Kovács, L. G.
Math. Zeitschr. 75, 277—294 (1961)

Groups with regular automorphisms of order four

By
L. G. Kovács

1. Introduction

An automorphism of a group G is called regular if it moves every element of G except the identity. Burnside proved that a finite group G has a regular automorphism of order two if and only if G is an abelian group of odd order, and then the only such automorphism maps every element onto its inverse ([2], p. 230). More recently several authors considered the question: what groups can admit regular automorphisms of order a prime p? B. H. Neumann [8] and M. Nagata [7] extended the original theorem about the case $p=2$ to various classes of infinite groups. The case $p=3$ appeared in the second edition of Burnside's book ([3], pp. 90—91), and this theorem was also generalized by Neumann [9]. For arbitrary p, the results so far can be summarized as follows. If a locally finite or locally nilpotent group G has a regular automorphism of prime order p, then G is nilpotent and its class is bounded in terms of p (G. Higman [4] and J. Thompson [10]).

In this paper we consider groups which admit regular automorphisms of order four. For this case one cannot obtain results like those quoted above. Finite nilpotent groups of every class have regular automorphisms of order four; hence locally nilpotent but non-nilpotent groups have them too. There are also non-nilpotent finite soluble groups, both metabelian and non-metabelian ones, that admit such automorphisms. (Examples are given in the last section of the paper.) What we can prove is that *if G is either a locally nilpotent or a periodic locally soluble group, and if G has a regular automorphism of order four, then the second derived group of G is contained in the centre of G.**) The examples just mentioned show this conclusion to be in a sense the best possible.

An outline of the proof is the following.

Let G be an arbitrary group with a regular automorphism α of order four, and let us consider the action of α^2. It is clear that α^2 need not be regular: the elements of G fixed by α^2 form a subgroup $T(G)$. The restriction of α to this subgroup is a regular automorphism of order two and so, under any of the conditions that we later adopt, $T(G)$ is abelian and its elements are all inverted by α. It is more difficult to see what happens to the elements

*) [*Added in proof,* 31. 12. 1960.] Dr. B. Huppert has kindly informed me of a new theorem of D. Gorenstein and I. N. Herstein, according to which a finite group with a regular automorphism of order four is always soluble. Hence in our result the condition "periodic locally soluble" could be replaced by "locally finite".

on which α^2 acts as if it were regular, that is, which are inverted by α^2. The set $S(G)$ formed by these elements need not be a subgroup. However, if we assume that every element of G has a unique square root in G, then G is the product of $T(G)$ and $S(G)$. This then implies that the subgroup $H(G)$ generated by $S(G)$ contains the derived group of G.

Here we turn to the investigation of $H(G)$. Since $H(G)$ may coincide with G, and in any case α can be restricted to $H(G)$, we may consider $H(G)$ without further reference to G. Thus let H be an arbitrary group which has a regular automorphism α of order four, such that the elements inverted by α^2 generate H. Assuming that H is locally nilpotent, commutator computations give that H is nilpotent of class at most three. Then we prove (by induction on the order of H) that the same conclusion holds if instead of local nilpotency we assume finiteness and solubility, and hence also if H is taken to be only periodic and locally soluble.

Returning to G with the condition on roots, we show that if $H(G)$ is nilpotent, then the second derived group of G is contained in the centre of G. Here again some computational work is involved, but after this the main theorems follow immediately. Namely, let G be an arbitrary group with a regular automorphism α of order four. If G is periodic and locally soluble, then the condition on roots is automatically satisfied. If G is locally nilpotent, then it can be embedded in a locally nilpotent group which satisfies the condition on roots and admits an extension of α as a regular automorphism of order four. So the previous theorems can be applied, and in either case we get that the second derived group of G is contained in the centre of G.

The major part of this paper was presented to the University of Manchester for the degree of M. Sc. It is a great pleasure to record my gratitude to Dr. B. H. Neumann, both for suggesting this topic to me and for his generous advice and guidance throughout. I also thank the University College of North Staffordshire for the Research Studentship which enabled me to carry out this work.

2. Notations and quotations

We follow the terminology and notation of Kurosh [5], apart from the exceptions and with the additions given in the first part of this section.

Let G be any group; then $H \subseteq G$ means that H is a subset, and $H \leqq G$ means that H is a subgroup of G. When H is a normal subgroup of G and a, b are elements of G belonging to the same coset of H, then the congruence $a \equiv b \bmod H$ may be used as a convenient way of recording this situation. We need the usual abbreviations $a^b = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$ in the following extended forms. For two subsets A, B of G, the symbol A^B denotes the set of all elements a^b with $a \in A$, $b \in B$. The n-th commutator $[h_1, \ldots, h_n]$ of the elements $h_1, \ldots, h_n$ of G is defined recursively by $[h_1, \ldots, h_n] = [[h_1, \ldots, h_{n-1}], h_n]$. If $H_1, \ldots, H_n$ are subsets of G, then $[H_1, \ldots, H_n]$ stands for the subgroup generated by all the commutators $[h_1, \ldots, h_n]$ with $h_1 \in H_1, \ldots,$ $h_n \in H$

The derived length of a soluble group G is the smallest integer n such that $G^{(n)}$ — the n-th derived group of G — is the unit subgroup E. Metabelian groups are the soluble groups of derived length two. We write the lower central chain of G as $G=\Gamma_1(G)\geqq\Gamma_2(G)\geqq\cdots$, and the upper central chain of G as $E=Z_0(G)\leqq Z_1(G)\leqq\cdots$.

In the course of computations, the following identities are used without further reference:

$$a^b=a[a,b],$$

$$[b,a]=[a,b]^{-1},$$

$$[a,b^{-1}]=[b,a]^{b^{-1}},$$

$$[a^{-1},b^{-1}]=[a,b]^{b^{-1}a^{-1}},$$

$$[ab,cd]=[a,d]^b[b,d][a,c]^{bd}[b,c].$$

Another important identity is

$$[a,b^{-1},c]^b[b,c^{-1},a]^c[c,a^{-1},b]^a=1; \tag{2.1}$$

we make a point of quoting it by number whenever it is applied.

Next we collect some definitions and results on groups with unique square roots. We take most of them from BAUMSLAG's thesis [*1*], but our notation is slightly different and the quotations rarely verbatim: as we use only special cases of his concepts and results, we do not quote them in their full generality. The numbering of the statements follows that in [*1*]; for instance, (B.5.1) stands for (a possibly simplified form of) (5.1) in [*1*].

Let G be any group, and $g\in G$. If $x^2=g$ $(x\in G)$, we write $x=g\sigma$ and say that x is a square root of g. If σ is a single-valued function (on the subset of G where it is defined at all), then we call G a Π-group, or write for short that $G\in[\Pi]$. If $G\in[\Pi]$ and if σ is defined on the whole of G, then G is called a Σ-group; for short, $G\in[\Sigma]$.

(B.5.1) *A periodic group G is a Σ-group if and only if G has no element of order two.*

(B.5.3) *If $G\in[\Pi]$, if η is any endomorphism of G, and if g is any element of G such that $g\sigma$ exists, then $(g\eta)\sigma$ also exists and $(g\eta)\sigma=(g\sigma)\eta$.*

(B.5.6) *If $G\in[\Pi]$ and $g,h\in G$, then $[g^{2^n},h^{2^m}]=1$ (n, m non-negative integers) is equivalent to $[g,h]=1$.*

Again, let G be any group. A subset H of G is closed (in G) if $x^2\in H$ $(x\in G)$ implies that $x\in H$. (We usually omit the reference to G as it is always clear from the context which group plays its role.) The intersection of closed subsets is closed. Let $cl(H)$ be the intersection of all closed subgroups of G that contain H; this is the minimal closed subgroup that contains H, and it is called the closure of H in G. A constructive definition of $cl(H)$ is the following. Let H_1 be the subgroup generated by H, ..., let H_{i+1} be the subgroup generated by all x $(\in G)$ such that $x^2\in H_i$, ... Then $cl(H)$ is the union of

$H_1, \ldots, H_i, \ldots$. — If H is normal, characteristic in G, then $cl(H)$ is also normal, characteristic in G, respectively. We shall use $\Delta_i(G)$ to denote the closure of $\Gamma_i(G)$ in G.

Any closed subgroup of a Σ-group is itself a Σ-group.

An ideal is a normal subgroup N of G such that G/N is a Π-group. Every ideal is closed.

(B.9.1) *If H, K are subsets of a Π-group and if $[H, K]=E$, then also $[cl(H), cl(K)]=E$.*

(B.9.2) *All terms of the upper central chain of a Π-group are ideals.*

(B.9.4) *A locally nilpotent group is a Π-group if and only if it has no element of order two.*

(B.9.5) *Any closed normal subgroup of a locally nilpotent group is an ideal.*

(B.10.2) *If G is a Π-group, and if H is an ideal and K a subgroup of G such that $[H, K]=E$ and $cl(K) \geqq H$, then $cl(KH/H)=cl(K)/H$.*

M. LAZARD has proved the following statement (in a more general form; [6], pp. 180, 186):

(2.2) *If G is a locally nilpotent group without elements of order two, then there exists a locally nilpotent Σ-group G^* such that* (i) *G^* contains G,* (ii) *to every element g^* of G^* there is a number $n=n(g^*)$ for which $(g^*)^{2^n}$ belongs to G, and* (iii) *every automorphism of G can be extended in one and only one way to an automorphism of G^*.*

Now we turn to automorphisms. Let us repeat that an automorphism α of a group G is called regular if $g\alpha=g$ $(g \in G)$ implies that $g=1$. For brevity we shall say that G is an A_n-group if G has a regular automorphism whose order divides n; sometimes we shall write $(G, \alpha) \in [\mathsf{A}_n]$ to express that α is such an automorphism of G. We shall also use $(G, \alpha) \in [\mathsf{A}_n \Pi]$, $(G, \alpha) \in [\mathsf{A}_n \Sigma]$ if simultaneously $(G, \alpha) \in [\mathsf{A}_n]$ and $G \in [\Pi]$, $(G, \alpha) \in [\mathsf{A}_n]$ and $G \in [\Sigma]$, respectively.

We need a remark of G. ZAPPA [11]:

(2.3) *If α is a regular automorphism of a finite group G, then every element g of G can be written in the form $g=x^{-1}x\alpha$ with some $x \in G$.*

Again we repeat that if $(G, \alpha) \in [\mathsf{A}_4]$, then $T(G)$ is the set of all t $(\in G)$ such that $t\alpha^2=t$, and $S(G)$ is the set of all s $(\in G)$ such that $s\alpha^2=s^{-1}$. Clearly $T(G)\alpha=T(G)$, $S(G)\alpha=S(G)$; and $T(G)$ is a subgroup of G.

Finally, two of the results mentioned in the introduction have to be stated precisely for reference in the sequel.

(2.4) (NEUMANN [8]) *If $(G, \alpha) \in [\mathsf{A}_2 \Sigma]$, then G is abelian and $g\alpha=g^{-1}$ for every g in G.*

(2.5) (NAGATA [7], HIGMAN [4]) *If G is a locally nilpotent group and if $(G, \alpha) \in [\mathsf{A}_2]$, then G is abelian and $g\alpha=g^{-1}$ for every g in G.*

3. Preparations

We start with a purely technical remark.

(3.1) *If α is a regular automorphism of a finite group G, then an equality of the form $a\alpha=a^g$ $(a, g\in G)$ can hold only for $a=1$.* Indeed, according to (2.3), $a\alpha=a^g$ yields $a\alpha=a^{x^{-1}x\alpha}$ with some x in G, whence $(a^{x^{-1}})\alpha=a^{x^{-1}}$, and, as α is regular, this can only happen if $a^{x^{-1}}=1$, $a=1$.

Next we present four lemmas on A_4- and Σ-groups.

(3.2) Lemma. *If $(G, \alpha)\in[\mathsf{A}_4]$ and G is locally finite, then G is a Σ-group.*

Proof. Take an arbitrary element g of G and consider its α-orbit, that is, the set consisting of the elements g, $g\alpha$, $g\alpha^2$, $g\alpha^3$. The subgroup K generated by these elements is finite and α-invariant; as such, it is the union of a finite number of disjoint α-orbits. Since the α-orbit of any non-identity element of G consists either of two or of four elements, it follows that the number of non-identity elements in K is even, the order of K is odd. Hence g itself is of odd order and, as g was arbitrary, the same holds for every element of G. Thus (B.5.1) proves the lemma.

(3.3) Lemma. *If $(G, \alpha)\in[\mathsf{A}_4\Sigma]$, then $T(G)$ is abelian and $t\alpha=t^{-1}$ for every t in $T(G)$.*

Proof. If $t\in T(G)$ then, using (B.5.3), we see that $(t\sigma)\alpha^2=(t\alpha^2)\sigma=t\sigma$, so $t\sigma$ also belongs to $T(G)$. This shows that $T(G)$ is a closed subgroup of G; as such, it is itself a Σ-group. If β denotes the restriction of α to $T(G)$, then $(T(G), \beta)\in[\mathsf{A}_2\Sigma]$, and so (2.4) implies the statement of the lemma.

(3.4) Lemma. *If $(G, \alpha)\in[\mathsf{A}_4\Sigma]$, then every element g of G can be written, in one and only one way, as $g=ts$ with $t\in T(G)$, $s\in S(G)$.*

Proof. Let us first assume that $g=ts$, $t\in T(G)$, $s\in S(G)$. Then $g^{-1}\alpha^2=(s^{-1}t^{-1})\alpha^2=st^{-1}$, so $g^{-1}\alpha^2g=s^2$, $s=(g^{-1}\alpha^2g)\sigma$; $s^{-1}=(s^2)^{-1}\sigma=(g^{-1}g\alpha^2)\sigma$, $t=gs^{-1}=g(g^{-1}g\alpha^2)\sigma$. This shows that no element of G can be written in the form $g=ts$ in more than one way.

On the other hand, using again that $(x\sigma)^{-1}=(x^{-1})\sigma$ with $x=g^{-1}\alpha^2g$, we always have that $g=g\cdot(g^{-1}g\alpha^2)\sigma\cdot(g^{-1}\alpha^2g)\sigma$; so we have to prove only that $g(g^{-1}g\alpha^2)\sigma=t\in T(G)$ and $(g^{-1}\alpha^2g)\sigma=s\in S(G)$ for every g in G. (B.5.3) enables us to argue that

$$\big((g^{-1}\alpha^2g)\sigma\big)\alpha^2=\big((g^{-1}\alpha^2g)\alpha^2\big)\sigma=(g^{-1}g\alpha^2)\sigma=\big((g^{-1}\alpha^2g)\sigma\big)^{-1},$$

so indeed $(g^{-1}\alpha^2g)\sigma=s\in S(G)$; using this,

$$(gs^{-1})\alpha^2(gs^{-1})^{-1}=g\alpha^2ssg^{-1}=g\alpha^2(g^{-1}\alpha^2g)g^{-1}=1,$$

so that $(gs^{-1})\alpha^2=gs^{-1}$, $gs^{-1}=t\in T(G)$.

(3.5) Lemma. *If $(G, \alpha)\in[\mathsf{A}_4\Sigma]$, then the subgroup $H(G)$ generated by $S(G)$ contains the derived group G' of G.*

Proof. First one shows that $H(G)$ is normal in G. If $s\in S(G)$ and $t\in T(G)$, then $(s^t)\alpha^2=(s^{-1})^t=(s^t)^{-1}$, so that $s^t\in S(G)$; whence $S(G)^{T(G)}=S(G)$. As (3.4) gives $G=T(G)\,S(G)$, we have that $S(G)^G=S(G)^{T(G)\,S(G)}=S(G)^{S(G)}\subseteq H(G)$,

and so indeed $H(G)^G = H(G)$. From (3.4) it follows also that $G = T(G)H(G)$, whence

$$G/H(G) = T(G)\,H(G)/H(G) \cong T(G)/\big(T(G) \cap H(G)\big),$$

which is abelian according to (3.3). This proves that $H(G)$ contains G'.

On locally nilpotent A_4-groups we have two similar lemmas. The first of these can be proved in the same way as (3.3), except that one has to use (2.5) instead of (2.4).

(3.6) Lemma. *If $(G, \alpha) \in [\mathsf{A}_4]$ and if G is locally nilpotent, then $T(G)$ is abelian and $t\alpha = t^{-1}$ for every t in $T(G)$.*

(3.7) Lemma. *If $(G, \alpha) \in [\mathsf{A}_4]$ and if G is locally nilpotent, then G has no element of order two.*

Proof. Take any element x of G such that $x^2 = 1$, and consider the subgroup K generated by x, $x\alpha$, $x\alpha^2$ and $x\alpha^3$. Then K is generated by a finite number of elements whose orders divide 2, therefore K is nilpotent, finite, and its order is a power of 2. On the other hand, α restricted to K is still a regular automorphism of order dividing 4, so that (3.2) and (B.5.1) imply that the order of K is odd. Hence the order of K can be only 2^0, proving that $x = 1$.

In the following two lemmas we turn to direct preparation for Section 4. There, in course of investigating a certain nilpotent A_4-group, we shall need homomorphic images of lower nilpotency class which admit regular automorphisms induced by the particular automorphism of the original group. Here we prove that such images are provided, for instance, by the homomorphisms corresponding to the closures $\Delta_i(G)$ of the members $\Gamma_i(G)$ of the lower central series of the group; also, that these closures $\Delta_i(G)$ form a central series of the group which inherits an analogue of the well-known property $[\Gamma_i(G), \Gamma_j(G)] \leqq \Gamma_{i+j}(G)$ of the lower central series of the group.

(3.8) Lemma. *If G is a nilpotent group without elements of order two, then $[\Delta_i(G), \Delta_j(G)] \leqq \Delta_{i+j}(G)$ (for every $i, j = 1, 2, \ldots$).*

Proof. This is done by induction on the nilpotency class c of G. For $c = 1$ the statement is obvious. If $c > 1$, we assume that the statement holds whenever the class of the group considered is less than c. — We have $G = Z_c(G)$, $\Gamma_{c+1}(G) = \Delta_{c+1}(G) = \Delta_{c+2}(G) = \cdots = E$, $\Gamma_c(G) \leqq Z_1(G)$. From (B.9.4) we know that $G \in [\Pi]$; by (B.9.2), $Z_1(G)$ is an ideal, so it is a fortiori closed. Hence $\Delta_c(G) = cl\,(\Gamma_c(G)) \leqq cl\,(Z_1(G)) = Z_1(G)$, so $[\Gamma_i(G), \Delta_c(G)] = E$; $\Delta_c(G)$ is closed and normal in G, so by (B.9.5) it is an ideal; finally, if $i \leqq c$ then $\Delta_i(G) = cl\,(\Gamma_i(G)) \geqq cl(\Gamma_c(G)) = \Delta_c(G)$. Thus (B.10.2) may be applied: $\Delta_i\,(G/\Delta_c(G)) = cl(\Gamma_i\,(G/\Delta_c(G))) = cl\,(\Gamma_i(G)\,\Delta_c(G)/\Delta_c(G)) = cl\,(\Gamma_i(G))/\Delta_c(G) = \Delta_i(G)/\Delta_c(G)$ for $i \leqq c$, and this can be extended to any i by writing it in the form $\Delta_i\,(G/\Delta_c(G)) = \Delta_i(G)\,\Delta_c(G)/\Delta_c(G)$. Now $G/\Delta_c(G)$ is of class $c - 1$, so the induction hypothesis gives that

$$[\Delta_i\,(G/\Delta_c(G)),\, \Delta_j\,(G/\Delta_c(G))] \leqq \Delta_{i+j}\,(G/\Delta_c(G)),$$
$$[\Delta_i(G)\,\Delta_c(G)/\Delta_c(G),\, \Delta_j(G)\,\Delta_c(G)/\Delta_c(G)] \leqq \Delta_{i+j}(G)\,\Delta_c(G)/\Delta_c(G),$$
$$[\Delta_i(G)\,\Delta_c(G),\, \Delta_j(G)\,\Delta_c(G)] \leqq \Delta_{i+j}(G)\,\Delta_c(G);$$

whence, using that $\Delta_c(G) \leqq Z_1(G)$, one obtains that $[\Delta_i(G), \Delta_j(G)] \leqq \Delta_{i+j}(G)\Delta_c(G)$. For $i+j \leqq c$ the right hand side is simply $\Delta_{i+j}(G)$ (since $\Delta_{i+j}(G) \geqq \Delta_c(G)$), so for this case the statement is verified. If however $i+j > c$, then $\Delta_{i+j}(G) = E$, and from $[\Gamma_i(G), \Gamma_j(G)] \leqq \Gamma_{i+j}(G) = E$ it follows by (B.9.1) that also $[\Delta_i(G), \Delta_j(G)] = E$. This completes the proof.

(3.9) Lemma. *Assume that* $(G, \alpha) \in [\mathsf{A}_4]$ *and that* G *is nilpotent. Then* α *induces regular automorphisms* α_i *on the* $G/\Delta_{i+1}(G)$, *for every* $i = 1, 2, \ldots$.

Proof. This again is done by induction on the nilpotency class c of G. If $c=1$, we have nothing to prove: $G/\Delta_{i+1}(G) = G$, $\alpha_i = \alpha$ for all i. If $c > 1$, let us assume that the statement of the lemma is true whenever the class of the group considered is less than c. The cases $i = c$, $c+1, \ldots$ are again obvious.

Take first $G/\Delta_c(G)$; since $\Delta_c(G)$ is characteristic, the mapping α_{c-1} defined by $(g\Delta_c(G))\alpha_{c-1} = (g\alpha)\Delta_c(G)$ is indeed an automorphism of $G/\Delta_c(G)$. If $(g\alpha)\Delta_c(G) = g\Delta_c(G)$, then $g\alpha = gd$ with some $d \in \Delta_c(G)$. (3.7) enables us to recall from the proof of (3.8) that $\Delta_c(G) \leqq Z_1(G)$. Thus $d \in Z_1(G)$, and so $(d\, d\alpha\, d\alpha^2\, d\alpha^3)\alpha = d\, d\alpha\, d\alpha^2\, d\alpha^3$; as α is regular, this means that $d\, d\alpha\, d\alpha^2 d\alpha^3 = 1$. Using this relation and the fact that also $d\alpha$, $d\alpha^2$, $d\alpha^3 \in Z_1(G)$, we get for $h = d^{-1}\alpha^2 \cdot d^{-2}\alpha \cdot d^{-3}$ that

$$h^{-1} h\alpha = d^3 d^2 \alpha\, d\alpha^2 d^{-1} \alpha^3 d^{-2} \alpha^2 d^{-3} \alpha = d^4 (d\, d\alpha\, d\alpha^2\, d\alpha^3)^{-1} = d^4.$$

On the other hand, $g\alpha = gd$ and $d \in Z_1(G)$ yield $g^4\alpha = g^4 d^4$. Thus $d^4 = g^{-4} g^4 \alpha = h^{-1} h\alpha$, therefore $(hg^{-4})\alpha = hg^{-4}$; as α is regular, we have in fact that $g^4 = h \in \Delta_c(G)$. Since $\Delta_c(G)$ is closed, this can only hold if g is in $\Delta_c(G)$. So we have proved that α_{c-1} is regular on $G/\Delta_c(G)$.

For $i < c-1$ we refer to the induction hypothesis. This applies to $G/\Delta_c(G)$ and α_{c-1}: the automorphisms induced by α_{c-1} on the $G/\Delta_c(G)/\Delta_{i+1}(G/\Delta_c(G))$ are regular. Recalling from the proof of (3.8) that $\Delta_{i+1}(G/\Delta_c(G)) = \Delta_{i+1}(G)/\Delta_c(G)$, one sees that these are in fact automorphisms of the $G/\Delta_c(G)/\Delta_{i+1}(G)/\Delta_c(G) \cong G/\Delta_{i+1}(G)$, induced by α through α_{c-1}, and this was to be proved.

4. A class of locally nilpotent A_4-groups

Following our set course, we now proceed with the study of an arbitrary group H which has a regular automorphism α of order four, such that the elements inverted by α^2 generate the whole of H. It seems natural to start by imposing strong conditions on H. For instance, if we assume that H is finite and nilpotent, can we get a bound for its nilpotency class? The answer is in the affirmative, and with the help of the lemmas presented in the previous section the whole argument carries over to the locally nilpotent case. We give it at once in this general form.

(4.0) Theorem. *If* H *is a locally nilpotent group, if* $(H, \alpha) \in [\mathsf{A}_4]$, *and if the elements that are inverted by* α^2 *generate* H, *then* H *is nilpotent of class at most three.*

The proof splits into several steps and occupies the rest of the section, where we always assume that the premisses of the theorem hold.

(4.1.1) The first step is a reduction. Since H is generated by $S(H)$, $\Gamma_4(H)$ is generated by all commutators of the form $[s_1, \ldots, s_4]$ $(s_1, \ldots, s_4 \in S(H))$ together with their conjugates in H. So if we prove that $[s_1, \ldots, s_4]=1$ for any four elements $s_1, \ldots, s_4$ of $S(H)$, then in fact we prove the theorem. To achieve this, we consider four arbitrary (but for the rest of the proof fixed) elements $s_1, \ldots, s_4$ of $S(H)$, and then the subgroup K generated by $s_1, \ldots, s_4, s_1\alpha, \ldots, s_4\alpha$, and prove that K is nilpotent of class at most three.

(4.1.2) If β denotes the restriction of α to K, then $(K, \beta) \in [\mathsf{A}_4]$. As a finitely generated subgroup of H, K is nilpotent of some class c. Also, $S(K)=S(H) \cap K$, so K is generated by $S(K)$. From (3.7) and (B.9.4) it follows that K is a Π-group.

We want to prove that $c \leqq 3$; it is convenient to restrict ourselves to the case $c \geqq 3$ and then prove that $c=3$.

(4.1.3) In the rest of the proof, we write briefly Γ_i, Δ_i, and Z_i for $\Gamma_i(K)$, $\Delta_i(K)$, and $Z_i(K)$. From (3.8) it follows that the Δ_i form a central series of K; as the length of such a series cannot be less than c, and as $\Delta_{c+1}=E$, all the $\Delta_1, \Delta_2, \ldots, \Delta_{c+1}$ are distinct. According to (3.9), β induces regular automorphisms β_i on the factor groups $K_i=K/\Delta_{i+1}$ $(i \leqq c)$. Each K_i is nilpotent of class exactly i, and from (B.9.5) it follows that each K_i is a Π-group.

Almost all the steps that follow are concerned with the derivation of congruences between commutators of arbitrary elements of $S(K)$. We use throughout without further mention that congruences modulo Γ_i automatically hold modulo Δ_i, and also that the Δ_i are characteristic in K.

Let then x, y, u, v denote arbitrary elements of $S(K)$; in this case also $x\beta, y\beta, u\beta, v\beta \in S(K)$.

(4.2.1) $[x, y]\beta \equiv [x, y]^{-1} \bmod \Delta_3$.

Proof. Since $[x, y]\beta^2=[x^{-1}, y^{-1}] \equiv [x, y] \bmod \Delta_3$, we have that $([x, y]\Delta_3)\beta_2^2 =[x, y]\Delta_3$; therefore $[x, y]\Delta_3 \in T(K_2)$. According to (4.1.3), (3.6) applies to $T(K_2)$ and gives that $([x, y]\Delta_3)\beta_2=([x, y]\Delta_3)^{-1}$, whence $([x, y]\beta)\Delta_3=[x,y]^{-1}\Delta_3$ and so (4.2.1) holds.

(4.2.2) $[x, y\beta] \equiv [x\beta, y] \bmod \Delta_3$.

Proof. $[x, y\beta]\beta=[x\beta, y^{-1}] \equiv [x\beta, y]^{-1} \bmod \Delta_3$; on the other hand, by (4.2.1), $[x, y\beta]\beta \equiv [x, y\beta]^{-1}$. Comparing these we get that $[x, y\beta]^{-1} \equiv [x\beta, y]^{-1}$, so indeed $[x, y\beta] \equiv [x\beta, y] \bmod \Delta_3$.

(4.3.1) $[a, b, c] \equiv [a, c, b][b, c, a]^{-1} \bmod \Delta_4$, for every $a, b, c \in K$.

This follows from (2.1).

(4.3.2) $[x, y, u\beta] \equiv [x, u, y\beta][y, u, x\beta]^{-1} \bmod \Delta_4$.

Proof. Apply (4.3.1) with $a=x$, $b=y$, $c=u$, and map the result by β:

$$[[x, y]\beta, u\beta] \equiv [[x, u]\beta, y\beta]\,[[y, u]\beta, x\beta]^{-1} \bmod \Delta_4.$$

Now use (4.2.1) and (3.8):

$$[[x, y]^{-1}, u\beta] \equiv [[x, u]^{-1}, y\beta]\,[[y, u]^{-1}, x\beta]^{-1} \bmod \Delta_4,$$

$$[x, y, u\beta]^{-1} \equiv [x, u, y\beta]^{-1}\,[y, u, x\beta] \bmod \Delta_4.$$

Taking inverses on both sides, we obtain the statement.

Similar reasoning leads to the following three congruences:

(4.3.3) $[x, y, u\beta] \equiv [x, u\beta, v]\,[y, u\beta, x]^{-1} \bmod \Delta_4$.

(4.3.4) $[x, u, y\beta] \equiv [x, y\beta\; u]\,[y, u\beta, x] \bmod \Delta_4$.

(4.3.5) $[y, u, x\beta] \equiv [x, y\beta, u]^{-1}\,[x, u\beta, y] \bmod \Delta_4$.

(4.3.6) $[x, y, u\beta] \equiv [x, y\beta, u] \bmod \Delta_4$.

Proof. By (4.3.2),

$$[x, y, u\beta]^2 \equiv [x, y, u\beta]\,[x, u, y\beta]\,[y, u, x\beta]^{-1} \bmod \Delta_4.$$

Substituting the terms on the right hand side from (4.3.3), (4.3.4), and (4.3.5), after cancellations one gets that $[x, y, u\beta]^2 \equiv [x, y\beta, u]^2 \bmod \Delta_4$. So

$$([x, y, u\beta]\,\Delta_4)^2 = ([x, y\beta, u]\,\Delta_4)^2$$

and, since $K_3 \in [\Pi]$, this implies that

$$[x, y, u\beta]\,\Delta_4 = [x, y\beta, u]\,\Delta_4.$$

(4.3.7) $[x, y\beta, u\beta]^{-1} \equiv [x, y, u] \bmod \Delta_4$.

This follows from (4.3.6), using (3.8).

(4.4.1) $[x, y, u, v] \equiv [x, y, v, u] \bmod \Delta_5$.

Proof. By (4.2.1), there exist $z, w \in \Delta_3$ such that $[x, y]\beta = [x, y]^{-1}z$ and $[u, v]\beta = [u, v]^{-1}w$. Using (3.8) we obtain that

$$[x, y, [u, v]]\,\beta = [[x, y]^{-1}z, [u, v]^{-1}w] \equiv [[x, y]^{-1}, [u, v]^{-1}] \equiv [x, y, [u, v]] \bmod \Delta_5.$$

Hence $([x, y, [u, v]]\,\Delta_5)\beta_4 = [x, y, [u, v]]\,\Delta_5$, and as β_4 is regular on K_4, this can only hold if $[x, y, [u, v]] \in \Delta_5$. Similarly one verifies that $[u, v^{-1}, [x, y]] \in \Delta_5$. Now apply (2.1) mod Δ_5 with $a=[x, y]$, $b=u$, $c=v$:

$$[x, y, u^{-1}, v]^u\,[u, v^{-1}, [x, y]]^v\,[v, [x, y]^{-1}, u]^{[x, y]} = 1$$

gives that

$$[x, y, u^{-1}, v]\,[v, [y, x], u] \equiv 1 \bmod \Delta_5,$$

whence

$$[x, y, u, v]^{-1}\,[y, x, v, u]^{-1} \equiv 1, \qquad [x, y, u, v]^{-1}\,[x, y, v, u] \equiv 1 \bmod \Delta_5.$$

(4.4.2) $[x, y, u, v] \in \Delta_5$.

Proof. $[x, y, u, v]\beta = [[x, y]\beta, u\beta, v\beta]$

$$\begin{aligned}
&\equiv [x, y, u\beta, v\beta]^{-1} \bmod \Delta_5, && \text{by (4.2.1) and (3.8)},\\
&\equiv [x, y\beta, u, v\beta]^{-1} \bmod \Delta_5, && \text{by (4.3.6) and (3.8)},\\
&\equiv [x, y\beta, v\beta, u]^{-1} \bmod \Delta_5, && \text{by (4.4.1)},\\
&\equiv [x, y, v, u] \bmod \Delta_5, && \text{by (4.3.7) and (3.8)},\\
&\equiv [x, y, u, v] \bmod \Delta_5, && \text{by (4.4.1)}.
\end{aligned}$$

So $([x, y, u, v]\Delta_5)\beta_4 = [x, y, u, v]\Delta_5$; as β_4 is regular on K_4, this can only hold if $[x, y, u, v]\Delta_5 = \Delta_5$.

(4.4.3) *K is nilpotent of class three.*

Proof. By (4.4.2), all the 4-th commutators of the generators $s_1, \ldots, s_4$, $s_1\alpha, \ldots, s_4\alpha$ of K belong to Δ_5; since Δ_5 is normal in K, it contains also their conjugates. Thus Δ_5 contains Γ_4, whence also $\Delta_4 = cl(\Gamma_4) \leqq \Delta_5$, so that in fact $\Delta_4 = \Delta_5$. As the central series $K = \Delta_1 > \cdots > \Delta_{c+1} = E$ was properly descending, this implies that $\Delta_4 = \Delta_5 = E$, a fortiori that $\Gamma_4 = E$, $c \leqq 3$. We assumed that $c \geqq 3$, so in fact $c = 3$.

In view of (4.4.1), this completes the proof of the theorem.

5. A class of (locally) finite and soluble A_4-groups

In the preceding section we started investigating a special class of A_4-groups: a group H belonged to this class if it had a regular automorphism α of order four, such that the elements that were inverted by α^2 generated the whole group H. There we worked under the additional assumption that H was locally nilpotent. Now we pursue the study of *finite* groups of this class, weakening the condition of nilpotency to solubility. Here we have to apply a very finite method: induction on the order of the group. Thus, though the problem is still of local character and admits a generalization in this direction, we cannot, as we did in the preceding section, deal with the generalized case at once.

Let us assume therefore, for the time being, that H is a finite group and that $(H, \alpha) \in [A_4]$. It will be convenient to write simply Γ_i instead of $\Gamma_i(H)$, and when it can be done without danger of ambiguity we shall also use T and S instead of $T(H)$ and $S(H)$. As usual, curly brackets denote (sub)-groups generated by the elements, or by the elements of the sets, listed between them. Let us recall the preparatory results concerning this case:

by (3.2) and (B.5.1),	$H \in [\Sigma]$, and the order of H is odd; so,
by (3.3),	T is abelian, and $t\alpha = t^{-1}$ for every t in T;
by (3.4),	$H = TS$;
by (2.3),	every element h of H can be written in the form $h = x^{-1}x\alpha$ with some x in H;
by (3.1),	$h\alpha = h^x$ (for $h, x \in H$) can only hold if $h = 1$.

Then we need four lemmas.

(5.1) Lemma. $[s^t, s\alpha] = [t, s, s\alpha] = 1$ *for every* $t \in T$, $s \in S$.

Proof. Let t, s be arbitrary elements of T, S, respectively. Then

$$[s^t, s\alpha]\,\alpha = [(s\alpha)^{t^{-1}}, s^{-1}] = [s, (s\alpha)^{t^{-1}}]^{s^{-1}} = [s^t, s\alpha]^{t^{-1}s^{-1}},$$

so by (3.1) we have $[s^t, s\alpha] = 1$. In particular, $[s, s\alpha] = 1$. Thus

$$1 = [s^t, s\alpha] = [s[s,t], s\alpha] = [s, s\alpha]^{[s,t]}[s, t, s\alpha] = [[t, s]^{-1}, s\alpha],$$

but then also $[t, s, s\alpha] = 1$.

(5.2) Lemma. $[t, s, s, s] = [t, s, s, s\alpha] = 1$ *for every* $t \in T$, $s \in S$.

Proof. Take any t from T and any s from S. Define $H_1 = \{s, s^t\}$, $H_2 = H_1\alpha = \{s\alpha, (s\alpha)^{t^{-1}}\}$, $H_3 = \{H_1, H_2\}$, $T_i = H_i \cap T$, $S_i = H_i \cap S$ $(i = 1, 2, 3)$. Then $H_i\alpha^2 = H_i$ for each i; since the order of H is odd, so is the order of H_i, and then $H_i \in [\Sigma]$. As in (3.4), one proves that $H_i = T_i S_i$. From (5.1) it follows that $[H_1, H_2] = E$, giving that $H_1 \cap H_2 \leqq Z_1(H_3)$. Since $T_1 = T_1\alpha = T_2$, $T_1 \leqq H_1 \cap H_2 \leqq Z_1(H_3)$; in particular, T_1 is normal in H_1. Consider now H_1/T_1. Define $(hT_1)\beta = (h\alpha^2)T_1$ for each h in H_1; since $T_1\alpha^2 = T_1$ and $H_1\alpha^2 = H_1$, β is an automorphism of H_1/T_1. Assume $(hT_1)\beta = hT_1$, that is, $h\alpha^2 = ht$ for some $h \in H_1$, $t \in T_1$. Then $t = h^{-1}h\alpha^2$; but we have seen in the proof of (3.4) that elements of this form belong to S, so $t \in T \cap S = E$. It follows from this that $h\alpha^2 = h$, $h \in T_1$, showing that β is regular. But then (3.2) and (2.4) imply that H_1/T_1 is abelian, $H_1' \leqq T_1$. Similarly $H_2' \leqq T_2 = T_1$; as we have already seen that $[H_1, H_2] = E$, we get also that $H_3' = \{H_1', H_2'\} \leqq T_1 \leqq Z_1(H_3)$. Now

$$[[t, s]^{-1}, s] = [s, t, s] = [s, s]^{[s,t]}[s, t, s] = [s[s, t], s] = [s^t, s] \in H_3' \leqq Z_1(H_3),$$

but then also $[t, s, s] \in Z_1(H_3)$, and our statement follows.

(5.3) Lemma. *Let N be any normal subgroup of H such that $T \cap Z_1(N) \neq E$. If s is an arbitrary element of S, and if $N_s = \{N, s, s\alpha\}$, then $Z_1(N_s) \neq E$.*

Proof. Take any nontrivial element t from $T \cap Z_1(N)$. As $Z_1(N)$ is normal in H, $[t, s, s] \in Z_1(N)$; further, by (5.2), $[t, s, s, s] = [t, s, s, s\alpha] = 1$, so in fact $[t, s, s] \in Z_1(N_s)$. If $[t, s, s] = 1$, then we consider $[t, s]$; this is also in $Z_1(N)$, and, by (5.1), $[t, s, s\alpha] = 1$, so that $[t, s] \in Z_1(N_s)$. Finally, if $[t, s] = 1$, then also $1 = [t, s]\alpha = [t^{-1}, s\alpha]$, whence $[t, s\alpha] = 1$, proving that $t \in Z_1(N_s)$. Therefore one of $[t, s, s]$, $[t, s]$, and t is a nontrivial element in $Z_1(N_s)$.

(5.4) Lemma. *If N is any subset of H such that $N^S \subseteq S$, then* $[N, S] = E$.

Proof. For any n in N and any s in S, $n^s \in S$, so that $(n^s)\alpha^2 = (n^s)^{-1}$; on the other hand, $n = n^1 \in S$ implies that $(n^s)\alpha^2 = (n^{-1})^{s^{-1}} = (n^{s^{-1}})^{-1}$. So $(n^s)\alpha^2 = (n^s)^{-1} = (n^{s^{-1}})^{-1}$, whence $n^s = n^{s^{-1}}$, so that $n^{s^2} = n$, $[n, s^2] = 1$. Thus by (B.5.6) also $[n, s] = 1$.

Now we are ready to prove the second theorem.

(5.5) Theorem. *Let H be a finite soluble group, $(H, \alpha) \in [\mathsf{A}_4]$, and assume that the elements that are inverted by α^2 generate H. Then H is nilpotent of class at most three.*

We prove this by induction on the order of H. The unit group provides a starting point, so we only have to give the inductive step. For this we assume that the order of H is greater than 1, and that the theorem holds whenever the order of the group considered is less than the order of H.

Again we split the proof into several steps.

(5.5.1) *If N is a normal subgroup of H such that $N\alpha = N$ and $N \neq E$, then $\Gamma_4(H)$ is contained in N.*

Proof. Define $(hN)\beta = (h\alpha)N$ for every $hN \in H/N$; clearly β is an automorphism of H/N. If $(hN)\beta = hN$, that is, if $h\alpha = hn$ for some h in H and n in N, then $h^{-1}h\alpha = n$. If γ means the restriction of α to N, then $(N, \gamma) \in [\mathsf{A}_4]$. According to (2.3), $n = m^{-1}m\alpha$ with some m in N; but then $n = h^{-1}h\alpha = m^{-1}m\alpha$, whence $(hm^{-1})\alpha = hm^{-1}$; as α is regular, this can only hold if $hm^{-1} = 1$, $h = m \in N$. Thus β is regular on H/N; further, $S(H/N) = S(H)N/N$ generates H/N, so that the inductive hypothesis applies to H/N: $\Gamma_4(H/N) = E$. But $\Gamma_4(H/N) = \Gamma_4(H)N/N$, so in fact $\Gamma_4(H) \leqq N$.

(5.5.2) *If $Z_1(H') = E$, then $H' = E$, so that $\Gamma_4(H) = E$.*

Proof. Suppose that, on the contrary, $H' \neq E$, but $Z_1(H') = E$. Then also $H'' \neq E$. As α can be restricted to H', and $T(H') = T(H) \cap H'$, $S(H') = S(H) \cap H'$, (3.4) gives that $H' = (T(H) \cap H')(S(H) \cap H')$. Since $T(H) \cap H'$ is abelian but H' is not, $S(H) \cap H' \neq E$. Also H'' is contained in $\{S(H) \cap H'\}$, by (3.5). Consider now $\{S(H) \cap H'\}$. This is a finite soluble group, in fact a proper subgroup of H (being contained in H'). Further, it is an α-invariant subgroup, and so α can be restricted to $\{S(H) \cap H'\}$ to show that it is an A_4-group. Finally, this group is generated by elements that are inverted by α^2. Hence the induction hypothesis implies that $\{S(H) \cap H'\}$ is nilpotent; since H'' is contained in $\{S(H) \cap H'\}$, it is nilpotent too, so that $Z_1(H'') \neq E$. As $Z_1(H'')$ is characteristic in H, (5.5.1) ensures that $\Gamma_4 = \Gamma_4(H)$ is contained in $Z_1(H'')$. On the other hand we know that $H'' \leqq \Gamma_4$, therefore $\Gamma_4 = Z_1(H'') = H'' \neq E$.

Now α can be restricted to Γ_4, so that Γ_4 is an A_4-group and $T(\Gamma_4) = T(H) \cap \Gamma_4$, $S(\Gamma_4) = S(H) \cap \Gamma_4$; therefore, by (3.4), $\Gamma_4 = (T(H) \cap \Gamma_4)(S(H) \cap \Gamma_4)$. If $T(H) \cap \Gamma_4 = E$, then $\Gamma_4 = S(H) \cap \Gamma_4 \subseteq S(H)$; as Γ_4 is normal in H, $\Gamma_4^{S(H)} = \Gamma_4 \subseteq S(H)$ too, so that (5.4) says that $[\Gamma_4, S(H)] = E$. Since $S(H)$ generates the whole of H, this means that $[\Gamma_4, H] = E$; a fortiori $[\Gamma_4, H'] = E$, whence $\Gamma_4 \leqq Z_1(H')$; but $\Gamma_4 \neq E$ while $Z_1(H') = E$, a contradiction. It follows therefore that $T(H) \cap \Gamma_4 \neq E$.

This is the same as saying that $Z_1(H'') \cap T(H) \neq E$. For any s in $S(H) \cap \Gamma_3$, the application of (5.3) with $N = H'' = \Gamma_4$ gives that $Z_1(\{\Gamma_4, s, s\alpha\}) \neq E$. Now $\{\Gamma_4, s, s\alpha\}$ is normal in H (because $s^h = s[s, h]$, $[s, h] \in \Gamma_4$ for any h in H, etc.), it is also α-invariant, and $Z_1(\{\Gamma_4, s, s\alpha\})$ is characteristic in $\{\Gamma_4, s, s\alpha\}$; hence (5.5.1) can be applied to $Z_1(\{\Gamma_4, s, s\alpha\})$ to conclude that $Z_1(\{\Gamma_4, s, s\alpha\})$ must contain Γ_4. Of this we only need only that $[\Gamma_4, s] = E$ for any $s \in S(H) \cap \Gamma_3$, that is, that $[\Gamma_4, S(H) \cap \Gamma_3] = E$.

Consider next Γ_3. Again it follows that $\Gamma_3 = (T(H) \cap \Gamma_3)(S(H) \cap \Gamma_3)$. Since $[T(H) \cap \Gamma_4, T(H) \cap \Gamma_3] \leqq T(H)' = E$ and $[T(H) \cap \Gamma_4, S(H) \cap \Gamma_3] = E$, we know

that $[T(H) \cap \Gamma_4, \Gamma_3] = E$. This shows that $T(H) \cap Z_1(\Gamma_3) \geqq T(H) \cap \Gamma_4 \neq E$. Choose now any s from $S(H) \cap H'$, and apply (5.3) with $N = \Gamma_3$ to obtain that $Z_1(\{\Gamma_3, s, s\alpha\}) \neq E$. Again we see that $\{\Gamma_3, s, s\alpha\}$ is normal in H and so on, finally (5.5.1) can be applied to prove that $\Gamma_4 \leqq Z_1(\{\Gamma_3, s, s\alpha\})$. This means that $[\Gamma_4, s] = E$ for any s in $S(H) \cap H'$, that is, $[\Gamma_4, S(H) \cap H'] = E$. But we have seen that $H' = (T(H) \cap H')(S(H) \cap H')$, so the relations $[T(H) \cap \Gamma_4, T(H) \cap H'] \leqq T(H)' = E$ and $[T(H) \cap \Gamma_4, S(H) \cap H'] = E$ show that $[T(H) \cap \Gamma_4, H'] = E$, $Z_1(H') \geqq T(H) \cap \Gamma_4 \neq E$. Thus we have arrived at a statement contradicting the initial hypothesis, and therefore (5.5.2) must hold.

(5.5.3) *If* $Z_1(H') \neq E$, *then* $\Gamma_4(H) = E$.

Proof. As $Z_1(H')$ is characteristic in H, (5.5.1) implies at once that $\Gamma_4 \leqq Z_1(H')$. Noting that α can be restricted to $Z_1(H')$, etc., one obtains that $Z_1(H') = (T(H) \cap Z_1(H'))(S(H) \cap Z_1(H'))$. If $T(H) \cap Z_1(H') = E$, then $Z_1(H') = S(H) \cap Z_1(H') \subseteqq S(H)$; since therefore also $Z_1(H')^{S(H)} = Z_1(H') \subseteqq S(H)$, (5.4) proves that $[Z_1(H'), S(H)] = E$. As $S(H)$ generates the whole of H, this means that $[Z_1(H'), H] = E$, whence $\Gamma_4 \leqq Z_1(H') \leqq Z_1(H)$, $\Gamma_5 = E$. — If on the other hand $T(H) \cap Z_1(H') \neq E$, then we can apply (5.3) to $N = H'$ and to any s in $S(H)$; thus $Z_1(\{H', s, s\alpha\}) \neq E$. Now $\{H', s, s\alpha\}$ is obviously normal in H, so that also $Z_1(\{H', s, s\alpha\})$ is normal in H; further, it is easy to see that $Z_1(\{H', s, s\alpha\})$ is α-invariant, and then (5.5.1) implies that Γ_4 is contained in $Z_1(\{H', s, s\alpha\})$. Of this we only need that $[\Gamma_4, s] = E$ for every s in $S(H)$; since $S(H)$ generates the whole of H, this means that Γ_4 is in the centre of H, whence $\Gamma_5 = E$. — In either case we found that $\Gamma_5 = E$, but then (4.0) implies that also $\Gamma_4 = E$.

So (5.5.2) and (5.5.3) prove that $\Gamma_4 = E$, H is nilpotent of class at most three. This completes the proof of the theorem.

Now let us turn to the infinite case.

(5.6) Theorem. *Let H be a periodic locally soluble group, let $(H, \alpha) \in [\mathsf{A}_4]$, and assume that the elements that are inverted by α^2 generate H. Then H is nilpotent of class at most three.*

Using that a periodic locally soluble group is always locally finite, this theorem can be derived from (5.5) by a routine argument (similar to the one used in the reduction part of the proof of (4.0)).

6. The general case

Let us finally utilize the rather special results of the preceding two sections. Their usefulness is apparent in the light of the fact that if $(G, \alpha) \in [\mathsf{A}_4 \Sigma]$ then G' is contained in the subgroup $H(G)$ generated by $S(G)$ and so G' inherits the nilpotency property of $H(G)$. This immediate consequence on G' is further improved in the following theorem.

(6.1) Theorem. *If $(G, \alpha) \in [\mathsf{A}_4 \Sigma]$, and if the subgroup $H(G)$ generated by the elements of G which are inverted by α^2 is nilpotent, then the second derived group of G is contained in the centre of G.*

The proof takes several steps. Instead of $H(G)$ we write briefly H.

(6.1.1) $\Delta_4(H)=E$. This is true because α can be restricted to H, so that (4.0) proves that $\Gamma_4(H)=E$; square roots being unique in H, it follows that also $\Delta_4(H)=E$.

Let Δ be the closure of $\Gamma_3(H)$ in G. As a closed subgroup of the Σ-group G, Δ is itself a Σ-group. H is clearly α-invariant and, according to (3.5), H is normal in G; as $\Gamma_3(H)$ is characteristic in H, it is also normal and α-invariant in G, and then so is Δ.

(6.1.2) $[\Delta, H]=E$.

Proof. Take Δ as the union of the ascending chain of subgroups

$$\Delta^{(1)}\leqq\cdots\leqq\Delta^{(i)}\leqq\cdots$$

defined inductively as follows. Let $\Delta^{(1)}$ be $\Gamma_3(H)$; if $\Delta^{(i)}$ is already defined, then let $\Delta^{(i+1)}$ be the subgroup generated by all x in G such that $x^2\in\Delta^{(i)}$. Now (6.1.1) implies that $[\Delta^{(1)}, H]=E$, and if $[\Delta^{(i)}, H]=E$, then (B.5.6) implies that also $[\Delta^{(i+1)}, H]=E$. So we have that $[\Delta^{(i)}, H]=E$ for all i, and so in fact $[\Delta, H]=E$.

For the rest of the proof of the theorem, x, y, u, v denote arbitrary elements of $S(G)$, while t and t' are arbitrary elements of $T(G)$.

(6.1.3) $[x, y]\alpha\equiv[x, y]^{-1} \bmod \Delta$.

Proof. Let us work out $[x, y]\alpha^2$:

$$\begin{aligned}[x, y]\alpha^2 &= [x^{-1}, y^{-1}]=[x, y]^{y^{-1}x^{-1}}=[x, y][x, y, y^{-1}x^{-1}]\\ &=[x, y][x, y, x^{-1}][x, y, y^{-1}]^{x^{-1}}\\ &=[x, y]\left([x, y, x]^{-1}\right)^{x^{-1}}\left([x, y, y]^{-1}\right)^{y^{-1}x^{-1}}\\ &=[x, y][x, y, x]^{-1}[x, y, y]^{-1} \qquad \text{by (6.1.2)},\\ &=[x, y]\left([x, y, y][x, y, x]\right)^{-1};\end{aligned}$$

with $d=\left([x, y, y][x, y, x]\right)\sigma\in\Delta$ this can be written as $[x, y]\alpha^2=[x, y]d^{\,2}$. From the proof of (3.4) we know that $(g^{-1}g\alpha^2)\sigma\in S(G)$, $g(g^{-1}g\alpha^2)\sigma\in T(G)$ for every g in G. Applying this to $g=[x, y]$, we get that $d^{-1}\in S(G)$ and that $[x, y]d^{-1}\in T(G)$. Now (3.3) yields that $\left([x, y]d^{-1}\right)\alpha=\left([x, y]d^{-1}\right)^{-1}$, whence $[x, y]\alpha=d[x, y]^{-1}d\alpha$; using (6.1.2) we see that this is the same as $[x, y]\alpha=[x, y]^{-1}dd\alpha$, so that $[x, y]\alpha\equiv[x, y]^{-1} \bmod \Delta$.

(6.1.4) $[x, y\alpha]\equiv[x\alpha, y] \bmod \Delta$.

Proof. Since $y\alpha\in S(G)$, we may substitute $y\alpha$ for y in (6.1.3):

$$[x, y\alpha]\alpha\equiv[x, y\alpha]^{-1} \bmod \Delta.$$

But also

$$[x, y\alpha]\alpha=[x\alpha, y^{-1}]=\left([x\alpha, y]^{-1}\right)^{y^{-1}}=[x\alpha, y]^{-1}[[x\alpha, y]^{-1}, y^{-1}]\equiv[x\alpha, y]^{-1} \bmod \Delta,$$

so we have that $[x, y\alpha]^{-1} \equiv [x\alpha, y]^{-1} \bmod \Delta$. Taking inverses on both sides, we get the required congruence.

(6.1.5) $[a, b, c] = [a, c, b][c, b, a]$ *for every* $a, b, c \in H$.

This follows from (2.1) and (6.1.1).

Now one can work out relations similar to (4.3.2)—(4.3.7) using an exact analogue of the corresponding part of Section 4, consistently referring

to Δ	instead of Δ_3,
E	Δ_4,
equality	congruence mod Δ_4,
(6.1.2)	$[\Delta_3, K] \leqq \Delta_4$,
(6.1.3)	(4.2.1),
(6.1.4)	(4.2.2),
(6.1.5)	(4.3.1):

The relations so obtained, in particular those corresponding to (4.3.6) and (4.3.7), with further application of (6.1.2) and (6.1.4), yield the following line of equalities:

(6.1.6) $[x, y, u]\alpha = [x\alpha, y, u]^{-1} = [x, y\alpha, u]^{-1} = [x, y, u\alpha]^{-1}$.

(6.1.7) $[x^t, y^t, u] = [x, y, u]$.

Proof. While proving (6.1.3) we saw that, for a certain d in Δ, $[x, y]d^{-1} \in T(G)$. By (3.3), $T(G)$ is abelian and so $([x, y]d^{-1})^t = [x, y]d^{-1}$; whence $[x, y]^t = [x, y]d^{-1}d^t$. Put $d^{-1}d^t = d'$; as Δ is normal in G, this element belongs to Δ. Then, by (6.1.2),

$$[x^t, y^t, u] = [[x, y]^t, u] = [[x, y]d', u] = [x, y, u]^{d'}[d', u] = [x, y, u].$$

(6.1.8) $[x, y, u^t] = [x^t, y, u][x^{t^{-1}}, u, y][u, x, y^{t^{-1}}]$.

Proof. As $S(G)^{T(G)} = S(G)$ (see the proof of (3.5)), x^t, $x^{t^{-1}}$, $y^{t^{-1}}$, u^t all belong to $S(G)$. This makes it possible to apply the relations proved above:

$$\begin{aligned}
[x, y, u^t] &= [x, u^t, y][u^t, y, x], && \text{by (6.1.5)},\\
&= [x^{t^{-1}}, u, y][u, y^{t^{-1}}, x], && \text{by (6.1.7)},\\
&= [x^{t^{-1}}, u, y][u, x, y^{t^{-1}}][x, y^{t^{-1}}, u], && \text{by (6.1.5)},\\
&= [x^{t^{-1}}, u, y][u, x, y^{t^{-1}}][x^t, y, u], && \text{by (6.1.7)},\\
&= [x^t, y, u][x^{t^{-1}}, u, y][u, x, y^{t^{-1}}], && \text{by (6.1.1)}.
\end{aligned}$$

(6.1.9) $[x, y, u, t] = 1$.

Proof. Let us write down (6.1.8) with $u\alpha^{-1}$ $(\in S(G))$ in the place of u; mapping both sides by α, changing to inverses and using (6.1.6) we obtain that $[x, y, u^{t^{-1}}] = [x^t, y, u][x^{t^{-1}}, u, y][u, x, y^{t^{-1}}]$. Comparing this with the original form of (6.1.8) we get $[x, y, u^{t^{-1}}] = [x, y, u^t]$. Transform this equality by t and use (6.1.7) to obtain $[x, y, u] = [x, y, u^{t^2}]$. From (6.1.7) it follows

also that $[x, y, u]^{t^2}=[x, y, u^{t^2}]$. Comparison of the last two equalities shows that $[x, y, u]^{t^2}=[x, y, u]$, so that $[x, y, u, t^2]=1$. Then (B.5.6) proves that also $[x, y, u, t]=1$.

(6.1.10) $[\Delta, G]=E$.

Proof. As H is generated by $S(G)$, $\Gamma_3(H)$ is the normal closure (in H) of the subgroup generated by all commutators of the form $[x, y, u]$. From (6.1.2) and (6.1.9) we know that $[x, y, u]$ commutes with both of $S(G)$ and $T(G)$; as these two sets generate G, $[x, y, u]$ commutes with the whole of G. Consequently, the subgroup generated by the $[x, y, u]$ is in the centre of G, thus it is its own normal closure $\Gamma_3(H)$, whence $[\Gamma_3(H), G]=E$. Again (B.5.6) is needed to conclude that $[\Delta, G]=E$.

(6.1.11) $[x, y, t]=1$.

Proof. In the course of proving (6.1.3) we saw that, with a suitable d $(\in\Delta)$, $[x, y]d^{-1}\in T(G)$. Since by (3.3) $T(G)$ is abelian, $[[x, y]d^{-1}, t]=1$. But $[[x, y]d^{-1}, t]=[x, y, t]^{d^{-1}}[d^{-1}, t]=[x, y, t]$ by (6.1.10), and so $[x, y, t]=1$.

(6.1.12) $[x, y, [u, t]]=1$.

Proof. $[x, y, [u, t]]=[x, y, u^{-1}t^{-1}ut]$

$$\begin{aligned}&\doteq [x, y, t][x, y, u]^t[x, y, t^{-1}]^{ut}[x, y, u^{-1}]^{t^{-1}ut}\\ &=[x, y, u]^t[x, y, u^{-1}]^{ut}, \qquad \text{by (6.1.9) and (6.1.11)},\\ &=[x, y, u]^t\left([x, y, u]^{-1}\right)^{u^{-1}ut}=1.\end{aligned}$$

(6.1.13) $[u, t, [v, t'], T(G)]=E$.

Proof. Using (6.1.1), $[u, t, [v, t']]=[u^{-1}u^t, v^{-1}v^{t'}]$

$$\begin{aligned}&\doteq [u^{-1}, v^{t'}]^{u^t}[u^t, v^{t'}][u^{-1}, v^{-1}]^{u^t v^{t'}}[u^t, v^{-1}]^{v^{t'}}\\ &=[u^{-1}, v^{t'}][u^{-1}, v^{t'}, u^t][u^t, v^{t'}][u^{-1}, v^{-1}][u^{-1}, v^{-1}, v^{t'}][u^{-1}, v^{-1}, u^t]\times\\ &\quad\times[u^t, v^{-1}][u^t, v^{-1}, v^{t'}],\end{aligned}$$

as $u^t, v^{t'}\in S(G)$, this shows that $[u, t, [v, t']]$ is a product of commutators of the form $[x, y]$ and $[x, y, u]$. Since by (6.1.11) and (6.1.9) these all commute with $T(G)$, so does $[u, t, [v, t']]$.

(6.1.14) $[u, t, [v, t'], S(G)]=E$.

Proof. Take any x from $S(G)$ and expound $[u, t, [v, t'], x]=[u^{-1}u^t, v^{-1}v^{t'}, x]$ using (6.1.1); it becomes

$$[u, v, x][u, v^{t'}, x]^{-1}[u^t, v, x]^{-1}[u^t, v^{t'}, x].$$

Now if one splits up each term according to (6.1.5), uses (6.1.7) and (6.1.9), then after cancellations one is left with 1.

(6.1.15) *G' is generated by the commutators of the form $[u, v]$ together with those of the form $[u, t]$.*

Proof. (3.4) can be modified to show that any two elements g_1, g_2 of G can be written as $g_1 = ut$, $g_2 = vt'$ with some $u, v \in S(G)$, $t, t' \in T(G)$. Then

$$\begin{aligned}[g_1, g_2] &= [u\,t, v\,t'] = [u, t']^t [t, t'] [u, v]^{tt'} [t, v]^{t'} \\ &= [u^t, t'] [u, v]^{tt'} [t, v^{t'}], \qquad \text{(since ,by (3.3), } T(G)' = E\text{)}, \\ &= [u^t, t'] [u, v] [v^{t'}, t]^{-1}, \qquad \text{by (6.1.11)};\end{aligned}$$

as u^t and $v^{t'}$ belong to $S(G)$, one can see that $[g_1, g_2]$ is a product of commutators of the required form. Since every element of G' is a product of commutators like $[g_1, g_2]$, this proves the statement.

(6.1.16) $G'' \leqq Z_1(G)$.

Proof. Let us consider the subgroup K generated by the (ordinary second) commutators formed from the generators of G' which are given in (6.1.15). Every $[x, y, [u, v]]$ is trivial by (6.1.1), and every $[x, y, [u, t]]$ is trivial by (6.1.12), so that in fact K is generated by the $[u, t, [v, t']]$. On the other hand, according to (6.1.13) and (6.1.14), all the $[u, t, [v, t']]$ commute with $T(G)$ and $S(G)$. Since these two sets generate G (see (3.4)), this means that K is central in G and a fortiori normal in G'. But the normal closure of K in G' is just G'', and so we have that $G'' = K \leqq Z_1(G)$.

Thus we arrive at the main results of the paper.

(6.2) Theorem. *If G is a locally nilpotent group, and if $(G, \alpha) \in [\mathsf{A}_4]$, then the second derived group of G is contained in the centre of G.*

Proof. According to (3.7), we can apply (2.3). This says that there exists a locally nilpotent Σ-group G^* such that (i) G^* contains G, (ii) to every element g^* of G^* there is a number $n = n(g^*)$ for which $(g^*)^{2^n} \in G$, and (iii) α can be extended in a unique way to an automorphism β of G^*. It is easy to see that β is regular: if $g^*\beta = g^*$, then the equalities $(g^*\beta)^{2^n} = \big((g^*)^{2^n}\big)\beta = \big((g^*)^{2^n}\big)\alpha = (g^*)^{2^n}$ show that, since α itself is regular, $(g^*)^{2^n} = 1$; as G^* is a Σ-group, this means that $g^* = 1$. Further, we verify that the order of β divides 4: the relations $(g^*\beta^4)^{2^n} = \big((g^*)^{2^n}\big)\beta^4 = \big((g^*)^{2^n}\big)\alpha^4 = (g^*)^{2^n}$ give that $g^*\beta^4 = g^*$, for every $g^* \in G^*$. Finally, β can be restricted to $H(G^*)$ and then (4.0) proves that $H(G^*)$ is nilpotent. So we can apply (6.1) to G^*, and the conclusion is obviously inherited by G.

(6.3) Theorem. *If $(G, \alpha) \in [\mathsf{A}_4]$, and if G is periodic and locally soluble, then the second derived group of G is contained in the centre of G.*

Proof. As a periodic locally soluble group, G is locally finite, so that (3.2) shows that G is a Σ-group. Clearly $H(G)$ is also periodic and locally soluble, and α can be restricted to $H(G)$, so that (5.6) implies that $H(G)$ is nilpotent. Thus we can refer to (6.1), and the proof is complete.

7. Examples

In this last section we give some examples to show that the results presented in this paper are in a sense the best possible. We confine ourselves to the description of the examples, omitting the straightforward but sometimes tedious verifications.

(7.1) First we give the smallest non-metabelian A_4-group. Let us start with a free nilpotent-of-class-two group K_0 on the five generators $s_0, s_1, \ldots, s_4$. Imposing exponent three and the relations $[s_i, s_j][s_{j+k}, s_{i+k}]=1$ $(i, j, k=0, \ldots, 4;\ j+k$ and $i+k$ to be taken mod 5), we get a factor group K_1 of K_0. It can be seen that K_1' is generated by $[s_0, s_1]$ and $[s_0, s_2]$, and that these commutators are independent, so that the order of K_1 is 3^7. Now take an isomorphic copy K_2 of K_1 generated in a corresponding way by $r_0, r_1, \ldots, r_4$. Let H be the generalized direct product of K_1 and K_2 amalgamating $[s_0, s_1]$ with $[r_0, r_1]^{-1}$ and $[s_0, s_2]$ with $[r_0, r_2]^{-1}$. Then H is of order 3^{12}, and it can be seen that the mapping τ defined by $s_i\tau=s_{i+1}$, $r_i\tau=r_{i-1}$ is an automorphism of H of order 5. Finally, let us take the splitting extension G of H by an element t corresponding to τ, such that $t^5=1$, $s_i^t=s_{i+1}$ and $r_i^t=r_{i-1}$. Clearly G is of order $5\cdot 3^{12}$; further, it is soluble and has derived length three, but it is not nilpotent. A regular automorphism α of order four can be defined on G by $s_i\alpha=r_i$, $r_i\alpha=s_i^{-1}$, and $t\alpha=t^{-1}$.

(7.2) Repeating exactly the above construction but replacing 5 by some power of 3, one can get non-metabelian A_4-groups which are nilpotent of arbitrarily high class. The direct product of an infinity of such groups of increasing class provides a locally nilpotent but not nilpotent A_4-group.

References

[1] Baumslag, G.: Some aspects of groups with unique roots. Ph. D. thesis, University of Manchester, 1958. (To appear in Acta Math.)

[2] Burnside, W.: Theory of groups of finite order, 1st ed. Cambridge 1897.

[3] Burnside, W.: Theory of groups of finite order, 2nd ed. Cambridge 1911.

[4] Higman, G.: Groups and rings having automorphisms without non-trivial fixed elements. J. London Math. Soc. **32**, 321—334 (1957).

[5] Kurosh, A. G.: Theory of groups. New York 1955—1956.

[6] Lazard, M.: Sur les groupes nilpotents et les anneaux de Lie. Ann. Sci. Ecole Norm. Sup. (3) **71**, 101—190 (1954).

[7] Nagata, M.: Note on groups with involutions. Proc. Japan Acad. **28**, 564—566 (1952).

[8] Neumann, B. H.: On the commutativity of addition. J. London Math. Soc. **15**, 203—208 (1940).

[9] Neumann, B. H.: Groups with automorphisms that leave only the unit element fixed. Arch. Math. **7**, 1—5 (1956).

[10] Thompson, J.: Finite groups with fixed-point-free automorphisms of prime order. Proc. Nat. Acad. Sci. U.S.A. **45**, 578—581 (1959).

[11] Zappa, G.: Sugli automorfismi uniformi nei gruppi di Hirsch. Ricerche di Mat. **7**, 3—13 (1958).

University College of North Staffordshire, Keele, Staffs. (England)

(Eingegangen am 1. Juli 1960)