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On infinite rank integral representations of groups and orders of finite lattice type

By

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Abstract. Let $\Lambda = \mathbb{Z}G$ be the integer group ring of a group, G, of prime order. A main result of this note is that every Λ-module with a free underlying abelian group decomposes into a direct sum of copies of the well-known indecomposable Λ-lattices of finite rank. The first part of the proof reduces the problem to one about countably generated modules, and works in a wider context of suitably restricted modules over orders of finite lattice type of a quite general type. However, for countably generated modules, use is seemingly needed of the classical theory of Λ-lattices.

1. Introduction. A classical theorem on integral representations of finite groups states that every lattice over the integer group ring, $\mathbb{Z}C(p)$, of a group, $C(p)$, of prime order, p, is isomorphic to a direct sum of copies of the $2h + 1$ different (up to isomorphism) indecomposable lattices, the number h being the ideal class number of the field of p-th roots of unity. A main result of this note is to extend this result to the following theorem describing certain *infinitely generated* $\mathbb{Z}C(p)$-modules.

Theorem 1.1. *The $\mathbb{Z}C(p)$-modules whose underlying abelian groups are free are isomorphic to direct sums of indecomposable lattices.*

The proof uses two quite different sorts of arguments. Firstly, in Section 2, it is shown that it suffices to prove the result for *countably generated* $\mathbb{Z}C(p)$-modules. The argument establishing this reduction works in a much wider context, for suitably restricted modules over a quite general class of orders of finite lattice type. Secondly, the proof of the theorem for countably generated $\mathbb{Z}C(p)$-modules given in Section 3 depends in two different ways on the Diederichsen-Reiner structure theory for $\mathbb{Z}C(p)$-lattices summarised, for example, in Theorem 34.31 in [6].

Section 2 contains a theory of *generalised lattices* over an *R-order* of *finite lattice type*. In this paper, R denotes a Dedekind domain, an *R-order* is an associative ring, $\mathcal{O}$, which contains R as a central subring, and which is finitely generated and projective as an

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R-module, and a *generalised lattice* over such an order is defined to be a (right) $\mathcal{O}$-module which is projective as an R-module. Thus a lattice in the usual sense (for example, in [6]) is just a finitely generated generalised lattice. Finally, an R-order which has only finitely many different isomorphism classes of indecomposable lattices is said to be of *finite lattice type*. Let $\mathcal{O}$ be such an R-order. The main result in Section 2, Theorem 2.1, states that each generalised lattice is isomorphic to *a direct summand* of a direct sum of lattices, and hence, Corollary 2.5, to a direct sum of countably generated generalised lattices. However, the analogue of Theorem 1.1, that generalised lattices are isomorphic to direct sums of lattices, can only be shown under further assumptions. It is the case if R is a complete discrete valuation ring (Corollary 2.6), or if $\mathcal{O}$ is a right hereditary ring such that $K \otimes_R \mathcal{O}$ is a semisimple algebra over the field, K, of fractions of R (Corollary 2.7); in this latter case, the generalised lattices are just the projective $\mathcal{O}$-modules. Familiar examples of the last type are the rings of integers in finite algebraic extension fields of the rationals, viewed as $\mathbb{Z}$-orders, and the result for one such order is actually used in the proof of Theorem 1.1.

The integer group ring, $\mathbb{Z}G$, of *any* finite group, G, has finite lattice type if and only if all Sylow subgroups of G are cyclic and of cube free order ([6]). For these groups, the above corollaries show that the generalised lattices must be direct summands of direct sums of lattices, and hence, direct sums of countably generated generalised lattices, the fact used in the proof in Section 3 of Theorem 1.1. However, the groups of prime order are the only ones for which the generalised lattices over $\mathbb{Z}$ are currently known to be direct sums of lattices. The situation for a group ring, $\hat{\mathbb{Z}}_pG$, over the ring $\hat{\mathbb{Z}}_p$ of p-adic integers is much simpler; this order has finite lattice type if and only if the Sylow p-subgoups of G are cyclic of order p or p^2, and for these, Corollary 2.6 shows that generalised lattices are direct sums of lattices.

Chapters 3 and 4 in [6] discuss the theory of lattices over orders of the type considered in this note. The approach used in Section 2 takes ideas from the paper [2], on orders of finite lattice type, and also from the proofs in [1] and in [9] that all modules over an artin algebra of finite representation type are direct sums of finitely generated indecomposable modules.

Remarks 1.2. (1) For $p = 2$, the question behind Theorem 1.1 was first raised in the context of Lie algebras by Roger Bryant in 2000 and solved by two of us in 2001. We have been informed that the same question arose in 2002 in work on C^*-algebras by A. Kumjian and N. C. Phillips, and was settled independently of our solution by D. J. Benson (see [5]).

(2) At the ICRA meeting at the Fields Institute in Toronto, Canada, in 2002, Yuri Drozd pointed out that the problem of generalised lattices over an order can be re-formulated as a bimodule problem to which the matrix reduction procedures of the Kiev School should be applicable. This interesting suggestion has not yet been pursued further, so far as we are aware.

2. General theorems. As in the introduction, R denotes a Dedekind domain and R-orders are assumed to be finitely generated and projective as R-modules. By definition, the *Auslander lattice* of an R-order *of finite lattice type* is the direct sum of one lattice from each of the finitely many different isomorphism classes of indecomposable lattices.

The following is the main structure theorem for generalised lattices over R-orders of finite lattice type.

Theorem 2.1. *Each generalised lattice over an R-order of finite lattice type is isomorphic to a direct summand of a direct sum of copies of the Auslander lattice of the order.*

The proof requires some notation and preliminary lemmas. Let $\mathcal{O}$ be an R-order of finite lattice type, A be its Auslander lattice, and Γ be the endomorphism ring of A as a right $\mathcal{O}$-module. Then A is a left Γ-, right $\mathcal{O}$-bimodule and determines the left exact functor

$$F : \mathrm{Mod}(\mathcal{O}) \longrightarrow \mathrm{Mod}(\Gamma), \;\; M \mapsto \mathrm{Hom}_{\mathcal{O}}(A, M),$$

from right $\mathcal{O}$-modules to right Γ-modules. Since $F(A) = \Gamma_{\Gamma}$ and F commutes with arbitrary direct sums, it induces category isomorphisms

$$\mathrm{Add}(A) \xrightarrow{\sim} \mathrm{Proj}(\Gamma) \;\text{ and }\; \mathrm{add}(A) \xrightarrow{\sim} \mathrm{proj}(\Gamma),$$

where $\mathrm{Add}(A)$ ($\mathrm{add}(A)$) is the full subcategory of $\mathrm{Mod}(\mathcal{O})$ with objects the direct summands of direct sums of copies of A (the direct summands of finite direct sums of copies of A), and where $\mathrm{Proj}(\Gamma)$ ($\mathrm{proj}(\Gamma)$) is the full subcategory of $\mathrm{Mod}(\Gamma)$ with objects the projective modules (the finitely generated projective modules).

Since A is the Auslander lattice of $\mathcal{O}$, $\mathrm{add}(A)$ is just the subcategory $\mathrm{latt}(\mathcal{O})$ of all right $\mathcal{O}$-lattices. Since R is hereditary, $\mathrm{latt}(\mathcal{O})$ is closed under submodules and hence under kernels, and since F is left exact, $\mathrm{proj}(\Gamma)$ is also closed under kernels; thus, finitely presented right Γ-modules have projective dimension 2 at most. Since Γ is both left and right noetherian, an argument given, for example, in Section 4.1 of [10] then proves the following known result.

Lemma 2.2. *Γ has both right and left global dimensions* 2*, at most. In particular,* $\mathrm{Proj}(\Gamma)$ *is closed under kernels.*

Next, let $\mathrm{Latt}(\mathcal{O})$ be the full subcategory of all generalised lattices over $\mathcal{O}$. It clearly contains the subcategory $\mathrm{Add}(A)$, and the proof of the theorem amounts to showing that they are actually the same. However, two more lemmas are useful.

Lemma 2.3. *F maps* $\mathrm{Latt}(\mathcal{O})$ *into* $\mathrm{Proj}(\Gamma)$.

Proof. For any right $\mathcal{O}$-module L, the natural $\mathcal{O}$-morphism

$$i_L : L \cong \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, L) \hookrightarrow \mathrm{Hom}_R(\mathcal{O}, L)$$

is injective and splits as an R-morphism, one left inverse being the evaluation map $f \mapsto f(1)$. Also

$$\mathrm{Hom}_R(\mathcal{O}, L) \cong L \otimes_R \mathcal{O}^*,$$

where

$$\mathcal{O}^* = \mathrm{Hom}_R(\mathcal{O}, R) \in \mathrm{latt}(\mathcal{O}) = \mathrm{add}(A).$$

Now assume that L is a generalised lattice, so that L_R is projective. Then also $L \otimes_R \mathcal{O}^*$ is projective as an R-module, and since i_L splits as an R-morphism, $L' = \mathrm{Coker}(i_L)$ is also a generalised lattice. We obtain an exact sequence

$$0 \to L \to L \otimes_R \mathcal{O}^* \to L' \otimes_R \mathcal{O}^*,$$

in which, since $\mathcal{O}^* \in \mathrm{add}(A)$ and L_R and L'_R are projective, the last two terms are in $\mathrm{Add}(A)$. Thus L is the kernel of a map in $\mathrm{Add}(A)$, so that $F(L)$ is kernel of a map in $\mathrm{Proj}(\Gamma)$. The lemma now follows from Lemma 2.2. □

Lemma 2.4. *Each $\mathcal{O}$-module L may be imbedded in a short exact sequence*

$$0 \to M \to X \otimes_R A \xrightarrow{p} L \to 0,$$

in which X_R is free and on which the functor F is exact.

Proof. Take X to be the free R-module with basis $F(L)$, and p to be the map $h \otimes_R a \mapsto h(a)$ for each $h \in F(L)$ and $a \in A$. The surjectivity of p is a consequence of the fact that A is a generator of $\mathrm{Mod}(\mathcal{O})$. □

Proof of Theorem 2.1. It suffices to show that each generalised lattice, L, is in $\mathrm{Add}(A)$. Iteration of the last lemma shows that there is an exact sequence of $\mathcal{O}$-modules,

$$Z \otimes_R A \xrightarrow{r} Y \otimes_R A \xrightarrow{q} X \otimes_R A \xrightarrow{p} L \to 0,$$

in which X, Y, Z are free R-modules, and such that the sequence of Γ-modules and morphisms

$$F(Z \otimes_R A) \xrightarrow{F(r)} F(Y \otimes_R A) \xrightarrow{F(q)} F(X \otimes_R A) \xrightarrow{F(p)} F(L) \to 0,$$

is exact. Lemma 2.3 shows that $F(L)$ is projective, and the first three modules in the sequence are isomorphic to the free Γ-modules $Z \otimes_R \Gamma$, $Y \otimes_R \Gamma$ and $X \otimes_R \Gamma$, respectively. Therefore there exists an exact sequence of Γ-morphisms

$$F(Z \otimes_R A) \xleftarrow{\nu} F(Y \otimes_R A) \xleftarrow{\mu} F(X \otimes_R A) \xleftarrow{\lambda} F(L) \leftarrow 0,$$

such that $F(p)\lambda = 1$, $\lambda F(p) + F(q)\mu = 1$, and $\mu F(q) + F(r)\nu = 1$. Recall that F induces an isomorphism $\mathrm{Add}(A) \to \mathrm{Proj}(\Gamma)$ of module categories. Thus the maps μ and ν are of the form $\mu = F(m)$ and $\nu = F(n)$ in such a way that the sequence

$$Z \otimes_R A \xleftarrow{n} Y \otimes_R A \xleftarrow{m} X \otimes_R A$$

is exact and $mq + rn = 1$. Let $e = qm$. Then e is an idempotent on $X \otimes_R A$. Since $pe = 0$, $\mathrm{Im}(e) \subsetneqq \mathrm{Ker}(p)$. On the other hand

$$\mathrm{Ker}(p) = \mathrm{Im}(q) = \mathrm{Im}(q(mq + rn)) = \mathrm{Im}(eq) \subset \mathrm{Im}(e).$$

Hence $\mathrm{Ker}(p) = \mathrm{Im}(e)$, which implies that p maps the summand $\mathrm{Im}(1 - e)$ of $X \otimes_R A$ isomorphically onto L. Since $X \otimes_R A$ is in $\mathrm{Add}(A)$, so also is L, which fact completes the proof of the theorem. $\square$

Here are some easy corollaries. Since the Auslander lattice is finitely generated, the well-known result of Kaplansky's (see [8] or Theorem 2.47 in [7]) that any direct summand of any direct sum of countably generated modules is a direct sum of countably generated modules immediately gives the following corollary.

Corollary 2.5. *The generalised lattices over an R-order of finite lattice type are direct sums of countably generated generalised lattices.*

The next two corollaries apply to more specialised orders.

Corollary 2.6. *Let R be a complete discrete valuation ring. The generalised lattices over an R-order of finite lattice type are isomorphic to direct sums of indecomposable lattices.*

Proof. By Proposition 6.10 in [6], the hypothesis on R implies that the endomorphism rings of indecomposable lattices over any R-order are local rings. For an R-order of finite lattice type, Theorem 2.1 therefore implies that each generalised lattice is a direct summand of a direct sum of copies of finitely generated modules with local endomorphism rings. By, for example, Corollary 2.55 in [7], a summand of such a direct sum is again a direct sum of copies of finitely generated modules with local endomorphism rings, which proves the corollary. $\square$

Let K be the field of fractions of R.

Corollary 2.7. *Let $\mathcal{O}$ be a right hereditary R-order such that $K \otimes_R \mathcal{O}$ is a semisimple K-algebra, and assume that $\mathcal{O}$ has only finitely many isomorphism classes of right ideals. Then its generalised lattices are its projective modules. In particular, if $\mathcal{O}$ is a Dedekind domain with finite ideal class group, then its non-finitely generated generalised lattices are free $\mathcal{O}$-modules.*

Proof. Since $K \otimes_R \mathcal{O}$ is semisimple, any lattice over $\mathcal{O}$ can be imbedded as a submodule of a finitely generated projective $\mathcal{O}$-module (see Exercise 1 in Section 23 of [6]). So, since $\mathcal{O}$ is right hereditary, its lattices are the finitely generated projective modules, and these are direct sums of finitely many copies of right ideals (see, for example, Proposition 4.3 in [6]). Since, up to isomorphism, there are only finitely many right ideals, one concludes that $\mathcal{O}$ has finite lattice type. Moreover, its Auslander lattice is projective, so Theorem 2.1

implies that the generalised lattices are also projectives, these again then being direct sums of copies of right ideals in $\mathcal{O}$. The last sentence of the corollary follows from the fact (see, for example, [3]) that non-finitely generated projective modules over any Dedekind domain are free. $\square$

3. The proof of Theorem 1.1. In this section, the integer group ring of a cyclic group of prime order, p, will be taken in the form

$$\Lambda = \mathbb{Z}[t]/(1 - t^p).$$

It is of course a $\mathbb{Z}$-order. The proof to follow of Theorem 1.1 splits naturally into two parts. In Subsection 3.2, Theorem 3.4 shows that any generalised Λ-lattice, L, has a direct decomposition

$$L = L_0 \oplus L_c \oplus L_a$$

into a *trivial* summand, L_0, which is annihilated by the operator $1-t$, a *cyclotomic* summand, L_c, which is annihilated by the operator $1 + t + \cdots + t^{p-1}$, and a summand L_a which is *acyclic* in the sense to be recalled below of cohomology theory. The trivial summand, L_0, is just a direct sum of copies of the trivial lattice of rank 1. The cyclotomic summand, L_c, is a generalised lattice for the ring of integers in the field of p-th roots of unity, so by Corollary 2.7 is isomorphic to a direct sum of ideals in that ring, and these ideals, viewed as Λ-modules, are indecomposable lattices. Thus, the second part of the proof of Theorem 1.1 consists in showing that the *acyclic* generalised lattices are direct sums of lattices, and this is done in Subsection 3.3 using Corollary 2.5 and an *ad hoc* inductive argument.

The Diederichsen-Reiner theory of Λ-lattices in Section 34 of [6] does, of course, show that any *lattice* is a direct sum of trivial, cyclotomic, and acyclic summands, and it further shows that the indecomposable acyclic summands are precisely the *projective* indecomposables, each of which has rank p. However, as pointed out already in the Introduction, these results will actually be needed to show that the acyclic generalised lattices are also direct sums of indecomposable *projective* lattices.

3.1. Notation and preliminary results. A special role will be played by the elements $1 - t$ and $1 + t + \cdots + t^{p-1}$ of the integer group ring $\Lambda = \mathbb{Z}[t]/(1 - t^p)$ so, for brevity, set

$$s := 1 + t + \cdots + t^{p-1}.$$

Any (right) Λ-module, L, contains submodules

$$L^+ := \operatorname{Ker}(1 - t)_L \text{ and } L^- := \operatorname{Ker} s_L,$$

where λ_L denotes right multiplication by an element λ of Λ on L. The first of these submodules is trivial and the second cyclotomic. Since $s(1 - t) = 0$, there are inclusions

$$L^+ \supseteqq Ls \text{ and } L^- \supseteqq L(1 - t), \tag{1}$$

and L will be called *acyclic* when both these inclusions are equalities. Since the cohomology groups of $C(p)$ with coefficients in L are given by $H^0(C(p), L) = L^+$ and, for all $n \geqq 1$,

$$H^{2n}(C(p), L) = L^+/Ls \text{ and } H^{2n-1}(C(p), L) = L^-/L(1-t),$$

the terminology accords with the use — for example, in [10] — of the term *F-acyclic object* for an object X on which all right derived functors $R^n F$ of the functor F with $n \geqq 1$ vanish.

Lemma 3.1. *Let L be a generalised lattice. Then:* (a) $L^+ \cap L^- = 0$. (b) *In any additive subgroup, M, of L, the subgroups $L^+ \cap M$ and $L^- \cap M$ are direct summands of M.* (c) *If $L^+ = Ls$ and L^- is finitely generated, then L is finitely generated.*

Proof. For $x \in L^+ \cap L^-$, $x = xt$ and so $0 = xs = xp$. Since L is torsionfree, $x = 0$, which proves (a). For (b), note that $L^+ \cap M$ and $L^- \cap M$ are kernels of the restrictions to M of the abelian group endomorphisms $(1-t)_L$ and s_L, respectively. Since $L_{\mathbb{Z}}$ is free, so also are the images of these restricted maps, so they split over their kernels.

The proof of (c) makes use of the identical formula

$$s + (1-t)q = p, \tag{2}$$

where $q = (p-1) + (p-2)t + \cdots + 2t^{p-3} + t^{p-2}$.

Since L^+ is a pure subgroup of L, $L^+p = L^+ \cap Lp$, and so $L^+/L^+p \cong (L^+ + Lp)/Lp$. Now assume the conditions in (c) hold. Since $L^+ = Ls$, (1) and (2) show that $L^+ + Lp = L(1-t)q + Lp \subseteqq L^- + Lp$. Hence L^+/L^+p is isomorphic to a subgroup of L^-/L^-p (this time using the purity of L^- in L). However both L^+ and L^- are free abelian groups, and L^- is finitely generated. Hence so is L^+. Finally, (1) and (2) show that $Lp \subseteqq L^+ + L^-$, so that Lp, and hence L itself, is finitely generated. □

Use will be made in the next section of the following result, reminiscent of a Stacked Bases Lemma.

Lemma 3.2. *Let S be a commutative ring with* 1 *and I a maximal ideal in S. If N is a submodule of the free S-module M, and $N \supseteqq MI$, then M has a direct decomposition $M = A \oplus B$ such that $N = AI \oplus B$.*

Proof. Let X be a free basis of M. Since $MI \subseteqq N$, M/N is a vector space over the field S/I, so there is a subset Y of X which maps one-to-one, modulo N, onto a basis of M/N. Let A be the submodule generated by Y, so that $M = A + N$. For each element $x \in X \backslash Y$, choose $z_x \in N$ such that $x - z_x \in A$, and let B be the submodule of N generated by the elements z_x. It is easy then to verify that A and B satisfy the lemma. □

Remark 3.3. It seems possible that Lemma 3.2 remains true, at least if S is a Dedekind domain, for M projective rather than free. If so, the last part of the proof of the decomposition theorem in the next section would simplify significantly.

3.2. A decomposition theorem.

Theorem 3.4. *A generalised* Λ*-lattice,* L*, has a direct decomposition* $L = L_0 \oplus L_c \oplus L_a$ *in which* L_0 *is trivial,* L_c *is cyclotomic, and* L_a *is acyclic.*

Proof. The first step is to show that

$$L = L_0 \oplus M, \text{ where } L_0 \subseteqq L^+ \text{ and } M^+ = Ms. \tag{3}$$

Since $t = 1$ on L^+, there are inclusions $L^+ \supseteqq Ls \supseteqq L^+p$ to which Lemma 3.2 may be applied with $S = \mathbb{Z}$, $I = p\mathbb{Z}$, $M = L^+$ and $N = Ls$, to obtain direct decompositions

$$L^+ = L_0 \oplus A \text{ and } Ls = L_0 p \oplus A.$$

This is, of course, a Λ-module decomposition, since every subgroup of L^+ is a trivial Λ-module.

By Lemma 3.1, there is an abelian group direct decomposition of L of the form

$$L = L^+ \oplus U = L_0 \oplus A \oplus U$$

in which U is not necessarily a submodule of L. Let $u \in U$. Then

$$us = x_u p + a_u = x_u s + a_u,$$

for some $x_u \in L_0$ and $a_u \in A$. Since U is a free abelian group, each element u of a chosen basis of U may be replaced by $\dot{u} := u - x_u$, and the $\dot{u}$'s are then a basis of a new complement, $\dot{U}$, of L^+ in L such that $\dot{U}s \subseteqq A$. For any $\dot{u} \in \dot{U}$, let

$$\dot{u}(1-t) = x + a + \ddot{u} \quad \text{where} \quad x \in L_0, a \in A, \ddot{u} \in \dot{U}.$$

Multiplication by s gives

$$0 = xp + ap + \ddot{u}s \in L_0 \oplus A \oplus \dot{U},$$

so, since $\ddot{u}s \in A$, $xp = 0$. Now L is torsionfree, so $x = 0$ and $\dot{u}t \in M := A \oplus \dot{U}$, which shows that $L = L_0 \oplus M$ is a module direct decomposition such that $M^+ \subseteqq A \subseteqq Ms$. Since $L^+ = L_0 \oplus M^+$ and $Ls = L_0 p \oplus Ms$, it follows from the initial choice of L_0 and A that $M^+ = A = Ms$. This completes the proof of (3).

Following this step, it suffices to prove the theorem under the assumption—made from now on—that

$$L^+ = Ls.$$

One possibility is that L^- is finitely generated. If so, part (c) of Lemma 3.1 shows that L itself is finitely generated, so the theorem follows from the Diederichsen-Reiner theory.

The other possibility is that L^- is not finitely generated. Since it is cyclotomic, it is a generalised lattice over the $\mathbb{Z}$-order $\Gamma = \Lambda/(s)$ of all algebraic integers in the field of pth roots of unity, so by the last assertion in Corollary 2.7, is a *free* Γ-module. Thus, Lemma 3.2

may be applied to L^-, with $S = \Gamma$, with I the ideal generated by the irreducible element $1 - t + (s)$ of Γ, and with M and N the Γ-modules L^- and $L(1-t)$, respectively. The lemma yields direct sum decompositions of Γ-modules, and hence of Λ-modules,

(4) $$L^- = L_c \oplus B \text{ and } L(1-t) = L_c(1-t) \oplus B.$$

By Lemma 3.1, there is an abelian group direct decomposition

$$L = L^- \oplus V = L_c \oplus B \oplus V$$

in which V is not necessarily a submodule of L. Let $v \in V$. Then

$$v(1-t) = y_v(1-t) + b_v,$$

for some $y_v \in L_c$ and $b_v \in B$. Since V is a free abelian group, each element v of a selected basis may be replaced by $\dot{v} := v - y_v$ to obtain a new complement $\dot{V}$ of L^- in L such that $\dot{V}(1-t) \subseteqq B$. Hence $N := B \oplus \dot{V}$ is a Λ-module complement of L_c such that $N(1-t) \subseteqq B$. Now comparison of the formulae $L^- = L_c \oplus N^-$ and $L^-(1-t) = L_c(1-t) \oplus N(1-t)$ with (4) shows that $N^- = B = N(1-t)$. However, N inherits from L the property $N^+ = Ns$, so it is an acyclic generalised lattice, as the theorem asserts. $\square$

Remark 3.5. If Γ is a principal ideal domain, that is, if the ideal class number $h = 1$, the use of the Diederichsen-Reiner theory in the proof of this theorem can be avoided, for then L^- is always a free Γ-module.

3.3. Acyclic generalised lattices. The proof of Theorem 1.1 can now be completed by proving it for acyclic generalised lattices. The actual result to be proved is the following more precise statement.

Theorem 3.6. *Each acyclic generalised Λ-lattice is isomorphic to a direct sum of indecomposable projective lattices.*

Some preliminary definitions and results are needed. First, there is the following straightforward consequence of the Diederichsen-Reiner classification of Λ-lattices in Theorem 34.31 in [6].

Lemma 3.7. *A Λ-lattice, F, of rank at most p is either an indecomposable projective lattice, or $F = F^+ \oplus F^-$.*

Next, recall that in relative homological algebra a lattice, M, over an R-order $\mathcal{O}$ is said to be *relatively injective* if any short exact sequence,

$$0 \to M \to X \to Y \to 0,$$

of $\mathcal{O}$-modules which splits as an exact sequence of R-modules is split as a sequence of $\mathcal{O}$-modules.

Lemma 3.8. *Finitely generated projective modules over the group ring RG over R of a finite group G (viewed as an R-order) are relatively injective.*

Proof. It suffices to show that RG is relatively injective. Let $f : RG \to X$ be an RG-module monomorphism for which there is an R-module epimorphism $h : X \to RG$ such that $hf = 1_{RG}$. If e denotes the R-module endomorphism of RG such that $e(1) = 1$ and $e(g) = 0$ for $g \in G\backslash\{1\}$, the homomorphism $h^* : X \to RG$ given by

$$h^*(x) = \sum_{g \in G} (eh(xg))g^{-1} \text{ for each } x \in X$$

is easily seen to be an RG-homomorphism satisfying $h^* f = 1_{RG}$. □

Corollary 3.9. *A finitely generated projective submodule, F, of a generalised Λ-lattice, L, which is pure as an additive subgroup of L is a Λ-module direct summand of L.*

Proof. The lemma shows that F is relatively injective for the $\mathbb{Z}$-order Λ. As an abelian group, it is a finitely generated pure subgroup of the free abelian group L, so is a direct summand of $L_{\mathbb{Z}}$. Therefore it is also a summand of L_{Λ}. □

Lemma 3.10. *In any generalised Λ-lattice, L,*

$$0 = \bigcap_{m \geqq 0} Lp^m = \bigcap_{n \geqq 0} L^-(1-t)^n.$$

Proof. The first equality holds since L is free as an abelian group. The second is then a consequence of the easily verified formula $(1-t)^{p-1} \equiv s (\mathrm{mod}\ p)$. □

Proposition 3.11. *Each element, x, of an acyclic generalised Λ-lattice, L, is contained in a finitely generated projective direct summand of L.*

Proof.

Case 1. Assume $x \in L^-$. By Lemma 3.10, we may assume that $x \notin L^-(1-t)$; however, since L is acyclic, there is some $y \in L\backslash L^-$ such that $x = y(1-t)$. Let F be the pure closure in L of the lattice $y\Lambda$. It has the same rank as $y\Lambda$, which is at most p, but cannot decompose as $F = F^+ \oplus F^-$ because, otherwise, $x \in F(1-t) = F^-(1-t)$, contrary to the assumption that $x \notin L^-(1-t)$. By Lemma 3.7, F is an indecomposable projective lattice, so by Corollary 3.9 is a summand of L.

Case 2. Assume $x \in L^+$. By Lemma 3.10, we may assume that $x \notin L^+s = L^+p$, but since L is acyclic, there is some $y \in L\backslash L^+$ such that $x = ys$. Let F be the pure closure in L of the lattice $y\Lambda$. It has the same rank as $y\Lambda$, which is at most p, but cannot decompose as $F = F^+ \oplus F^-$ because, otherwise, $x \in Fs = F^+s$, contrary to the assumption that $x \notin L^+s$. By Lemma 3.7, F is an indecomposable projective lattice, so by Corollary 3.9 is a summand of L.

Case 3. Any $x \in L$. By Case 1, L has a direct decomposition $L = F_1 \oplus L_1$ in which $x(1-t)$ is contained in the finitely generated projective module F_1, and L_1 is an acyclic generalised lattice. Let $xs = x_0 + x_1$, with $x_0 \in F_1{}^+$ and $x_1 \in L_1{}^+$. By Case 2, L_1 has a direct decomposition $L_1 = F_2 \oplus L_2$ in which x_1 is contained in the finitely generated projective module F_2. By (2), it follows that $xp \in F = F_1 \oplus F_2$, and since F is a direct summand of L and L is torsionfree, $x \in F$. Since F is also a finitely generated projective module, this proves the proposition. $\square$

Proof of Theorem 3.6. By Corollary 2.5 it suffices to consider an acyclic generalised lattice, L, which is countably generated. Given a sequence $x_1, x_2, \ldots$ of generators of L, one can construct inductively a sequence of projective sublattices $F_1, F_2, \ldots$ of L such that, for each $n \geqq 1$, L has a direct decomposition $L = F_1 \oplus F_2 \oplus \ldots \oplus F_n \oplus L_n$, and the summand $F_1 \oplus F_2 \oplus \ldots \oplus F_n$ contains $x_1, x_2, \ldots, x_n$. The proposition shows this can be done for $n = 1$. Suppose $n \geqq 1$, and that $F_1, F_2, \ldots, F_n$ have been constructed with the required properties. Let $x_{n+1} = y + z$ with $y \in F_1 \oplus F_2 \oplus \ldots \oplus F_n$ and $z \in L_n$. Since L_n is acyclic, the proposition implies that L_n has a direct decomposition of the form $F_{n+1} \oplus L_{n+1}$ in which F_{n+1} is a projective lattice containing the element z. This extends the construction to $n + 1$, as required. By construction the infinite sum $\sum_{n \geqq 1} F_n$ is direct, and contains a generating set of L, so it must equal L. Finally, the construction ensures that each F_n is a finitely generated projective lattice.

Note (added 18 February 2004). We are grateful to the referee for drawing our attention to the facts that [5] has appeared by now, and that we (like [5]) could also have used Theorem 6.1 of [4] to advantage. Namely, instead of Theorem 3.6 proved with the help of Corollary 2.5, one can aim directly for the claim that *every acyclic generalised* Λ*-lattice is projective*, and invoke [3] to deduce that therefore such a generalised lattice is either free or finitely generated. To sketch a proof of this claim, let L be an acyclic generalised Λ-lattice. By Theorem 6.1 of [4], in order to prove that L is projective, it suffices to show that L/pL is projective as $(\mathbb{Z}/p\mathbb{Z})C(p)$-module. Recall that every $(\mathbb{Z}/p\mathbb{Z})C(p)$-module is a direct sum of quotients of the regular module (the argument is the same as for $(\mathbb{Z}/p^k\mathbb{Z})$-modules). From this, or directly, it is easy to see that a $(\mathbb{Z}/p\mathbb{Z})C(p)$-module, M, is free if and only if $\operatorname{Ker}(1-t)_M \leqq Ms$. It remains to check that $M = L/pL$ passes this test. Suppose that $l + pL \in \operatorname{Ker}(1-t)_M$, that is, $l(1-t) = px$ for some $x \in L$. Then $px \in L^-$; as L^- is a group direct summand in L, we also have $x \in L^-$; as L is acyclic, $L^- = L(1-t)$, so $x = y(1-t)$ for some $y \in L$; and then $(l - py)(1-t) = 0$. This shows that $l - py \in L^+$; using $L^+ = Ls$ (the other half of the acyclicity assumption), we conclude that $l - py = zs$ for some $z \in L$, and then $l + pL = (z + pL)s \in Ms$ as required.

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