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A note on regular rings.

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A note on regular rings.

To the memory of my beloved teacher Professor Tibor Szele.

By LÁSZLÓ KOVÁCS in Debrecen.

It is well known, that in a regular[1]) ring R every left (right) ideal I is idempotent (i. e. $I^2 = I$). The question, whether or not this is a characteristic property of the regular rings, seems to be of some interest. In this little note we give three theorems related to this problem. In Theorem 1 we characterize the regular rings by a condition for one-sided ideals, which in the commutative case is equivalent to the idempotency of the ideals in R. In Theorem 2 we solve an analogous problem, the result shows that the regular rings without nonzero nilpotent elements are characterized by the idempotency of their quasi-ideals. Applying this result and a theorem of A. KERTÉSZ we get Theorem 3, which is a criterion for decomposibility of rings into a direct sum of division rings. — I am indebted to A. KERTÉSZ for his valuable help.

The concept of the quasi-ideal was introduced by O. STEINFELD in [5]. A submodule M of the ring R is said to be a quasi-ideal if $RM \cap MR \subseteq M$. Elementary facts connected with this concept: a quasi-ideal is a subring, but not every subring is a quasi-ideal; the intersection of one-sided ideals is always a quasi-ideal; in the presence of a one-sided unity every quasi-ideal is intersection of one-sided ideals, etc. These and further results on quasi-ideals are to be found in [6] and [7].

In an arbitrary ring R

(1) $$JL \subseteq J \cap L$$

holds for any right-ideal J and any left-ideal L of R. Our first theorem characterizes the rings in which equality holds in (1) in every case. Namely, we prove the following

Theorem 1. *An arbitrary ring R is regular if and only if*

(2) $$JL = J \cap L$$

holds for every right-ideal J and left-ideal L of R.

[1]) In the sense of J. VON NEUMANN [4]. — Numbers in brackets refer to the bibliography at the end of this note.

First we assume (2) and show that R is regular. Let a be an arbitrary element of R. The right-ideal generated by a is the set $[an+ar]$ of all elements of the form $an+ar$ (n integer, $r\in R$). By (2)

$$[an+ar]=[an+ar]\cap R=[an+ar]R=aR$$

and so $a\in aR$. Analogously $a\in Ra$ and hence

$$a\in aR\cap Ra=aR^2a,$$

i. e. $a=axa$, R is regular.

The converse statement is almost trivial. Let R be a regular ring, by (1) we have only to show that any element a of $J\cap L$ is in JL. From $a=axa$, $a\in J$, $xa\in L$ we conclude $a\in JL$.

In what follows we shall give a characterization of regular rings without nonzero nilpotent elements, in terms of quasi-ideals.

Theorem 2. *For an arbitrary ring R the following conditions are equivalent*:

α) *R is a regular ring without nonzero nilpotent elements*;

β) *every quasi-ideal of R is idempotent;*

γ) *for every right-ideal J and every left-ideal L of R*

$$JL=J\cap L\subseteqq LJ$$

holds;

δ) *R is regular and isomorphic to a subdirect sum of division rings.*[2])

α) *implies* β). Let M be a quasi-ideal of R and a an arbitrary element of M. Since M is a subring, $M^2\subseteqq M$ and so we have only to prove $M\subseteqq M^2$, i. e. $a\in M^2$. By the regularity of R we have $a=axa$. Here xa is an idempotent and so, since R has no nonzero nilpotents, xa is in the center of R. Using also $MR^2M\subseteqq MR\cap RM\subseteqq M$ we get

$$a=(ax)a(xa)=(ax)(xa)a=(ax^2a)a\in MR^2M\cdot M\subseteqq M^2,$$

qu. e. d.

β) *implies* γ). Let M and N denote quasi-ideals in R, then $M\cap N$ is also a quasi-ideal. By the idempotency of $M\cap N$ we have

$$M\cap N=(M\cap N)^2\subseteqq MN\cap NM.$$

On the other hand

$$MN\cap NM\subseteqq MR\cap RM\subseteqq M$$

analogously $MN\cap NM\subseteqq N$, and so $M\cap N=MN\cap NM$.

Now let J be a right-ideal, L a left-ideal in R. Since a one-sided ideal is always a quasi-ideal, we have $J\cap L=JL\cap LJ$, but $JL\subseteqq J\cap L$ and so $JL=J\cap L\subseteqq LJ$.

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2) The equivalence of α) and δ) has been proved by A. Forsythe and N. H. McCoy in [2]; we prove it here only for completness' sake.

γ) implies δ). Let R be a subdirect sum of the subdirectly irreducible rings $R_1, \ldots, R_\nu, \ldots$.[3]) Since every R_ν is a homomorphic image of R, all the R_ν's have property γ). Thus, by Theorem 1, R and all the R_ν's are regular. Suppose, that one of the R_ν's has divisors of zero, i. e. for some nonzero elements a and b of R_ν we have $ab=0$. Then by γ)

$$bR_\nu \cdot R_\nu a = bR_\nu \cap R_\nu a \subseteq R_\nu a \cdot bR_\nu = 0$$

and so

$$R_\nu bR_\nu \cap R_\nu aR_\nu = R_\nu bR_\nu \cdot R_\nu aR_\nu = 0$$

where, by $a=(ax)a(xa)\in R_\nu aR_\nu$ and $b=(by)b(yb)\in R_\nu bR_\nu$, none of these ideals is 0. This contradicts to the supposition that R_ν is subdirectly irreducible, and so we have that all the R_ν's are regular rings without divisors of zero, i. e. all the R_ν's are division rings.

That *δ) implies α)* does not need a proof.

Corollary. *A commutative ring R is regular if and only if every ideal is idempotent in R.*

This follows immediately from Theorem 2, since in a commutative ring every quasi-ideal is an ideal, and a commutative regular ring can have no nonzero nilpotents. Our statement is also a simple corollary of Theorem 1, since (2) in the case $J=L$ implies the idempotency of the ideals, and the proof of the converse statement is analogous to that of β) implies γ) in Theorem 2.

As an application of Theorem 2 we have

Theorem 3. *A ring R is a direct sum[4]) of division rings if and only if R satisfies the descending chain condition for principal (two-sided) ideals and every quasi-ideal of R is idempotent.*[5])

The proof is based on the following (not yet published) theorem of A. Kertész:

Let φ be a property defined for simple rings. A ring R is a direct sum of simple rings with property φ if and only if R satisfies the descending chain condition for principal ideals and contains a system of its maximal ideals M_ν, the intersection of which is 0 and for which all the factor rings R/M_ν have property φ.

Since a direct sum R of division rings is a regular ring without nonzero nilpotent elements, by Theorem 2 every quasi-ideal of R is idempotent;

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[3]) According to a theorem of G. Birkhoff [1] such a representation always exists.

[4]) By a direct sum we always mean a two-sided (ring-theoretical) discrete direct sum, but we use the term "subdirect sum" in the usual sense, i. e. for certain subrings of the complete direct sum.

[5]) Another criterion is given by A. Gertschikoff [3].

moreover it is evident, that R satisfies the descending chain condition for principal ideals. Conversely, if R satisfies the conditions of Theorem 3, then by Theorem 2 it is a subdirect sum of division rings, i. e. there exists a system of maximal ideals M_ν in R, the intersection of which is 0 and for which every factor ring R/M_ν is a division ring. So we have by the above theorem of A. KERTÉSZ (in our case property φ characterizes the division-rings) that R is a direct sum of division rings.

REMARK. I am indebted to Professor L. FUCHS who has kindly directed my attention to the fact that

A ring R is a finite direct sum of division rings if and only if it has no nilpotent quasiideals and satisfies the descending chain condition for quasi-ideals.

This is an immediate consequence of the WEDDERBURN—ARTIN structure theorem, and also of GERTSCHIKOFF's theorem [3].

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