On central Frattini extensions of finite groups

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To Professor A. Bovdi for his seventieth birthday

Abstract. An extension of a group $A$ by a group $G$ is thought of here simply as a group $H$ containing $A$ as a normal subgroup with quotient $H/A$ isomorphic to $G$. It is called a central Frattini extension if $A$ is contained in the intersection of the centre and the Frattini subgroup of $H$. The result of the paper is that, given a finite abelian $A$ and finite $G$, there exists a central Frattini extension of $A$ by $G$ if and only if $A$ can be written as a direct product $A = U \times V$ such that $U$ is a homomorphic image of the Schur multiplicator of $G$ and the Frattini quotient of $V$ is a homomorphic image of $G$.

1. Discussion

Given a finite abelian group $A$ and an arbitrary finite group $G$, consider all extensions

$$1 \rightarrow A \rightarrow H \rightarrow G \rightarrow 1$$

such that (the embedded copy of) $A$ lies in the intersection of the centre and the commutator subgroup:

$$A \leq Z(H) \cap H'.$$  \hspace{1cm} (2)

It is well known (for an elementary exposition and a bibliography, see WIEGOLD [5]) that such extensions exist if and only if $A$ is a homomorphic

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image of the (Schur) multiplicator of $G$. The aim of this note is to put on record a similar result, with a similarly elementary proof, concerning central Frattini extensions. These are extensions (1) satisfying

$$A \leq Z(H) \cap \Phi(H)$$

instead of (2), with $\Phi(G)$ denoting the Frattini subgroup of $G$. Suzuki’s book [4] discussed them at length under the name of irreducible central extensions, but stopped short of the following.

**Theorem.** Given a finite group $G$ and a finite abelian group $A$, there exists a central Frattini extension of $A$ by $G$ if and only if $A$ can be written as a direct product $A = U \times V$ such that $U$ is a homomorphic image of the multiplicator of $G$ and the Frattini quotient of $V$ is a homomorphic image of $G$.

Since the proof requires no new ideas, it is somewhat surprising that the result still does not seem to be in the literature.

Let $M$ denote the multiplicator of $G$; for each prime $p$, let $p^{f(p)}$ denote the order of the largest elementary abelian $p$-quotient of $G$; and write the exponent of $A$ as $\prod p^{f(p)}$. Let $B$ be the direct product of $M$ and $\sum p^{f(p)}$ cyclic groups, $f(p)$ of which have order $p^{e(p)}$. An alternative form of the condition in the theorem is that $A$ should be a homomorphic image of $B$.

A related question in the literature concerns Frattini extensions which are not necessarily central. Given $G$ and a prime $p$, consider extensions (1) with elementary abelian $p$-groups $A$ such that $A \leq \Phi(H)$, and view these $A$ as $(\mathbb{Z}/p\mathbb{Z})G$-modules, the action of $G$ coming from conjugation in $H$. Gaschütz [2] had shown that the $A$ of maximal order are all isomorphic to the second Heller translate of the 1-dimensional trivial $(\mathbb{Z}/p\mathbb{Z})G$-module, which Griess and Schmid [3] then called the Frattini module of $G$ with respect to $\mathbb{Z}/p\mathbb{Z}$. Gaschütz [2] also showed that the other $A$ which occur in such extensions are precisely the homomorphic images of this Frattini module. It follows from our theorem that the largest $G$-trivial quotient of the Frattini module of $G$ with respect to $\mathbb{Z}/p\mathbb{Z}$ is the largest exponent-$p$ quotient of $M \times (G/G')$. (This was implicit in an aside in [3], immediately before Theorem 2, which invoked the Universal Coefficient Theorem.)

The results quoted here from [2] were extended in [1] to Frattini extensions with kernels that are not elementary abelian (and even to profinite
Frattini extensions). A special case of that work shows that if we consider Frattini extensions by $G$ with abelian kernels $A$ of exponent dividing some given positive integer $e = \prod p^{e(p)}$, the kernels of maximal order are all isomorphic as $G$-modules, and as groups they are just direct products of cyclic groups of order $e$. Thus one might speak of the Frattini module of $G$ with respect to $\mathbb{Z}/e\mathbb{Z}$, and note that the theorem also determines the largest $G$-trivial quotient of this module.

**Corollary.** The largest $G$-trivial quotient of the Frattini module of $G$ with respect to $\mathbb{Z}/e\mathbb{Z}$ is the direct product of $M/M^e$ and $\sum_p f(p)$ cyclic groups, $f(p)$ of which have order $p^{e(p)}$. □

(Here $M/M^e$ denotes the largest exponent $e$ quotient of $M$.)

2. Proofs

We shall need some preparatory results about finite abelian groups.

**Lemma 1.** If $A$ is a finite abelian group and $A/E$ an elementary abelian quotient, then $A$ has a direct decomposition $A = \prod C_i$ with cyclic factors $C_i$ such that $E = \prod (C_i \cap E)$.

**Proof.** Induction on the order of $A$, exploiting that in a finite abelian group the maximum of the orders of cyclic subgroups equals the maximum of the orders of cyclic quotients. Let $A/D$ be a cyclic quotient of maximal order. As $|a| \leq |A/D|$ for all $a \in A$, if a coset $aD$ generates $A/D$ then $A = C \times D$ with $C = \langle a \rangle$. If $DE = A$, such an $a$ can be chosen from $E$, and then $E = C \times (D \cap E)$. Otherwise choose a direct complement $F/E$ for $DE/E$ in $A/E$ (this is where we use that $A/E$ is elementary): then $DF = A$ and so $C$ can be chosen within $F$. In this case, $C \cap DE \leq F \cap DE = E$ whence $C \cap DE = C \cap E$ and $DE = (C \cap E) \times D$, $E = (C \cap E) \times (D \cap E)$ follow. The last conclusion being available in either case, an application of the inductive hypothesis (with $D$ and $D \cap E$ in place of $A$ and $E$) completes the proof. □

**Corollary 1.** If $A$ is a finite abelian group and $A/E$ is an elementary abelian quotient, then $A$ has a direct decomposition $A = U \times V$ such that $E = U \times \Phi(V)$. 

Proof. Denote by $U$ the product of the $C_i$ that lie in $E$, and by $V$ the product of the other $C_i$. □

**Corollary 2.** If $A$ is a finite abelian group and $B$ is any subgroup of $A$, then $A$ has a direct factor $V$ such that $BV = A$ and $B \cap V \leq \Phi(V)$.

Proof. Apply Corollary 1 with $E$ defined by $E/B = \Phi(A/B)$. Then $A = UV = EV$ so $A/B = (E/B)(BV/B) = \Phi(A/B)(BV/B) = BV/B$ (because the Frattini subgroup is omissible), while $B \cap V \leq E \cap V = \Phi(V)$. □

**Lemma 2.** Given two finite abelian groups $V$, $W$, there exists an abelian Frattini extension of $V$ by $W$ if and only if $V/\Phi(V)$ is a homomorphic image of $W$.

Proof. As usual, if $X$ is any group, we write $\exp X$ for the exponent of $X$ and $d(X)$ for the minimum of the cardinalities of the generating sets of $X$. We can assume that we are dealing with abelian $p$-groups. If there is an extension $X$ of the kind envisaged, then $V \leq \Phi(X)$ and $X/V \cong W$ imply that $d(V) \leq d(X) = d(W)$. Conversely, if $d(V) \leq d(W)$ and $W$ is written as $P/Q$ with $P$ a direct product of $d(W)$ cyclic groups of orders $(\exp V)(\exp W)$, then $Q$ has a subgroup $R$ with $Q/R \cong V$, and $P/R$ can serve as $X$. □

Proof of the Theorem. Suppose first that there is a group $H$ such that $A \leq Z(H) \cap \Phi(H)$ and $H/A \cong G$. Set $B = A \cap H'$ and use Corollary 2 to obtain $A = U \times V = BV$ and $B \cap V \leq \Phi(V)$. Then $U \cong A/V \leq Z(H/V) \cap (H/V)'$ and $(H/V)/(A/V) \cong G$, so (by the property of the multiplicator mentioned at the beginning of this paper) $U$ is a homomorphic image of the multiplicator of $G$. Further, $H/H'$ is an abelian Frattini extension of $H'/H'$ by $G/G'$, $H'/H' \cong V/(B \cap V)$, and $V$ has the same Frattini quotient as $V/(B \cap V)$, so it follows from Lemma 2 that $V/\Phi(V)$ is a homomorphic image of $G$.

Conversely, suppose that $A = U \times V$ where $U$ is a homomorphic image of the multiplicator of $G$ and $V/\Phi(V)$ is a homomorphic image of $G$. By that property of the multiplicator, there is a group $T$ with a normal subgroup $U$ such that $U \leq Z(T) \cap T'$ and $T/U \cong G$, and then $T/T' \cong G/G'$. By Lemma 2, there is an abelian group $X$ with a subgroup $V$ such that $V \leq \Phi(X)$ and $X/V \cong G/G'$. The quotient of $T \times X$ over $T'V$ has a direct decomposition $TV/T'V \times T'X/T'V$ with both direct factors
isomorphic to $G/G'$. Let $H/T'V$ be the diagonal subgroup formed along any isomorphism between the two direct factors, so it is a common direct complement to each of those: $TH = HX = TX$ and $TV \cap H = H \cap T'X = TV \cap T'X = T'V$. In particular, it follows that

$$T \cap H = (T \cap TV) \cap H = T \cap (TV \cap H) = T \cap T'V = T'.$$

Thus the projection of the direct product $T \times X$ onto its second direct factor maps $H$ onto $X$ with kernel $T'$, and it maps $T'V$ onto $V$. As $V \leq \Phi(X)$, we conclude that every maximal subgroup of $H/T'$ must contain $T'V/T'$.

As $TX = HX$ and $X$ is central, we have $T' = (TX)' = (HX)' = H'$. Recall that $U \leq T'$, so $A = UV \leq T'V \leq H$. As $U$ is central in $T$ and $X$ is abelian, $A$ is obviously central in $H$. If a maximal subgroup $K$ of $H$ failed to contain $A$, we would have $H = AK$ and hence $K' = H' = T'$; but then $K/T'$ would be a maximal subgroup of $H/T'$ not containing $T'V/T'$. This proves that $A \leq Z(H) \cap \Phi(H)$. It remains to note that

$$H \cap UX = H \cap (T'X \cap UX) = (H \cap T'X) \cap UX = T'V \cap UX = UV = A,$$

so $H/A = H/(H \cap UX) \cong HUX/UX = TX/UX \cong T/U \cong G$. □

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