countable set is again $I = \{1, 2, 3, \ldots\}$ and we denote by Σ the set of all subsets σ of I that satisfy the following three conditions:

(i) $1 \in \sigma$;
(ii) if $i \in \sigma$ and $i < j < 2i$, then $j \notin \sigma$;
(iii) if $i \in \sigma$, then $2i \in \sigma$ or $2i+1 \in \sigma$.

Then every $\sigma \in \Sigma$ clearly also has the properties

(iv) σ is infinite;
(v) if $1 < i \in \sigma$, then $[\frac{1}{2}i] \in \sigma$;
(vi) if $1 < i \in \sigma$, then $i+1 \notin \sigma$;

and it is not difficult to verify that the $\sigma \in \Sigma$ are characterized also by (iv), (v), (vi). We also note, for later reference, another property of $\sigma \in \Sigma$, namely

(vii) if $j \geq 1$, then there is one and only one $i \in \sigma$ in the range $j \leq i \leq 2j-1$.

If σ, σ' are elements of Σ, and if $1 < i \in \sigma \cap \sigma'$, then also $[\frac{1}{2}i] \in \sigma \cap \sigma'$, by (v), and no j strictly between $[\frac{1}{2}i]$ and i belongs to either σ or σ', by (ii) and (vi); hence σ and σ' coincide on $\{1, 2, \ldots, i\}$. It follows that if $\sigma \cap \sigma'$ is infinite, then $\sigma = \sigma'$, or, differently put, different elements of Σ have finite intersections.

Next we allocate to each $\sigma \in \Sigma$ a sequence $\epsilon = \{\epsilon_1, \epsilon_2, \epsilon_3, \ldots\}$ of zeros and ones: if $\sigma = \{i_1, i_2, i_3, \ldots\}$ with $1 = i_1 < i_2 < i_3 < \ldots$, we put

$$\epsilon_n = i_{n+1} - 2i_n.$$

As $i_1 = 1$ and
$$i_{n+1} = 2^n + \epsilon_1 2^{n-1} + \epsilon_2 2^{n-2} + \ldots + \epsilon_n,$$

the correspondence between the $\sigma \in \Sigma$ and the sequences ϵ is one-to-one, and it follows that the cardinal of Σ is $\mathfrak{c}$.

We now choose a finite group A that contains two elements a, a' which are not consonant but have equal prime power order. For the present we shall take A to be the symmetric group of degree 3,

$$A = \text{gp}(a, b;\ a^2 = b^3 = (ab)^2 = 1);$$

and we again put $a' = ab$. But the construction can equally well be carried out with a different group A, and we shall later vary A.

The group G we define will again be an interdirect power of A contained in the cartesian power $P = A^I$ (and containing the direct power P^*). Specifically, G is to be the group generated by P^* and certain further elements g_σ, one to each $\sigma \in \Sigma$, which are defined as follows.

(i) If $i \in \sigma$ and $2i \in \sigma$, then $g_\sigma(i) = a$;
(ii) if $i \in \sigma$ and $2i+1 \in \sigma$, then $g_\sigma(i) = a'$;
(iii) if $i \notin \sigma$, then $g_\sigma(i) = 1$.

It follows that the support of g_σ is

$$\sigma(g_\sigma) = \sigma.$$

Now if $f, f' \in P$, then clearly

$$\sigma([f, f']) \subseteq \sigma(f) \cap \sigma(f').$$

It follows that

$$[g_\sigma, g_{\sigma'}] \in P^*;$$

and it is not difficult to see that G/P^* is an elementary abelian 2-group with basis $\{g_\sigma P^*\}_{\sigma \in \Sigma}$. We note that the order of G is $\mathfrak{c}$, and also that the elements of order 3 form, with the unit element, a subgroup T of P^* which is the—clearly unique—Sylow 3-subgroup of G.

LEMMA 4·1. *Every consonant set of elements of G is countable.*

Proof. Every element of G can be written in the form

$$f = f^* g_{\sigma_1} g_{\sigma_2} \cdots g_{\sigma_r}, \tag{4·2}$$

where $f^* \in P^*$. We may take $\sigma_1, \sigma_2, \ldots, \sigma_r$ all different here, and then r depends on the element f only, and so do $\sigma_1, \sigma_2, \ldots, \sigma_r$. The support $\sigma(f^*)$ is finite, and so are the intersections $\sigma_m \cap \sigma_n$ for $m \neq n$. Thus the set

$$\sigma(f^*) \cup \bigcup_{m<n} \sigma_m \cap \sigma_n$$

is finite, and there is a least integer $k(f)$ that exceeds all its elements. It is not difficult to see that $k(f)$ depends on f only, not on its representation (4·2); for $k(f)$ is the least integer k with the property that if $i \geqslant k$ and if $i \in \sigma(f)$, that is if $f(i) \neq 1$, then there is a unique σ_n among $\sigma_1, \sigma_2, \ldots, \sigma_r$ such that $i \in \sigma_n$, and

$$f(i) = g_{\sigma_n}(i).$$

To each σ_n, $1 \leqslant n \leqslant r$, there is, by property (vii), one and only one i in the range $k(f) \leqslant i \leqslant 2k(f)-1$ such that $i \in \sigma_n$; we denote this by $j_n(f)$; and we put

$$J(f) = \{j_1(f), j_2(f), \ldots, j_r(f)\}.$$

We note in passing that the $j_n(f)$ are all distinct, and thus $r \leqslant k(f)$. We also note that J depends on f only, not on the order of the factors in (4·2).

There are only countably many possible values of $k(f)$, and to each of them there are only finitely many sets (namely the subsets of $\{k(f), k(f)+1, \ldots, 2k(f)-1\}$) that can serve as $J(f)$. Let F be an uncountable subset of G. Then there is at least one integer, k_0 say, and at least one set $J_0 \subseteq \{k_0, k_0+1, \ldots, 2k_0-1\}$ which are the $k(f)$ and $J(f)$, respectively, of uncountably many $f \in F$. Thus the set

$$F_0 = \{f \mid f \in F . k(f) = k_0 . J(f) = J_0\}$$

is uncountable.

Let $f_0 \in F_0$. As P^* is countable, there are elements in F_0 that are not congruent to f_0 modulo P^*; let f'_0 be one of them. We represent f_0, f'_0 in the form

$$f_0 = f_0^* g_{\sigma_1} g_{\sigma_2} \cdots g_{\sigma_r}, \tag{4·3}$$

$$f'_0 = f_0^{*\prime} g_{\sigma'_1} g_{\sigma'_2} \cdots g_{\sigma'_r}. \tag{4·3'}$$

Note that the number r of factors outside P^* is the same for both, because $r = |J_0|$; note also that J_0 is not empty, as otherwise f_0 and f'_0 would be congruent modulo P^*. There is at least one σ_n that does not occur among $\sigma'_1, \sigma'_2, \ldots, \sigma'_r$; for otherwise the sets $\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ and $\{\sigma'_1, \sigma'_2, \ldots, \sigma'_r\}$ would be equal, and f_0 and f'_0 would be congruent modulo P^* and commutators, hence modulo P^*, contrary to assumption. We may assume that σ_1 does not occur among the $\sigma'_1, \sigma'_2, \ldots, \sigma'_r$, because changing the order of the factors g_{σ_n} only changes the initial f_0^* in (4·3). Now as

$$J(f_0) = J_0 = J(f'_0),$$

there is a number n, $1 \leqslant n \leqslant r$, such that $j_n(f'_0) = j_1(f_0)$, and again we lose no generality by assuming $n = 1$. Thus $\sigma_1 \neq \sigma'_1$, but there is an i, namely

$$i = j_1(f_0) = j_1(f'_0) \geqslant k_0,$$

which is common to σ_1 and σ'_1. Let i^* be the greatest number in the finite set $\sigma_1 \cap \sigma'_1$. Then

$$i^* \geqslant k_0 = k(f_0) = k(f'_0),$$

and therefore

$$f_0(i^*) = g_{\sigma_1}(i^*),$$

$$f'_0(i^*) = g_{\sigma'_1}(i^*).$$

As one of σ_1, σ'_1 contains $2i^*$, while the other contains $2i^*+1$ (because i^* is the greatest element of their intersection), one of $g_{\sigma_1}(i^*)$, $g_{\sigma'_1}(i^*)$ is a and the other is a'. It follows that $f_0(i^*)$ and $f'_0(i^*)$ are not consonant, and thus f_0 and f'_0 are not consonant either. Thus every uncountable subset F of G contains pairs of elements that are not consonant, and the lemma follows.

THEOREM 4·4. *There is a metabelian group G of exponent 6 and of order $\mathfrak{c}$ all of whose Sylow p-subgroups are countable. It has a (normal, hence unique) Sylow 3-subgroup T which has no complement in G. Every Sylow 2-subgroup S of G has a complement U in G which is normal in G and contains elements of order 2.*

Proof. The group G we have here constructed is—as an interdirect power of a metabelian group of exponent 6—metabelian and of exponent 6; we have already seen that its order is $\mathfrak{c}$; and its Sylow p-subgroups are countable by lemma 4·1. As all elements of order 3 belong to the unique Sylow 3-subgroup T, any complement of T would have to be of exponent 2, and so would be contained in a Sylow 2-subgroup S: but these are countable only, and so ST is only a countable subgroup of G. On the other hand, given a Sylow 2-subgroup S of G, we can construct a complement U of it as follows: As G/T is an elementary abelian 2-group, every subgroup of it is a direct factor. We choose U so as to contain T and so that U/T is a complementary direct factor of ST/T in G/T. Then S and U together generate G; but U contains the derived group T of G and hence is normal in G; it follows that

$G = SU$. Finally, as U/T and ST/T are complements, $S \cap U$ is contained in T. But $S \cap T = \{1\}$, whence also $S \cap U = \{1\}$, and the theorem follows.

We could have carried out the same construction with other primes than 2 or 3, or with 2 and 3 interchanged. Specifically, let H be the group obtained by the same construction when the symmetric group A of degree 3 is replaced by the tetrahedral group

$$B = \mathrm{gp}\,(a, b;\, a^3 = b^2 = (ab)^3 = 1),$$

where again $a' = ab$. Denote by K the direct product

$$K = G \times H. \tag{4·5}$$

As B is also metabelian and of exponent 6, so then is H, and K. The Sylow p-subgroups of K are the direct products of Sylow p-subgroups of G and of H, hence countable. To show that they are not complemented, we first prove the following more general lemma.

Lemma 4·6. *Let G, H be arbitrary groups with subgroups $T \leqslant G$ and $U \leqslant H$, and let $K = G \times H$ and $V = T \times U$. If V is complemented in K, then T is complemented in G.*

Proof. Let W be a complement of V in K; thus $VW = K$ and $V \cap W = \{1\}$. Denote by π and ρ the projections of K onto G and H, respectively. We show that

$$X = (U\rho^{-1} \cap W)\pi$$

is a complement of T in G. Clearly X is a subgroup of G. Let $g \in G$ be arbitrary; then there are elements $v \in V$, $w \in W$ such that $g = vw$. Write $v = tu$ and $w = xy$, where $t = v\pi$, $u = v\rho$, $x = w\pi$, $y = w\rho$. Then $t \in T$ and $u \in U$; also $uy = g\rho = 1$, whence $y \in U$, and $w \in U\rho^{-1} \cap W$; thus $x \in X$. Further, $tx = g\pi = g$, and it follows that $TX = G$. Finally let $x^* \in T \cap X$. Then there is an element $w^* \in U\rho^{-1} \cap W$ such that $w^*\pi = x^* \in T$ and $w^*\rho \in U$; then $w^* \in T \times U = V$; but as also $w^* \in W$ and $V \cap W = \{1\}$, it follows that $w^* = 1$ and $x^* = 1$. Hence $T \cap X = \{1\}$, and X is a complement of T in G, as required.

We return to the particular group K of (4·5), and consider a Sylow 3-subgroup V of it; now $V = T \times U$ where T is the unique Sylow 3-subgroup of G and U is some Sylow 3-subgroup of H. As T is not complemented in G, it follows from the lemma that V is not complemented in K. Correspondingly the Sylow 2-subgroups of K are not complemented either. Thus we have proved

Theorem 4·7. *There is a metabelian group K of exponent 6 and of order $\mathfrak{c}$ all of whose Sylow p-subgroups are countable, and none of whose Sylow 2-subgroups or 3-subgroups is complemented.*

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