Lie Powers of the Natural Module for $GL(2)$

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1. INTRODUCTION

Let $L$ be a free Lie algebra of finite rank $r$ over a field $\mathbb{F}$. For each positive integer $n$, denote the degree $n$ homogeneous component of $L$ by $L^n$. The group of graded algebra automorphisms of $L$ may be identified with $GL(r, \mathbb{F})$ in such a way that $L^1$ becomes the natural module, and then the $L^n$ are referred to as the Lie powers of this module. Understanding the $GL(r, \mathbb{F})$-module structure of the $L^n$ may be thought of as an essential part of understanding $L$.

For the case when the characteristic of $\mathbb{F}$ is 0, the $L^n$ are semisimple and the multiplicities of the various simple modules in the $L^n$ are given by a formula of Wever [24]. We are concerned here with the case when $\mathbb{F}$ has prime characteristic, $p$; then very few of the $L^n$ are semisimple, and the problem has a different complexion. If $n$ is not divisible by $p$, then $L^n$ is a direct summand of the $n$-fold tensor power of the natural module $L^1$, so the indecomposable direct summands of $L^n$ may be taken as known from that context. This has been exploited in Erdmann [12] and in Donkin and

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Erdmann [9] to obtain a formula for the relevant Krull–Schmidt multiplicities under the assumption that \( F \) is infinite. In this paper we consider the simplest case at the other extreme, when \( r = 2 \) and \( F = \mathbb{F}_p \), the field of \( p \) elements. The advantage here is that the Sylow \( p \)-subgroups of \( GL(2, p) \) are cyclic and therefore there are only finitely many indecomposable modules to consider: in fact, the indecomposables have been completely described in Glover [13], so the answer can again be sought simply in terms of Krull–Schmidt multiplicities. In spite of this, we can report only partial success: we can give these multiplicities only if either \( p = 2 \) (Theorem 3.1) or \( p = 3 \) (Theorem 6.1), or if \( n \) is not divisible by \( p \) (see Section 8). When \( p = 3 \), the key lies in an intermediate result (Theorem 5.2) whose main point is that \( L^n \) as module for a Sylow 3-subgroup of \( GL(2, 3) \) has no one-dimensional direct summand (except for \( n = 2 \)). This depends very closely on Bryant and Stöhr [6].

A much harder question concerns the homogeneous components of the free Lie algebra of rank \( r \) over \( \mathbb{Z} \): can one say anything about their structure as modules for \( GL(r, \mathbb{Z}) \)? The first step in this direction is implicit in Bryant and Stöhr [5]: for the case \( r = 2 \), the results of that paper readily yield the structure of these homogeneous components as modules for an indecomposable subgroup of order 2 in \( GL(2, \mathbb{Z}) \) (all such subgroups are conjugate). In Corollary 7.3 and Theorem 7.4, we extend this result to the maximal finite subgroups of \( GL(2, \mathbb{Z}) \) (which fall into two conjugacy classes). While in general there is no Krull–Schmidt theorem for the integral representations of these larger finite groups, all turns out to be well from this point of view for the representations that we have to deal with, in the sense that our result can be given in terms of multiplicities that are genuine invariants.

The formulas of Wever and of Erdmann involve characters and Brauer characters of the symmetric group \( S_n \). The task of evaluating these at the relevant elements of \( S_n \) (namely at powers of a cyclic permutation of degree \( n \)) is a challenge in itself, and has been the subject of recent papers by Erdmann [11] and Zhuravlev [26]. However, here we only need them for the characters corresponding to partitions with at most two parts. For this case, the decomposition numbers are available from James [16], and we are fortunate in that the numbers we want can be obtained without seeing individual values of characters (of symmetric groups).

We are indebted to Csaba Schneider for computing several large examples which pointed to Theorem 5.2. He used a LiePQ program that is not publicly available yet, and Magma, see [2]. We are also grateful to Karin Erdmann for drawing our attention to several relevant references, and to Roger Bryant for many illuminating discussions.

The organization of the paper is as follows. Section 2 gathers some preliminary material. The case of \( GL(2, 2) \) is dealt with in Section 3, the
next three sections are occupied by the case of $GL(2, 3)$, while finite subgroups of $GL(2, \mathbb{Z})$ are considered in Section 7. Finally, in Section 8, we discuss the $GL(2, p)$-modules $L^n$ when $p$ is arbitrary but $n$ not divisible by $p$.

2. PRELIMINARIES

Let $\mu$ denote the Möbius function and $n$ any positive integer. It is an elementary exercise to deduce the well known formula

$$\sum_{d\mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise}. \end{cases} \quad (2.1)$$

In turn, one proves just as easily the following less well known variants:

$$\sum_{d\mid n, \ 2\nmid d} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is a power of } 2 \\ & (\text{including the case } n = 1), \\ 0 & \text{otherwise}; \end{cases} \quad (2.2)$$

$$\sum_{d\mid n, \ 2\nmid d} \mu(d) = \begin{cases} -1 & \text{if } n \text{ is a power of } 2 \text{ but } n \neq 1, \\ 0 & \text{otherwise}; \end{cases} \quad (2.3)$$

$$\sum_{d\mid n} \mu(d)(-1)^{n/d} = \begin{cases} -1 & \text{if } n = 1, \\ 2 & \text{if } n = 2, \\ 0 & \text{otherwise}; \end{cases} \quad (2.4)$$

$$\sum_{d\mid n, \ 3\nmid d} \mu(d)(-1)^{n/d} = \begin{cases} -1 & \text{if } n \text{ is a power of } 3, \\ 2 & \text{if } n/2 \text{ is a power of } 3, \\ 0 & \text{otherwise}. \end{cases} \quad (2.5)$$

The relevance of the Möbius function to our context stems from two dimension formulas due to Witt (Theorem 5.11 in Magnus, Karrass, and Solitar [21]). Let $L^n$ denote the homogeneous component of degree $n$ in the free Lie algebra of rank 2 (over any field), and let $L^n_m$ denote the span of the Lie monomials which have $m$ factors equal to one free generator and $n - m$ factors equal to the other, so $L^n = \bigoplus_m L^n_m$. The formulas in question state that

$$\dim L^n = \frac{1}{n}\sum_{d\mid n} \mu(d)2^{n/d} \quad \text{and}$$

$$\dim L^n_m = \frac{1}{n}\sum_{d\mid (m, n)} \mu(d)\left(\frac{n/d}{m/d}\right), \quad (2.6)$$
where \((m, n)\) stands for the greatest common divisor of \(m\) and \(n\). (They apply equally over \(\mathbb{Z}\), provided one reads dimension as the \(\mathbb{Z}\)-rank of the \(\mathbb{Z}\)-free group in question.) We shall want to use that

\[
\sum_{2 \mid m} \dim L_m^n = \frac{1}{n} \sum_{2 \mid m} \sum_{d \mid (m, n)} \mu(d) \left( \frac{n/d}{m/d} \right)
\]

\[
= \frac{1}{n} \sum_{d \mid n, 2 \mid d} \mu(d) \sum_{2 \mid k} \left( \frac{n/d}{k} \right)
\]

\[
= \frac{1}{n} \sum_{d \mid n, 2 \mid d} \mu(d) 2^{(n/d)-1}.
\]

It will be convenient to define, for each prime \(p\),

\[
\psi(n) = \frac{1}{n} \sum_{d \mid n} \mu(d) 2^{n/d} = \psi(n, p) + \psi(n, p'),
\]

(2.7)

where \(\psi(n, p)\) stands for \(\frac{1}{n} \sum \mu(d) 2^{n/d}\) with \(d\) ranging over those divisors of \(n\) which are divisible by \(p\), and \(\psi(n, p')\) stands for the same expression with \(d\) ranging over the divisors of \(n\) that are prime to \(p\). We also write this as

\[
\psi(n, p) = \frac{1}{n} \sum_{d \mid n, p \mid d} \mu(d) 2^{n/d} \quad \text{and} \quad \psi(n, p') = \frac{1}{n} \sum_{d \mid n, p \nmid d} \mu(d) 2^{n/d},
\]

(2.8)

setting a pattern for further abuses of notation. In these terms,

\[
\dim L^n = \psi(n),
\]

(2.9)

\[
\sum_{2 \mid m} \dim L_m^n = \frac{1}{n} \psi(n, 2') = \frac{1}{2} \psi(n) - \frac{1}{2} \psi(n, 2),
\]

(2.10)

From (2.9) we may note that \(\psi(n, 2')\) is always positive. On the other hand, \(\psi(n, 2)\) is never positive: this is not hard to see directly, and will be implied by Theorem 3.1.

We shall also need the formula for the Brauer characters of finite general linear groups afforded by the Lie powers of the natural module. This is the exact analogue of the character formula proved in characteristic 0 by Brandt in [3] and by Wever in [24]; it has been recognized for a long time that the proof in the latter paper, which was based on the dimension formulas (2.6) of Witt, has a wider application. Let \(\lambda_n\) denote the Brauer character in question; the result is that, for each \(p'\)-element \(g\) of such a group of characteristic \(p\),

\[
\lambda_n(g) = \frac{1}{n} \sum_{d \mid n} \mu(d) (\lambda_1(g^d))^{n/d}.
\]

(2.11)
This holds without any assumption (other than the characteristic) on the field or on the free rank of the Lie algebra, though of course we shall only need it in the rank 2 case.

3. THE CASE OF $GL(2, 2)$

In this section, $L$ is a free Lie algebra of rank 2 over the field $\mathbb{F}_2$.

Consider $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, elements of order 3 and 2 in $GL(2, 2)$. The characteristic roots of $s$ on $L^1$ are the two primitive cube roots of 1 (in the field of order 4), whence $\lambda_1(s) = -1 \in \mathbb{C}$. Of course, $\lambda_1(1) = 2$, so by (2.11) and (2.7) we get that

$$\lambda_n(s) = \psi(n, 3) + \frac{1}{n} \sum_{d|n, 3|d} \mu(d)(-1)^{n/d}. \quad (3.1)$$

Over any field of characteristic 2, the group $GL(2, 2)$ has three isomorphism types of indecomposable modules: the one-dimensional trivial, the two-dimensional natural, and a two-dimensional which is not irreducible; on the latter, $s$ acts trivially and $t$ acts regularly; the value of its Brauer character at $s$ is 2. Let $a(n), b(n), c(n)$ denote the respective Krull–Schmidt multiplicities in $L^n$, so $a(n) + 2b(n) + 2c(n) = \dim L^n$. Also, $a(n) - b(n) + 2c(n) = \lambda_n(s)$, while $a(n) + b(n) + c(n)$ is the dimension of the fixed point space of $t$ in $L^n$. By Theorem 1 of Bryant and Stöhr [5], the latter is $\frac{1}{2} \psi(n, 2')$. Thus we have three simultaneous linear equations which determine the three multiplicities: we may express $a(n)$ from the first and last, $b(n)$ from the first and second, and then $c(n)$ from the last. In view of (3.1) and (2.5), the conclusion may be put as follows.

**Theorem 3.1.** In the Lie power $L^n$ of the natural module for $GL(2, 2)$, the Krull–Schmidt multiplicities of the indecomposable modules are the following. The multiplicity of the trivial module is $-\psi(n, 2)$. The multiplicity $b(n)$ of the natural module is given by

$$b(n) = \begin{cases} \frac{1}{2^n}(2^n + 1) & \text{if } n \text{ is a power of } 3, \\ \frac{1}{2^n}(2^n - 2^{n/2} - 2) & \text{if } n/2 \text{ is a power of } 3, \\ \frac{1}{3} \psi(n, 3') & \text{otherwise.} \end{cases}$$

The multiplicity of the third indecomposable module is $\frac{1}{2} \psi(n) + \frac{1}{2} \psi(n, 2) - b(n)$. 


All but one of the essential ingredients of this theorem have been available for over 50 years. Problem 11.47 in the Kourovka notebook [17] was posed with this in mind, and its recent solution in [5] finally provided the missing step. In turn, the extension of that work in [6] has provided the basis for our next theorem, which will play a similar role in later sections.

4. FURTHER PRELIMINARIES FOR $GL(2, 3)$

Let $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and let $H$ be the nonabelian subgroup of order 6 that $g$ and $h$ generate in $GL(2, 3)$. The key to understanding the action of $GL(2, 3)$ on the Lie powers of its natural module lies in understanding the action of $H$ on those modules. (This is because the subgroup $P$ generated by $g$ is a Sylow 3-subgroup, the normalizer of $P$ is the direct product of $H$ with the group of scalars, and the latter act like scalars on all relevant modules.) We begin preparations here by reviewing the representation theory of $H$ over an arbitrary field $\mathbb{F}$ of characteristic 3. This amounts to little more than setting up notation: the group $H$ is so small that all proofs are straightforward exercises.

For each integer $i$, let $U_i$ stand for a one-dimensional $\mathbb{F}H$-module on which $g$ and $h$ act as the scalars 1 and $(-1)^i$, respectively. Note that $U_i \cong U_{i'}$ if and only if $i \equiv i' \pmod{2}$, while $U_i \otimes U_{i'} \cong U_{i+i'}$. The transitive permutation representation of $H$ on the set of its right cosets modulo $h$ will be referred to as the natural permutation representation of $H$. View it as a linear representation over $\mathbb{F}$ and call the corresponding (right) module $U(0, 3)$. One of the three permuted points is fixed by $h$: the corresponding basis vector is fixed by $h$ and generates $U(0, 3)$ as $\mathbb{F}H$-module. More generally, set $U(i, 3) = U_i \otimes U(0, 3)$: then $U(i, 3)$ is uniserial and its composition factors, in descending order, are $U_i, U_{i+1}, U_{i'}$. For $j = 1, 2$, let $U(i, j)$ denote the unique factor module of $U(i, 3)$ with composition length $j$ [so in particular $U(i, 1) \cong U_i$]. Note that $U(i, j)$ is generated by an eigenvector for $h$ with eigenvalue $(-1)^j$; the module is simple if and only if $j = 1$, and it is projective if and only if $j = 3$.

We shall find two less familiar observations also useful.

**Lemma 4.1.** An $\mathbb{F}H$-module $V$ is indecomposable if and only if it can be generated by an eigenvector for $h$; if this is the case and if the corresponding eigenvalue is $(-1)^i$, then $1 \leq \dim V \leq 3$ and $V \cong U(i, \dim V)$.

**Proof.** Each element of $H$ can be written as $h^k g^l$: therefore the $H$-orbit of an eigenvector for $h$ spans the same subspace as its $P$-orbit. Thus if an eigenvector for $h$ generates $V$ as $\mathbb{F}H$-module, it also generates it as $\mathbb{F}P$-module. Since the regular $\mathbb{F}P$-module is uniserial, so is every $\mathbb{F}P$-module generated by a single element. This proves that if $V$ is generated by an
eigenvector for \( h \), then \( V \) is uniserial. Since each uniserial module is a quotient of a projective indecomposable, the rest of the lemma follows from the preceding discussion, because here indecomposability implies uniseriality (see pp. 42–43 in Alperin [1] or Theorem VII.5.6 in Huppert and Blackburn [14]).

**Lemma 4.2.** For each positive integer \( n \), the tensor power \( U(0, 2)^{\otimes n} \) is the direct sum of a projective module and a nonprojective indecomposable module.

**Proof.** Induction on \( n \), using that \( U(0, 2) \otimes U(0, 2) = U(1, 1) \oplus U(0, 3) \).

Let us denote by \( L \), \( R \), and \( T \) the free Lie algebra, the free restricted Lie algebra, and the free associative (or tensor) algebra of rank 2 over \( \mathbb{F} \), each freely generated by \( x \) and \( y \), and write \( L^n, R^n, \) and \( T^n \) for their homogeneous components of degree \( n \). It will be convenient to take \( T \) (and then of course all its subalgebras) without a multiplicative identity element. As usual, we take the view that \( L \subset R \subset T \), so \( L^n \leq R^n \leq T^n \). In particular, the three vector spaces \( L^1, R^1, \) and \( T^1 \) are equal, with basis \( \{x, y\} \). We identify this space with \( U(0, 2) \) by letting \( H \) act on it in the obvious way:

\[
\begin{align*}
xg &= x + y, & xh &= x, \\
yg &= y, & yh &= -y.
\end{align*}
\]

Extend this to an action on all of \( T \) by algebra automorphisms, so each \( L^n, R^n, \) and \( T^n \) becomes an \( \mathbb{F}H \)-module. Our eventual aim is to obtain decompositions of the \( L^n \) into direct sums of indecomposable \( \mathbb{F}H \)-modules, but first we recall some simple facts.

**Lemma 4.3.** If \( 3 \nmid n \), then \( L^n \) is a direct summand of \( T^n \) as \( \mathbb{F}H \)-module.

**Proof.** This holds even with respect to the action of \( GL(2, \mathbb{F}) \) as the group of all graded automorphisms of \( T \). By Wever’s formula (see Theorem 5.16 in Magnus, Karrass, and Solitar [21]), there is a \( GL(2, \mathbb{F}) \)-invariant map from \( T^n \) to \( L^n \) which on \( L^n \) is just multiplication by the scalar \( n \), so if \( 3 \nmid n \) then the kernel of this map is a \( GL(2, \mathbb{F}) \)-invariant complement to \( L^n \) in \( T^n \).

We see from (2.6) and (2.4) that if \( n > 2 \) and \( 3 \nmid n \) then \( 3 \mid dim L^n \), and from Lemma 4.2 that every direct summand of \( T^n \) with dimension divisible by 3 is projective. Hence Lemma 4.3 yields the following.

**Corollary 4.4.** If \( n > 2 \) and \( 3 \nmid n \), then \( L^n \) is a projective \( \mathbb{F}H \)-module.

In low dimensions, we have

\[
\begin{align*}
L^2 &\cong U(1, 1), & L^3 &\cong U(1, 2), & L^4 &\cong U(1, 3), \\
L^5 &\cong U(0, 3) \oplus U(1, 3).
\end{align*}
\]
Direct calculation shows that the Lie monomials $[y, x, x, x, x, x]$, $[y, x, x, y, y]$, and $[[y, x, x, x], [y, x]]$ generate $L^6$ as $\mathbb{F}H$-module (in omitting brackets, we follow the left-norming convention $[u, v, w] = [[u, v], w]$).

Each of these monomials is an eigenvector for $h$, so we know from Lemma 4.1 that each of them generates an indecomposable module of dimension at most 3. Since $L^6$ is the sum of these modules and has dimension 9, each summand must have dimension 3 and the sum must be a direct sum. The eigenvalues corresponding to the three monomials are $(-1)^1, (-1)^3, (-1)^2$, so (using Lemma 4.1 once more) we conclude that

$$L^6 \cong U(0, 3) \oplus U(1, 3)^{\oplus 2}.$$  

Finally, it is readily checked that $\{x^3, (xg)^3, (xg^2)^3, y^3\}$ is a basis for $R^3$. From this we see that $R^3$ is the direct sum of the three-dimensional indecomposable module generated by $x^3$ and of the nontrivial one-dimensional submodule generated by $y^3$, so

$$R^3 \cong U(0, 3) \oplus U(1, 1).$$

5. THE CENTRAL ARGUMENT FOR $GL(2, 3)$

The next lemma is the key technical result at the heart of this paper. In preparation, we need to recall several general facts about free restricted Lie algebras. To avoid the impression that these depend on the special assumptions we have adopted here, we switch notation for the moment by changing to a different font; at the same time, we also suspend the assumption that the characteristic $p$ is 3. Let $U$ be a vector space over $\mathbb{F}$, $T$ be the tensor algebra on $U$, $L$ be the free Lie algebra and $R$ be the free restricted Lie algebra generated by $U$ in $T$. Given any vector space basis of $L$, the set of $p^k$th powers of elements of that basis (with $k = 0, 1, 2, \ldots$) form a basis of $R$. Recall also that

$$[R, R] \subseteq L.$$ \hspace{1cm} (5.1)

If $R$ is an ordered basis of $R$, then the associative products

$$w_1^{\alpha_1}w_2^{\alpha_2}\cdots w_l^{\alpha_l}$$ \hspace{1cm} (5.2)

with

$$l > 0,$$

$$w_i \in R \quad \text{and} \quad 1 \leq \alpha_i \leq p - 1 \quad \text{for} \quad i = 1, 2, \ldots, l,$$

and

$$w_1 < w_2 < \cdots < w_l,$$
are pairwise distinct and form a basis of $T$ (see [15, Chap. 5]). (Notice that we have excluded the empty word, to conform with our convention that $T$ has no multiplicative identity element.) By a well known theorem of Witt [25], any restricted Lie subalgebra of $R$ is itself free. Let $S$ be a restricted Lie subalgebra of $R$ and let $A$ denote the associative subalgebra of $T$ that is generated by $S$. If $V$ is the subspace spanned by a free generating set of $S$, then $A$ can be identified with the tensor algebra on $V$. (This can be seen as follows. Let $P$ be an ordered basis of $S$, and extend it to an ordered basis $R$ of $R$. The tensor algebra on $V$ may be considered to embed $S$, i.e., as a basis which consists of products (5.2) with the $w_i$ in $P$, and the identity map on $V$ extends to a homomorphism of this tensor algebra onto $A$. On the other hand, the products (5.2) in $T$ with the $w_i$ ranging over all of $R$ form a basis of $T$; those products (5.2) in $T$ that involve only elements from $P$ are pairwise distinct and form a set which, being contained in a basis of $T$, is linearly independent: so the given homomorphism onto $A$ from the tensor algebra on $V$ is an isomorphism.) Finally, we require a result of Kukin [19] about free generating sets for subalgebras of free restricted Lie algebras. A set $S$ of homogeneous elements in $R$ is called reduced if no element of $S$ is contained in the restricted Lie subalgebra generated by the other elements of $S$. It is a special case of Lemma 2 in [19] that if $S$ is reduced then it is a free generating set for the restricted Lie subalgebra that it generates.

We now return to $p = 3$ and the algebras $L$, $R$, $T$ that are freely generated by $x$ and $y$ and on which $H$ acts by graded algebra automorphisms. Let $S$ be the restricted Lie subalgebra of $R$ that is generated by $x^3$, $x^3y$, $x^3y^2$, and the $L^k$ with $k \geq 4$. Note that $S$ is $H$-invariant and, being generated by homogeneous elements of $R$, it is a graded subalgebra in the sense that $S = \bigoplus (R^k \cap S)$. It is also a free restricted Lie algebra, so it acquires a grading of its own once we choose a free generating set for it. However, this internal grading of $S$ will not be respected by $H$ unless the subspace spanned by the free generating set is an $H$-submodule. The first of our key lemmas shows how this can be achieved.

**Lemma 5.1.** There is a free generating set for $S$ as restricted Lie algebra which is a disjoint union $P \cup Pg \cup Pg^2$, with $P$ consisting of eigenvectors for $h$ that are homogeneous as elements of $R$. Moreover, $P \cap R^n \subseteq L^n$ whenever $n \geq 4$.

**Proof.** For $k \geq 3$, we plan to show the existence of a subset $P_k$ in $R^k$ such that $P = \bigcup_k P_k$ yields a generating set with the required property. Let $S(n)$ denote the restricted Lie subalgebra of $R$ that is generated by $x^3$, $x^3y$, $x^3y^2$, and the $L^k$ with $4 \leq k \leq n$. Further, let $P(n) = \bigcup_{i=3}^{n} P_i$ and $P(n) = P(n)g \cup P(n)g^2$. In order to show that $P \cup Pg \cup Pg^2$
freely generates $S$, it will suffice to show that each $\mathcal{R}(n)$ is reduced in the sense of Kukin and generates $S(n)$.

The existence of suitable $\mathcal{R}_k$ will be proved by induction on $k$. For $k = 3$ we put $\mathcal{R}_3 = \{x^3\}$; this will obviously do as an initial step. Let $n > 3$ and consider the restricted Lie algebra $S(n - 1)$; by the inductive hypothesis, this is freely generated by $\mathcal{R}(n - 1)$. The subalgebra $S(n)$ is generated by $S(n - 1)$ and $L^n$. We claim that $L^n/(L^n \cap S(n - 1))$ is a free $\mathbb{F}P$-module. Once this is established, our inductive step can be easily completed. Indeed, then $L^n/(L^n \cap S(n - 1))$ is a projective $\mathbb{F}H$-module, so $L^n$ is the direct sum of $L^n \cap S(n - 1)$ and certain projective indecomposable summands. For each of the latter, we can choose a module generator that is an eigenvector for $h$, and let the set of these elements be $\mathcal{R}_n$. It is easy to see then that $\mathcal{R}(n)$ is reduced in the sense of Kukin and that it generates $S(n)$.

The rest of this proof will be taken up by showing that $L^n/(L^n \cap S(n - 1))$ is free as $\mathbb{F}P$-module.

As a first step, note that by the inductive hypothesis the vector space $V$ spanned by $\mathcal{R}(n - 1)$ is a free $\mathbb{F}P$-module. Let $A$ denote the associative subalgebra of $T$ generated by $S(n - 1)$; then $A$ can be identified with the tensor algebra on $V$. Hence $A$ is a free $\mathbb{F}P$-module. As $A = \bigoplus_{k=1}^n A \cap T^k$, it follows that each $A \cap T^k$ is a free, and therefore injective, $\mathbb{F}P$-module.

Let $\mathcal{M}$ be an ordered homogeneous basis of $S(n - 1)$ that includes a basis of $S(n - 1) \cap L^n$. Since $S \cap T^k = S(n - 1) \cap T^k$ when $k < n$, and $S \cap T^n = (S(n - 1) \cap T^n) + L^n$, we may extend $\mathcal{M}$ to an ordered homogeneous basis $\mathcal{M}'$ of $S$ in such a way that $\mathcal{M}' \cap T^k = \mathcal{M} \cap T^k$ when $k < n$, $\mathcal{M}' \cap T^n$ includes a basis of $L^n$, and $(\mathcal{M}' \setminus \mathcal{M}) \cap T^n \subseteq L^n$.

The next step is to show that the elements $x, y^3, [x, y]$, and $[x, y]^3$ with $\alpha \geq 0$, together with the elements of $\mathcal{M}'$, form a basis $\mathcal{B}$ of $R$. To this end, set
\[
c_1 = [x, y, x] + [y, x, y], \quad c_2 = -[x, y, x] + [y, x, y]
\]
and use that $L$ has a basis consisting of $x, y, [x, y], c_1, c_2$ together with homogeneous elements of degree at least 4. Therefore $R$ has a basis consisting of $x^3, y^3, [x, y]^3, c_1^\alpha, c_2^\alpha (\alpha \geq 0)$ together with set $\mathcal{F}$ consisting of $3^\alpha$th powers of Lie elements of degree at least 4. Now
\[
x^3g = (x + y)^3 = x^3 + y^3 + c_1, \quad x^3g^2 = (x - y)^3 = x^3 - y^3 + c_2
\]
whence
\[
c_1 = x^3g - x^3 - y^3, \quad c_2 = x^3g^2 - x^3 + y^3
\]
and so
\[
c_1^{3i} = (xg)^{3i+1} - x^{3i+1} - y^{3i+1} + w_1, \\
c_2^{3i} = (xg^2)^{3i+1} - x^{3i+1} + y^{3i+1} + w_2,
\]
where \( w_1, w_2 \) are linear combinations of elements of \( \mathcal{F} \). Therefore in our basis of \( R \) we can replace \( c_1^{3^r}, c_2^{3^r} \) by \( (xg)^{3^{r+1}}, (xg^2)^{3^{r+1}} \), respectively. It remains to note that in the resulting basis of \( R \) the elements other than \( x, y^3, [x, y]^3 \) form a basis of \( S \), and therefore they can be replaced by \( \mathcal{M}' \).

Next, we extend the order of \( \mathcal{M}' \) to \( R \) by setting

\[
y<y<y^3<\ldots<x<[x, y]<[x, y]^3<[x, y]^9<\ldots
\]

and postulating that \( u<u' \) whenever \( u \in R \setminus \mathcal{M}' \) and \( u' \in \mathcal{M}' \). Then the elements

\[
y^\beta x^\gamma [x, y]^\delta w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_l^{\alpha_l}
\]

with

\[
\beta \geq 0, \quad 0 \leq \gamma \leq 2, \quad \delta \geq 0, \quad l \geq 0, \quad \beta + \gamma + \delta + l > 0,
\]

\[
w_i \in \mathcal{M}' \quad \text{and} \quad 1 \leq \alpha_i \leq 2 \quad \text{for } i = 1, 2, \ldots, l,
\]

and

\[
w_1 < w_2 < \cdots < w_l,
\]

form a basis of \( T \), and the elements of this form with degree \( k \) form a basis of \( T^k \). Notice that the elements (5.4) with \( \beta = \gamma = \delta = 0 \) form a basis of the associative subalgebra generated by \( S \); among these, those with degree \( k \) form a basis of \( A \cap T^k \) when \( k < n \), while those of degree \( n \) form a basis of \( L^n + (A \cap T^n) \).

Let \( W_k \) be the subspace of \( T^k \) spanned by the elements

\[
y^\beta x^\gamma [x, y]^\delta
\]

with \( \beta, \gamma, \delta \) as above and \( \beta + \gamma + 2\delta = k \). This is an \( FP \)-submodule, because \( y \) and \( [x, y] \) are fixed by \( g \) while for \( x \) we have

\[
xg = x + y, \quad x^2g = (x + y)^2 = x^2 + 2yx + y^2 + [x, y],
\]

so the image of an element (5.5) under \( g \) is always a linear combination of such elements.

For each \( k \) with \( 0 \leq k \leq n \), let \( T^n_k \) denote the span of the basis elements (5.4) in \( T^n \) such that \( \beta + \gamma + 2\delta = k \). Then \( T^n = \bigoplus_k T^n_k \) and it follows from the foregoing that each \( T^n_k \) is an \( FP \)-submodule: in fact,

\[
T^n_0 = L^n + (A \cap T^n),
\]

\[
T^n_k = W_k \otimes (A \cap T^{n-k}) \quad \text{if } 1 \leq k \leq n - 1,
\]

and

\[
T^n_n = W_n.
\]
For any real number \( r \), write \([r]\) and \([r]\) for the integers such that
\[
|r| \leq r < |r| + 1 \quad \text{and} \quad \lfloor r \rfloor - 1 < r \leq \lfloor r \rfloor.
\]

It is straightforward to count that the dimension of \( W_n \) is \( 3\lfloor n/2 \rfloor + 2 \) if \( n \) is odd and \( 3\lfloor n/2 \rfloor + 1 \) if \( n \) is even: in either case, this dimension is not divisible by 3 and so \( W_n \) is never free as \( P \)-module. We know from Lemma 4.2 that, as \( P \)-module, \( T^n \) is a direct sum of a free module and a nonprojective indecomposable module, so in any direct decomposition of \( T^n \) all but one of the summands are free. Since \( W_n \) and \( L_n + (A \cap T^n) \) are distinct summands in the decomposition \( T^n = \bigoplus_k T^n_k \), we may therefore conclude that \( L_n + (A \cap T^n) \) is free. We have already seen that \( A \cap T^n \) is injective, hence it follows that the quotient \( (L_n + (A \cap T^n))/(A \cap T^n) \) is free. We shall prove that \( L_n \cap A \subseteq L_n \cap S(n-1) \) whenever \( L_n \cap (A \cap T^n) = L_n \cap A = L_n \cap S(n-1) \), and then \( (L_n + (A \cap T^n))/(A \cap T^n) \cong L_n/(L_n \cap S(n-1)) \) follows. As it was the freeness of \( L_n/(L_n \cap S(n-1)) \) that we had to show, this will complete the proof.

It remains, then, to prove that \( L_n \cap A \subseteq L_n \cap S(n-1) \). The elements (5.4) with \( \beta = \gamma = \delta = 0 \) and \( w_1, \ldots, w_l \in M \) form a basis of \( A \). Among them, those which are not in \( M \) (that is, those which are products of at least two elements from \( M \)) are linearly independent not only modulo \( S(n-1) \) but also modulo \( R \). Hence a linear combination of elements of this basis of \( A \) falls into \( R \) if and only if all elements outside \( M \) appear with coefficient 0. So \( R^n \cap A = R^n \cap S(n-1) \), and then \( L_n \cap A \subseteq L_n \cap R^n \cap A \subseteq L_n \cap R^n \cap S(n-1) \subseteq L_n \cap S(n-1) \).

Toward our next theorem, let \( M \) denote the Lie subalgebra generated by \( \mathcal{F} \). Since \( \mathcal{F} \) is a free generating set of the restricted Lie algebra \( S \), it is also free as generating set of the Lie algebra \( M \), and \( M \) has a grading \( M = \bigoplus k M^k \) with \( M^1 \) the subspace spanned by \( \mathcal{F} \). Given that \( \mathcal{F} \) consists of homogeneous elements of \( R \), it follows that each \( M^k \) is a graded \( H \)-subspace of \( R \), that is,
\[
M^k = \bigoplus_n (M^k \cap R^n) \quad \text{for } k = 1, 2, \ldots,
\]
and then also \( M \cap R^n = \bigoplus_k (M^k \cap R^n) \). The construction of \( \mathcal{F} \) yields that \( M^1 \cap R^n \subseteq L \) whenever \( n \geq 3 \), and \( M^k \cap R^n \subseteq L \) because of (5.1) whenever \( k > 1 \); thus \( M \cap R^n \subseteq L^n \) if \( n > 3 \).

In fact, \( M \cap R^n = L^n \) when \( n > 3 \). To show the missing inclusion, we use that \( S \) has a basis consisting of the \( x^{3^{n+1}}, (xy)^{3^{n+1}}, (xy^2)^{3^{n+1}}, u^{3^{n}} \), where \( a \geq 0 \) and \( u \) ranges over a suitable set \( \mathcal{N} \) of homogeneous elements of degree at least 4 in \( M \).

Since
\[
(xg)^{3^{n+1}} = x^{3^{n+1}} + y^{3^{n+1}} + c_1 g + w_1,
\]
\[
(xg^2)^{3^{n+1}} = x^{3^{n+1}} - y^{3^{n+1}} + c_2 g + w_2,
\]

and so...
where \( c_1 \) and \( c_2 \) are as defined in (5.3) while \( w_1 \) and \( w_2 \) are linear combinations of 3\(^{\text{rd}}\)th powers of Lie elements of degree at least 4, and since the elements \( x^{3^{\text{rd}}}, y^{3^{\text{rd}}}, c_1^{3^\text{rd}}, c_2^{3^\text{rd}} \) are (as powers of Lie elements of degrees 1 and 3, respectively) linearly independent modulo the restricted Lie subalgebra of \( R \) generated by the \( L_k \) with \( k \geq 4 \), it follows that the basis elements \( x^{3^{\text{rd}}}, (xy)^{3^{\text{rd}}}, (xy^2)^{3^{\text{rd}}}, u^3 (\alpha > 0, u \in N) \) are linearly independent modulo \( L \). Thus, if \( u \in L \cap S \) and \( \deg u \geq 4 \), then all basis elements of \( S \) occurring with nonzero coefficient in the expansion of \( u \) are in \( N \), and hence \( u \in M \).

It now follows that

\[
L^n = \bigoplus_k (M^k \cap R^n) \quad \text{for } n = 4, 5, \ldots
\]  

(5.6)

Further, \( M^1 \) is free as \( F \)-module, and \( \mathcal{F} \) is an \( F \)-free generating set for \( M \). It follows from Theorem 6.4 of [6] that there exist subsets \( \mathcal{L}_1, \mathcal{L}_2, \ldots \) in \( M \) such that

\[
\mathcal{L}_k \cup \mathcal{L}_k \mathcal{g} \cup \mathcal{L}_k \mathcal{g}^2 \cup \bigcup_{\alpha \geq 0} \left\{ [s^{3^\alpha}, s^{3^\alpha} \mathcal{g}, s^{3^\alpha} \mathcal{g}^2], [s^{3^\alpha} \mathcal{g}, s^{3^\alpha} \mathcal{g}^2, s^{3^\alpha}] \mid s \in \mathcal{L}_{k/3^{\alpha+1}} \right\}
\]

is a disjoint union and is a basis of \( M^k \) whenever \( k > 0 \) (with all the Lie products shown nonzero and pairwise distinct). Differently put: \( M^k \) has a decomposition as direct sum of indecomposable \( F \)-modules, each element of \( \mathcal{L}_k \) generating a three-dimensional summand and each \( s \) in \( \mathcal{L}_{k/3^{\alpha+1}} \) generating a two-dimensional summand. (Theorem 6.4 of [6] named \( w(s^{3^\alpha})(1 - g) \) and \( w(s^{3^\alpha})(1 - g^2) \) instead of the two Lie products in our last displayed formula. In the terminology of that paper, at \( p = 3 \) we have \( w(x_1, x_2, x_3) = x_2x_1x_3 + x_3x_1x_2 \), so \( w(s)(1 - g) = -[s, sg, sg^2] \) and \( w(s)(1 - g^2) = [sg^2, s, sg] \). By the Jacobi identity, whenever these Lie products are linearly independent, \([s, sg, sg^2], [sg, sg^2, s]\) is another basis for the subspace they span. The same holds also with \( s^{3^\alpha} \) in place of \( s \), and this justifies the switch to the present version.)

Theorem 6.4 of [6] is more precise than the paraphrase above: it also implies that the \( \mathcal{L}_k \) can be chosen to consist of elements that are multihomogeneous in the following sense. For each element \( s \) of \( \mathcal{F} \) and for each associative product whose factors all come from \( \mathcal{F} \), define the \( s \)-degree of the product as the number of factors belonging to \( \{s, sg, sg^2\} \). For each map \( \phi: \mathcal{F} \to \{0, 1, 2, \ldots\} \), \( s \mapsto \phi(s) \), call a linear combination of such products \textit{multihomogeneous of degree} \( \phi \) if, for each \( s \) in \( \mathcal{F} \), all the products involved have \( s \)-degree \( \phi(s) \). The set of all multihomogeneous elements of degree \( \phi \) is an \( F \)-submodule called the multihomogeneous component of degree \( \phi \), and the associative subalgebra of \( T \) generated by \( \mathcal{F} \) is the direct sum of these components.
We shall need even more here, namely that the $\mathcal{L}_k$ can be chosen to consist of eigenvectors for $h$, without giving up any of their previously claimed properties. This is ensured by Proposition 6.5 of [6].

Once the $\mathcal{L}_k$ consist of eigenvectors for $h$, the summands in the direct decompositions of the $M^k$ discussed above are in fact $\mathbb{F}H$-submodules. To see this, one need merely note that if $s$ is an eigenvector for $h$ then $[s^3g, s^3g^2, s^3]$ is also an eigenvector for $h$. Moreover, if the eigenvalue on the former is $(-1)^i$, then the eigenvalue on the latter is $(-1)^{i+1}$, so the submodule generated by $s$ is isomorphic to $U(i, 3)$ while that generated by the commutator in question is isomorphic to $U(i + 1, 2)$.

If $s \in \mathcal{L}$, then the elements $s, sg, sg^2$ are homogeneous with respect to the grading of $R$ and the degrees of these three elements are equal to each other. It follows that each multihomogeneous element of $M$ is homogeneous with respect to the grading of $R$. Thus each indecomposable summand of the direct decomposition we have obtained for $M^k$ lies in some $M^k \cap R^n$. It follows that each $M^k \cap R^n$ is the direct sum of some of the indecomposable summands in the decomposition of $M^k$ we obtained with reference to $\mathcal{L}_k, \mathcal{L}_{k/3}, \ldots$, namely, of the summands obtained from $\mathcal{L}_k \cap R^n, \mathcal{L}_{k/3} \cap R^{n/3}, \ldots$. Put $\mathcal{L} = \bigcup_k \mathcal{L}_k$. In view of (5.6), we may conclude that if $n > 3$ then $L^n$ is the direct sum of the copies of $U(0, 3)$ and $U(1, 3)$ obtained from $\mathcal{L} \cap R^n$ and of the copies of $U(0, 2)$ and $U(1, 2)$ obtained from $\mathcal{L} \cap R^{n/3}, \mathcal{L} \cap R^{n/9}, \ldots$. Note that $\mathcal{L} \cap R^1$ and $\mathcal{L} \cap R^2$ are empty, while $\mathcal{L} \cap R^3$ consists of a single element.

**Theorem 5.2.** For $n > 2$, the $H$-module $L^n$ is isomorphic to

$$U(0, 2)^{\oplus \kappa(n, 0, 2)} \oplus U(1, 2)^{\oplus \kappa(n, 1, 2)} \oplus U(0, 3)^{\oplus \kappa(n, 0, 3)} \oplus U(1, 3)^{\oplus \kappa(n, 1, 3)},$$

where the multiplicities $\kappa(n, i, j)$ may be calculated by the recursive rules

$$\kappa(3, i, j) = \begin{cases} 1 & \text{if } (i, j) = (1, 2), \\ 0 & \text{otherwise}, \end{cases}$$

while if $n \geq 4$ then

$$\kappa(n, 0, 2) = \sum_\beta \kappa(n/3^\beta, 1, 3),$$

$$\kappa(n, 1, 2) = \begin{cases} 1 & \text{if } n = 9, \\ \sum_\beta \kappa(n/3^\beta, 0, 3) & \text{otherwise}, \end{cases}$$

both sums being over all positive $\beta$ such that $n/3^\beta$ is an integer larger than 2, and

$$\kappa(n, 0, 3) = \frac{1}{2}(2\psi(n) - \kappa(n, 0, 2) - \kappa(n, 1, 2)) - \frac{1}{2}\psi(n) + \frac{1}{2}\psi(n, 2),$$

$$\kappa(n, 1, 3) = \frac{1}{2}(2\psi(n) - \kappa(n, 0, 2) - \kappa(n, 1, 2)) - \frac{1}{2}\psi(n) - \frac{1}{2}\psi(n, 2).$$
We have deliberately left $\kappa(n, i, j)$ undefined when $j = 1$ or $n \leq 2$, though for $j = 1$ and $n > 2$ we could have defined it to be 0, to indicate what may be the most important part of this theorem: namely, that neither $U(0, 1)$ nor $U(1, 1)$ occurs as a direct summand in any $L^n$ with $n > 2$. The recursive formulas given here for the $\kappa(n, i, 3)$ will be improved to explicit ones in Theorem 7.2.

Proof of Theorem 5.2. We have already seen that $L^3 \cong U(1, 2)$, whence we know the $\kappa(3, i, j)$. For each integer $i$, let $\lambda(n, i)$ denote the number of elements of $L \cap R^n$ on which the eigenvalue of $h$ is $(-1)^i$. Note that $\lambda(1, i) = \lambda(2, i) = \lambda(3, 1) = 0$ while $\lambda(3, 0) = 1$. For $n > 3$, we know also that $\kappa(n, i, 3) = \lambda(n, i)$ and $\kappa(n, i, 2) = \sum_\beta \lambda(n/3^\beta, i - 1)$ (we have set $\alpha + 1 = \beta$), and this yields the recursive formulas for the $\kappa(n, i, 2)$. Let us agree that $L^n_m$ is the span of the Lie monomials with $m$ factors $x$ and $n - m$ factors $y$. The eigenspace of $h$ in $L^n$ corresponding to the eigenvalue $(-1)^i$ is then the direct sum of the $L^n_m$ with $n - m \equiv i \pmod 2$, so dimension count leads to two simultaneous linear equations for the $\kappa(n, i, 3)$, namely to

$$\kappa(n, 0, 2) + \kappa(n, 1, 2) + \kappa(n, 0, 3) + \kappa(n, 1, 3) + \kappa(n, i, 3) = \sum_{n - m \equiv i} \dim L^n_m.$$  

The right-hand sides here may be replaced by the expressions provided in (2.9) and (2.10), and then the recursive formulas claimed for the $\kappa(n, i, 3)$ form just one way of writing the solution of the resulting system of equations.

The way Theorem 6.4 and Proposition 6.5 were used in the analysis of the module structure of $M$ obviously has wider application. We record here a general conclusion which seems to have some interest in its own right, even if it does not belong to the main thrust of this paper.

**Theorem 5.3.** Let $M$ be a free Lie algebra over a field of characteristic 3, and suppose that $H$ acts on $M$ by graded algebra automorphisms in such a way that the first homogeneous component $M^1$ is free as $P$-module. Then each homogeneous component $M^k$ of $M$ is a direct sum of copies of the $U(i, j)$ with $2 \leq j \leq 3$, and the multiplicity of $U(i, 2)$ in $M^k$ is the sum of the multiplicities of $U(i + 1, 3)$ in the $M^{k/3^\beta}$ as $\beta$ ranges over the positive integers such that $k/3^\beta$ is an integer.

6. THE CONCLUSIONS FOR $GL(2, 3)$

We need to review the representation theory of $GL(2, 3)$ over the field $\mathbb{F}_3$. The first four Lie powers of the natural module are easily seen to be irreducible, with $L^2$ a nontrivial one-dimensional, $L^3 \cong L^1 \otimes L^2$, and
the tensor product of $L^2$ with the symmetric square of $L^1$. Of course then $L^2 \otimes L^2$ and $L^2 \otimes L^4$ are further irreducibles, so we have recognized six of them. Since this is the number of conjugacy classes of $3^2$-elements in $GL(2, 3)$, there can be no more: we have them all. (Compare (6.2) in Glover [13].) In particular, the irreducible Brauer characters of $GL$ are $\lambda_2$, $\lambda_1$, $\lambda_1 \lambda_2$ ($= \lambda_3$), $\lambda_1 \lambda_4$, $\lambda_4$. As can be seen from the discussion early in Section 4, in this order the restrictions to $H$ of the corresponding modules are $U(0, 1)$, $U(1, 1)$, $U(0, 2)$, $U(1, 2)$, $U(0, 3)$, $U(1, 3)$.

The two irreducibles listed last here are of course projective; the projective covers of the first four are all uniserial of composition length 3, with middle composition factor isomorphic to the tensor product of the top (or bottom) composition factor with $L^2$ (see (6.1) in Glover [13]). Each of these four projective indecomposables has a uniserial quotient of composition length 2. Those and the four nonprojective irreducibles together give eight nonprojective indecomposables: in view of (3.3) and (3.8) of [13], there can be no more, so we have all the indecomposables.

It also follows that the restrictions to $H$ of the projective covers of the two one-dimensionals are $U(0, 3)$ and $U(1, 3)$, and the restrictions of the composition length 2 quotients of these are $U(0, 2)$ and $U(1, 2)$. The restriction of $L^1$ has a quotient $U(0, 1)$ and a submodule $U(1, 1)$, and the projective cover of $L^1$ has both a quotient $L^1$ and a submodule $L^1$. The restriction of this projective cover must be a direct sum of copies of $U(0, 3)$ and $U(1, 3)$; since it has both a quotient $U(0, 1)$ and a submodule $U(1, 1)$, it must in fact be $U(0, 3) \oplus U(1, 3)$. The same is true for the projective cover of $L^1 \otimes L^2$. Finally, the quotient of the $H$-module $U(0, 3) \oplus U(1, 3)$ over any $U(0, 2)$ is a $U(0, 3) \oplus U(1, 1)$, while its quotient modulo any $U(1, 2)$ is a $U(0, 1) \oplus U(1, 3)$. This accounts for the restrictions of all the indecomposables.

We know from Theorem 5.2 that (except when $n = 2$) the restriction of $L^n$ has no one-dimensional direct summand. It follows that 4 of the 14 indecomposable $GL(2, 3)$-modules cannot occur as direct summands in an $L^n$ with $n \neq 2$. The central involution of $GL(2, 3)$ acts on $L^n$ as the scalar $(-1)^n$: this means that four of the remaining indecomposables can only occur in $L^n$ when $n$ is odd, and the other six only when $n$ is even.

Let us take first the case of $n$ odd with $n \geq 3$. The four relevant indecomposables are $L^1$, $L^1 \otimes L^2$, and their projective covers. The restrictions of $L^1$ and $L^1 \otimes L^2$ to $H$ are $U(0, 2)$ and $U(1, 2)$, and neither of these can occur as a direct summand in the restriction of any projective module. This proves that the Krull–Schmidt multiplicities of $L^1$ and $L^1 \otimes L^2$ in $L^n$ are $\kappa(n, 0, 2)$ and $\kappa(n, 1, 2)$, respectively. Consequently, the Brauer character of the largest projective direct summand of $L^n$ is $\lambda_n - \kappa(n, 0, 2)\lambda_1 - \kappa(n, 1, 2)\lambda_1 \lambda_2$. A simple application of the modular or-
thogonality relations then yields the Krull–Schmidt multiplicities of the two relevant projective indecomposables in $L^n$.

When $n$ is even but $n \geq 4$, again only two of the indecomposables involved in $L^n$ are nonprojective and the restrictions of those to $H$ are $U(0, 2)$ and $U(1, 2)$, so we can argue as before that their multiplicities in $L^n$ are $\kappa(n, 0, 2)$ and $\kappa(n, 1, 2)$. The Brauer characters of these two nonprojective indecomposables coincide: both are equal to $2n$. This enables us to write the Brauer character of the largest projective direct summand of $L^n$ as $\lambda_n = [\kappa(n, 0, 2) + \kappa(n, 1, 2)](\lambda_2^3 + \lambda_2)$, and to use orthogonality once more to compute the multiplicities of the projective indecomposables.

The actual calculations could be entirely omitted, but as they involve some choices, we indicate ours, so the reader may arrive at the same form of the conclusion. As representatives of the six conjugacy classes of $3'$-elements in $GL(2, 3)$, we take $1$ (the identity matrix), $-1$, $h$, and $r$, where $r$ is an element of order 8. Their conjugacy classes consist of $1, 1, 1, 6, 6, 6$ elements, respectively, and the powering maps can be read off the observation that $r^3$ is conjugate to $r$, $r^4 = -1$, $r^5 = -r$, $r^6$ is conjugate to $r^2$, and $r^7$ is conjugate to $-r$.

We have that $\lambda_n(1) = \psi(n)$ by (2.8); of course, $\lambda_n(-1) = (-1)^n\lambda_n(1)$; likewise, $\lambda_n(-r) = (-1)^n\lambda_n(r)$. It is immediate from (2.7), (2.9), and (2.10) that $\lambda_n(h) = \psi(n, 2)$.

The characteristic roots of $r$ are a fourth root of $-1$ and the cube of that, so (on lifting the first to $e^{2\pi i/8}$ in $\mathbb{C}$) we get $\lambda_1(r) = i\sqrt{2}$ and $\lambda_1(r^2) = 0$.

We may therefore write

$$\lambda_1(r^d) = \begin{cases} (-1)^{|d/4|}i\sqrt{2} & \text{if } d \equiv 1 \pmod{2}, \\ 0 & \text{if } d \equiv 2 \pmod{4}, \\ -2 & \text{if } d \equiv 4 \pmod{8}. \end{cases}$$

By (2.11), it follows that

$$\lambda_n(r) = \begin{cases} \frac{1}{2} \sum_{d|n} \mu(d) (-1)^{|d/4|} (i\sqrt{2})^{n/d} & \text{if } n \text{ is odd}, \\ \frac{1}{2} (-1)^{n/2} \psi(n/2, 2') & \text{if } n \text{ is even}, \end{cases}$$

$$\lambda_n(r^2) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ -\frac{1}{2} (-1)^{n/2} \psi(n/2, 2') & \text{if } n \text{ is even}. \end{cases}$$

In particular, $\lambda_3(r) = -1$, $\lambda_3(r^2) = 1$, $\lambda_4(r) = 1$, and $\lambda_4(r^2) = -1$.

We now have all the ingredients for applying the modular orthogonality relations, and the conclusions may be put as follows.

**Theorem 6.1.** First suppose that $n$ is odd and $n \geq 3$. The Krull–Schmidt multiplicity of the natural $GL(2, 3)$-module $L^1$ in its Lie power
$L^n$ is $\kappa(n, 0, 2)$, while that of $L^1 \otimes L^2$ is $\kappa(n, 1, 2)$. The multiplicity of the projective cover of $L^1$ is

$$\frac{1}{12} \psi(n) - \frac{2}{3} \kappa(n, 0, 2) + \frac{1}{5} \kappa(n, 1, 2) + \frac{1}{2n} \sum_{d|n} \mu(d)(-1)^{d/4}(-2)^{(n-d)/2d},$$

and that of the projective cover of $L^1 \otimes L^2$ is

$$\frac{1}{12} \psi(n) + \frac{1}{3} \kappa(n, 0, 2) - \frac{2}{3} \kappa(n, 1, 2) - \frac{1}{2n} \sum_{d|n} \mu(d)(-1)^{d/4}(-2)^{(n-d)/2d}.$$ 

There are no other indecomposables involved in $L^n$.

Next suppose $n$ is even and $n \geq 4$. The multiplicity in $L^n$ of the two-dimensional quotient of the projective cover of $L^2 \otimes L^2$ is $\kappa(n, 0, 2)$, while that of the two-dimensional quotient of the projective cover of $L^2$ is $\kappa(n, 1, 2)$. The multiplicity of the projective cover of $L^2 \otimes L^2$ is

$$\frac{1}{24} \psi(n) + \frac{1}{4} \psi(n, 2) + \frac{1}{16}(-1)^{n/2} \psi(n/2, 2') - \frac{1}{4} [\kappa(n, 0, 2) + \kappa(n, 1, 2)],$$

the multiplicity of the projective cover of $L^2$ is

$$\frac{1}{24} \psi(n) - \frac{1}{4} \psi(n, 2) - \frac{3}{16}(-1)^{n/2} \psi(n/2, 2') - \frac{1}{4} [\kappa(n, 0, 2) + \kappa(n, 1, 2)],$$

the multiplicity of the projective irreducible $L^2 \otimes L^4$ is

$$\frac{1}{8} \psi(n) + \frac{1}{4} \psi(n, 2) - \frac{1}{16}(-1)^{n/2} \psi(n/2, 2'),$$

and the multiplicity of the projective irreducible $L^4$ is

$$\frac{1}{8} \psi(n) - \frac{1}{4} \psi(n, 2) + \frac{3}{16}(-1)^{n/2} \psi(n/2, 2').$$

There are no other indecomposables involved in $L^n$.

[Recall that $\psi(n)$, $\psi(n, 2)$, and $\psi(n/2, 2')$ were defined in (2.7); for the determination of the $\kappa(n, i, j)$, refer back to Theorem 5.2 and ahead to Theorem 7.2 and Remark 7.3.]

7. Finite Subgroups of $GL(2, \mathbb{Z})$

In this section, $L$ stands for the free Lie ring of rank 2 over $\mathbb{Z}$. The natural action of $GL(2, \mathbb{Z})$ on the homogeneous component $L^1$ extends to an
action on all of \( L \), by graded Lie ring automorphisms, so each homogeneous component \( L^n \) becomes a \( \mathbb{Z} \)-free \( GL(2, \mathbb{Z}) \)-module.

Let \( s = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \) and \( t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), and consider the subgroup \( H \) generated in \( GL(2, \mathbb{Z}) \) by \( s \) and \( t \). This is a dihedral group of order 6, and upon reduction modulo 3 it becomes conjugate to the subgroup of \( GL(2, 3) \) that we called \( H \) before. Its integral representation theory is well understood: see Lee [20] or p. 752 in Curtis and Reiner [7]. There are 10 isomorphism types of indecomposables, and one can readily check that only 4 of those satisfy the two conditions that we know must hold for every direct summand of an \( L^n \) with \( n \neq 2 \): as a module for the subgroup \( \langle t \rangle \), it can have no trivial direct summand (because of the corollary in Bryant and Stöhr [5]), and upon reduction modulo 3 it must not acquire a one-dimensional direct summand (because of the present Theorem 5.2). It follows that an unrefinable direct decomposition of an \( L^n \) (with \( n \neq 2 \)) can involve only those four indecomposables. To identify the four in our present terms, note that \( L^1, L^3, L^4 \) remain distinct and irreducible even after reduction modulo 3, so they must be among the four. On the other hand, reduction modulo 2 turns \( H \) into \( GL(2, 2) \) and \( L^5 \) into a sum of three two-dimensional indecomposables, while reduction modulo 3 turns \( L^5 \) into the sum of two three-dimensional indecomposables; hence \( L^5 \) must also be indecomposable over \( \mathbb{Z} \). The multiplicities of \( L^1, L^3, L^4, \) and \( L^5 \) in \( L^n \) can be calculated from what Theorem 5.2 tells us happens after reduction modulo 3, and the result may be put as follows:

**Theorem 7.1.** If \( n \neq 2 \), then in any unrefinable direct decomposition of \( L^n \) as \( \mathbb{Z}H \)-module, \( \kappa(n, 0, 2) \) summands are isomorphic to \( L^1 \), \( \kappa(n, 1, 2) \) to \( L^3 \), \( \kappa(n, 1, 3) - \kappa(n, 0, 3) \) to \( L^4 \), and \( \kappa(n, 0, 3) \) to \( L^5 \); no other isomorphism types occur among the summands.

Note that this is a stronger statement than if we had merely said that \( L^n \) is isomorphic to the direct sum formed from these indecomposables with the given multiplicities, because there is no Krull–Schmidt theorem for integral representations of \( H \).

If we now consider in detail what happens after reduction modulo 2, and compare it with Theorem 3.1, we are lead to the following conclusion.

**Theorem 7.2.** The multiplicities \( \kappa(n, i, j) \) defined in Theorem 5.2 may also be calculated as

\[
\kappa(n, 0, 3) = \frac{1}{2} \psi(n) + \frac{1}{2} \psi(n, 2) - b(n),
\]

\[
\kappa(n, 1, 3) = \frac{1}{2} \psi(n) - \frac{1}{2} \psi(n, 2) - b(n),
\]

\[
\kappa(n, 0, 2) + \kappa(n, 1, 2) = 3b(n) - \psi(n).
\]
where, as before,

\[
b(n) = \begin{cases} 
\frac{1}{3n}(2^n + 1) & \text{if } n \text{ is a power of } 3, \\
\frac{1}{3n}(2^n - 2^{n/2} - 2) & \text{if } n/2 \text{ is a power of } 3, \\
\frac{1}{3}\psi(n, 3') & \text{otherwise}.
\end{cases}
\]

This represents an improvement on the recursive determination of the \(\kappa(n, i, 3)\) in Theorem 5.2. For the \(\kappa(n, i, 2)\), one also obtains closed formulas now, by substituting the new expressions of the \(\kappa(n, i, 3)\) into those of the first half of Theorem 5.2. One may note that the \(n\) even case of Theorem 6.1 only involves \(\kappa(n, 0, 2)\) and \(\kappa(n, 1, 2)\) via \(\kappa(n, 0, 2) + \kappa(n, 1, 2)\), so there Theorem 7.2 will suffice.

**Remark 7.3.** The closed formulas for the \(\kappa(n, i, 2)\) will involve double sums, and it may be worth seeking simpler or alternative versions of them. For example, consider the case when neither \(n\) nor \(n/2\) is a power of 3. Then, by Theorems 5.2 and 7.2, we have that

\[
\kappa(3m, 0, 2) = \sum_{\beta \geq 0} \kappa(m/3^\beta, 1, 3)
\]

\[
= \frac{1}{2} \sum_{\beta \geq 0} \psi(m/3^\beta) - \frac{1}{2} \sum_{\beta \geq 0} \psi(m/3^\beta, 2) - \frac{1}{2} \sum_{\beta \geq 0} \psi(m/3^\beta, 3').
\]

The first sum on the last right-hand side may be recognized as the dimension of the restricted Lie power \(R^m\): 

\[
\sum_{\beta \geq 0} \psi(m/3^\beta) = \sum_{\beta \geq 0} \dim L^{m/3^\beta} = \dim R^m. \tag{7.1}
\]

This gives \(\dim R^m\) as a double sum (double, because \(\psi(m/3^\beta)\) was defined as a sum). On the other hand, on p. 209 of [25] Witt gave a formula, which may also be easily verified from (7.1), involving only a single summation,

\[
\dim R^m = \frac{1}{m} \sum_{d|m} \mu(d) \phi(d) 2^{m/d}, \tag{7.2}
\]

where \(\phi\) is Euler’s function, \(d_3\) is the highest 3-power divisor of \(d\), and \(d = d_3 d_1\) (so \(3 \nmid d_3\)). Treating the other terms similarly, one finds that if neither \(m\) nor \(m/2\) is a power of 3, then

\[
\kappa(3m, 0, 2) = \frac{1}{2m} \sum_{d|m, 2 \nmid d} \mu(d_3) \phi(d_3) 2^{m/d} - \frac{1}{5m} \sum_{d|m} \mu(d_2) d_2 2^{m/d}. \tag{7.3}
\]

It is fascinating that the dimension of \(R^m\), and variations thereof, come in here in such an interesting way.
Next, let us write $C$ for the centre of $GL(2, \mathbb{Z})$. Given that $C$ acts on each $L^n$ by scalars, we only have to “rectify parity” to obtain full information on the $L^n$ as $(C \times H)$-modules. Let $K$ be the $(C \times H)$-module which is additively an infinite cyclic group and on which $H$ acts trivially while the involution generating $C$ acts as $-1$.

**Corollary 7.4.** If the $L^n$ are viewed as $\mathbb{Z}(C \times H)$-modules, in the case of odd $n$ the indecomposable direct summands are $L_1$, $L_3$, $K \otimes L_4$, and $L_5$, while the indecomposable direct summands of the $L^n$ with even $n (> 2)$ are $K \otimes L_1$, $K \otimes L_3$, $L_4$, and $K \otimes L_5$. Their multiplicities are what Theorem 7.1 gives for the restrictions to $H$.

Once more, note that the multiplicities are unambiguous, in spite of the absence of a Krull–Schmidt theorem for integral representations of $C \times H$.

It is also possible to obtain conclusive results when the $L^n$ are viewed as modules for the dihedral subgroup $D$ of order 8 generated in $GL(2, \mathbb{Z})$ by $h$ and $t$, where $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Although there are infinitely many isomorphism types of indecomposable $\mathbb{Z}$-free $\mathbb{Z}D$-modules that we might have to worry about, it turns out that very few of them can be involved in the $L^n$.

To start with, note that $h$ acts on each $L^n_m$ as a scalar, namely as $(-1)^{n-m}$. For a fixed $n$, let $M$ denote the sum of the $L^n_m$ with $n - m$ even, and let $N$ denote the sum of those with $n - m$ odd. Then $L^n = M \oplus N$, and $h$ acts as 1 on $M$ and as $-1$ on $N$. It follows from this alone that every direct summand $S$ of the $\mathbb{Z}$-module $L^n$ which admits $h$ is the direct sum of its intersections with $M$ and $N$ (if $u \in M$, $v \in N$, and $u + v \in S$, then also $u - v \in S$, and so $2u, 2v \in S$, whence $u, v \in S$ also follows, because the quotient modulo $S$ is torsion-free), and those intersections are direct summands of $M$ and $N$, respectively.

When $n$ is odd, $t$ swaps $M$ and $N$. Let $S$ be any direct summand of $L^n$ as $\mathbb{Z}D$-module: then $t$ swaps $M \cap S$ and $N \cap S$. This shows that the $\mathbb{Z}D$-structure of $S$ is the same as that of a direct sum of copies of $L^1$. In particular, if $S$ is indecomposable then it must be isomorphic to $L^1$.

When $n$ is even, each of $M$ and $N$ admits $t$ as well. Let $S$ be an indecomposable direct summand of $L^n$ as $\mathbb{Z}D$-module: then $M \cap S$ and $N \cap S$ are also $\mathbb{Z}D$-submodules, so one of them must be 0, that is, $S$ must lie either in $M$ or in $N$. It follows that $S$ must be indecomposable even as $\langle t \rangle$-module. It cannot be trivial as $\langle t \rangle$-module, because of the corollary in [5], so $t$ must act on $S$ either regularly or as the scalar $-1$. We have proved that, when $n$ is even, there are only four choices for the isomorphism type of an indecomposable direct summand of $L^n$.

Keeping to even $n$, note that $t$ swaps the $L^n_m$ in pairs, except for $L^n_{n/2}$ which it maps to itself. Thus $t$ acts freely on $M$ when $n/2$ is odd, and
on $N$ when $n/2$ is even. It follows that the cyclic direct summands of any unrefinable direct decomposition of the $\mathbb{Z}D$-module $L^n$ must all lie in $N$ when $n/2$ is odd, and in $M$ when $n/2$ is even. After reduction modulo 2 and restriction to $\langle t \rangle$, these become the only one-dimensional direct summands of the relevant version of $L^n$, so we can see from Theorem 3.1 that the multiplicity in question must have been $-\psi(n, 2)$. Using the $\mathbb{Z}$-ranks of $M$ and $N$ as given in (2.9) and (2.10), the other multiplicities can then also be calculated, and we end up with the following result.

**Theorem 7.5.** Five isomorphism types of indecomposable $\mathbb{Z}D$-modules can appear in direct decompositions of the $L^n$. Two of these are $L^1$ and $L^2$. The third is additively a cyclic group, $t$ acts on it as $-1$, and $h$ acts as 1. The fourth and fifth are additively of rank 2, $t$ acts on both of them regularly, while $h$ acts as 1 on the fourth and as $-1$ on the fifth.

If $n$ is odd, $L^n$ is the direct sum of $\frac{1}{2}\psi(n)$ copies of $L_1$.

If $n$ is even but $n/2$ is odd, then $L^n$ is the direct sum of

$-\psi(n, 2)$ copies of $L^2$,

$\frac{1}{2}\psi(n) + \frac{1}{2}\psi(n, 2)$ copies of the fourth, and

$\frac{1}{2}\psi(n) + \frac{1}{2}\psi(n, 2)$ copies of the fifth of these indecomposables.

If $n$ is divisible by 4, then $L^n$ is the direct sum of

$-\psi(n, 2)$ copies of the third,

$\frac{1}{4}\psi(n) + \frac{3}{4}\psi(n, 2)$ copies of the fourth, and

$\frac{1}{4}\psi(n) - \frac{1}{4}\psi(n, 2)$ copies of the fifth.

There are no other ways of writing the $L^n$ as direct sums of indecomposable $\mathbb{Z}D$-modules.

It has been known for a very long time (see Brown et al. [4] or Newman [22]) that each finite subgroup of $GL(2, \mathbb{Z})$ is conjugate either to a subgroup of $C \times H$ or to a subgroup of $D$. Thus Corollary 7.4 and Theorem 7.5 are sufficient to yield full information about the $L^n$ as modules for any finite subgroup of $GL(2, \mathbb{Z})$.

8. **ON $GL(2, \mathbb{Z})$ WITH $p > 3$**

We have no closed formulas for $p > 3$, but we do have methods for computing multiplicities when $p \nmid n$. The first was proposed in the unpublished thesis [23] of Schooneveldt and established in an unpublished report [18] of the first author; a brief outline will be given below. The one we currently favour relies on more recent results of Donkin and Erdmann, and on results of James belonging to a different context; this will be presented in more detail.
Let us return for a moment to the full generality of the beginning of this paper, where $L$ was the free Lie algebra of rank $r$ over an arbitrary field $\mathbb{F}$, and consider an arbitrary extension $\mathbb{E}$ of $\mathbb{F}$. The Lie algebra $L \otimes_{\mathbb{F}} \mathbb{E}$ is a free Lie algebra over $\mathbb{E}$, and its homogeneous components are the $L^n \otimes_{\mathbb{F}} \mathbb{E}$. On the other hand, $GL(r, \mathbb{F})$ is a subgroup of $GL(r, \mathbb{E})$, and we can view $L^n \otimes_{\mathbb{F}} \mathbb{E}$ as a $GL(r, \mathbb{F})$-module in two ways: as the module obtained by extension of scalars from the Lie power $L^n$ of the natural module for $GL(r, \mathbb{F})$, and as that obtained by restricting to $GL(r, \mathbb{F})$ the Lie power of the natural module for $GL(r, \mathbb{E})$. It is very useful to recognize that the two $GL(r, \mathbb{F})$ actions so defined on $L^n \otimes_{\mathbb{F}} \mathbb{E}$ are the same. For example, given any element $g$ in $GL(r, \mathbb{F})$ whose order is finite and prime to the characteristic of $\mathbb{F}$, one can choose $\mathbb{E}$ large enough to contain all characteristic roots of $g$, then choose a basis for $L_1 \otimes_{\mathbb{F}} \mathbb{E}$ which consists of eigenvectors for $g$, and use that this basis is a free generating set of the free Lie algebra $L \otimes_{\mathbb{F}} \mathbb{E}$: each $L^n \otimes_{\mathbb{F}} \mathbb{E}$ has a basis consisting of Lie monomials in these new free generators, each such monomial is an eigenvector for $g$, and so the second of Witt’s dimension formulas (2.6) immediately yields the character formula (2.11).

Schooneveldt’s idea was to use this process for the whole of $GL(r, \mathbb{F})$, not just for one element at a time. As we already recalled in the proof of Lemma 4.3, if $p \nmid n$ then $L^n$ is a direct summand of the $n$th tensor power of the natural module. For simplicity of description, consider the first $n$ tensor powers of the natural module for $SL(2, p^k)$ (with $p^k > n$). Schooneveldt conjectured that each involves precisely one composition factor that has not occurred in lower tensor powers, and precisely one indecomposable direct summand that has not occurred as direct summand in any lower tensor power. To put it in another way, consider the $n \times n$ matrix whose $i, j$ entry is the Jordan–Holder multiplicity of the simple module which first occurred in the $i$th tensor power, in the indecomposable which first occurred in the $j$th: he conjectured that this matrix is upper triangular. Given this, if two direct sums of direct summands of these tensor powers afford the same Brauer character, they have the same Krull–Schmidt multiplicities, and the latter are readily calculated from the former, using the inverse of that upper triangular matrix. (In fact, he conjectured more, namely that the relevant matrix is upper unitriangular, so calculating its inverse should present no problem.) For understanding the $n$th Lie power of the natural module for $SL(2, p)$, this would leave only the task of determining the restrictions to $SL(2, p)$ of the relevant indecomposable $SL(2, p^k)$-modules, for some $p^k > n$. Similar (though slightly more complicated) statements would hold for $GL(2, p)$ instead of $SL(2, p)$.

His conjectures were subsequently proved in [18], where a description of the relevant indecomposable $GL(2, p^k)$-modules was also given (in terms of a twisted tensor product formula, from which the unitriangular matrix in
question could be read off), as well as an algorithm for computing restrictions to $GL(2, p)$. This established the first method.

For a description of our current method, let $\mathbb{F} = \mathbb{F}_p$, let $\mathbb{E}$ be the algebraic closure of $\mathbb{F}$, and consider the natural module for $GL(r, \mathbb{E})$. The indecomposable direct summands of tensor powers of this module were identified in Erdmann [10] as certain tilting modules, and their multiplicities were given as degrees of irreducible Brauer characters of symmetric groups. When $p \nmid n$, the indecomposable direct summands of the $n$th Lie power also come from among these tilting modules; the multiplicities were given in Theorem 3.3 of Erdmann [12] (which is also Theorem 3.3 in Donkin and Erdmann [9]), again in terms of Brauer characters of symmetric groups. In general, one does not even know the dimensions of the tilting modules, but for $r = 2$ they are well understood. Apart from the first few cases, this understanding comes from a twisted tensor product formula of Donkin [8] (which is re-stated in Erdmann [11] as 1.4(b) and matches the twisted tensor formula given for $SL(2, p^k)$ in [18]). It is very convenient for the present purposes that one of the tensor factors in this formula becomes projective when restricted to $SL(2, p)$. Hence almost all the tilting modules restrict to projective $GL(2, p)$-modules; it is easy to identify those that do not, and it turns out that the restrictions of those are simple modules. Thus the multiplicities of the nonprojective indecomposable direct summands in these Lie powers can be determined. As in the proof of Theorem 6.1, one can then write down the Brauer character of $GL(2, p)$ afforded by the largest projective direct summand of the Lie power, and use the modular orthogonality relations to determine the multiplicities of the projective indecomposables.

We need not say any more here about the last step, but we have to justify and expand the earlier ones. We know from Section 3 that if $p = 2$ and $p \nmid n$ then $L^n$ is projective, so this is a case we can ignore now. To avoid having to point out degenerate exceptions, in the sequel we assume that $p > 2$. Let $D^i$ denote the one-dimensional module on which each matrix acts like the $i$th power of its determinant, and let $S^j$ denote the $j$th symmetric power of the natural module—let us not complicate notation by indicating the changing field or matrix group in question. For $r = 2$, the isomorphism types of the tilting modules are indexed by partitions $(n - i, i)$ of $n$ with at most two parts: in what follows, we adopt the notation of [11, Sect. 1] and write them as $T(n - i, i)$.

**Lemma 8.1.** The restriction of a tilting module $T(n - i, i)$ to $GL(2, p)$ is nonprojective if and only if $0 \leq n - 2i \leq p - 2$, and then the restriction is isomorphic to $D^i \otimes S^{n-2i}$.

**Proof.** As one can see from Glover [13, particularly (6.2), (5.2), and (3.3)], the nonprojective indecomposable direct summands of the $n$th tensor
power of the natural module for $GL(2, p)$ are precisely the $D^i \otimes S^{n-2i}$ with $0 \leq n - 2i \leq p - 2$. Note that as the index $(GL(2, p) : SL(2, p))$ is prime to $p$, the restriction of $T(n - i, i)$ to $GL(2, p)$ is projective if and only if its restriction to $SL(2, p)$ is projective. In Erdmann [11], the restriction of $T(n - i, i)$ to $SL(2, p)$ is written as $T(n - 2i)$. Comparing Lemma 1.5 of [11] with (5.2) of [13], we see from (3.3) of [13] that the restriction of Erdmann’s $T(p - 1)$ to $SL(2, p)$ is projective. Therefore, by her Lemma 1.5, so is the restriction of every $T(n - 2i)$ with $n - 2i \geq p - 1$. The comparison of [11, Lemma 1.5] with [13, (5.2)] also shows that if $n - 2i \leq p - 1$ then the restriction of $T(n - 2i)$ to $SL(2, p)$ is the $(n - 2i)$th symmetric power of the natural module. The only direct summand of the $n$th tensor power of the natural $GL(2, p)$-module with this restriction to $SL(2, p)$ is $D^i \otimes S^{n-2i}$, so if $0 \leq n - 2i \leq p - 1$ then the restriction of $T(n - i, i)$ to $GL(2, p)$ must be $D^i \otimes S^{n-2i}$.

Let $\beta^{(n-i,i)}$ denote the irreducible $p$-Brauer character of the symmetric group $S_n$ indexed by the partition $(n - i, i)$. By Theorem 3.3 of [12] and by [9], we may now conclude the following.

**Lemma 8.2.** If $p \nmid n$, then each nonprojective indecomposable direct summand of $L^n$ is isomorphic to a $D^i \otimes S^{n-2i}$ with $0 \leq n - 2i \leq p - 2$, and the Krull–Schmidt multiplicity of such a $D^i \otimes S^{n-2i}$ in $L^n$ is

$$\frac{1}{n} \sum_{d|n} \mu(d) \beta^{(n-i,i)}(\sigma^n/d),$$

where $\sigma$ is any cyclic permutation of length $n$.

Given that even the values $\beta^{(n-i,i)}(1)$ have only recently been made explicit (in Erdmann [11]), one may question whether this formula can be used to obtain numerical answers when $n$ is large. We are grateful to Karin Erdmann for suggesting that we should use the result of James [16] on decomposition numbers.

Recalling the assumption that $p > 2$, consider the ordinary irreducible characters $\chi^{(n-j,i)}$ of $S_n$ (with $0 \leq j \leq n/2$). It was shown in [16] that the restriction of such a character to the set of $p'$-elements of $S_n$ is a sum

$$\sum_{d|n/2} d_i^{(n)} \beta^{(n-i,i)},$$

where the decomposition numbers $d_i^{(n)}$ are nonnegative integers which may be calculated by a process that we shall describe later. Importantly, as we shall see in (8.2), the matrix $d_i^{(n)}$ they form (for a fixed $n$, with rows and columns indexed by $i$ and $j$ in the range from 0 to $\lfloor n/2 \rfloor$) is unitriangular. Denote the inverse of this matrix by $c_i^{(n)}$. Then $\beta^{(n-i,i)}$ can be thought of as the restriction of $\sum_{j=0}^{n} c_i^{(n)} \chi^{(n-j,i)}$, so

$$\frac{1}{n} \sum_{d|n} \mu(d) \beta^{(n-i,i)}(\sigma^n/d) = \frac{1}{n} \sum_{d|n} \sum_{j} \mu(d) c_i^{(n)} \chi^{(n-j,i)}(\sigma^n/d).$$
We shall prove below that
\[
\frac{1}{n} \sum_{d|n} \mu(d) \chi^{(n-j,i)}(a^{n/d}) = \dim L^n_j - \dim L^n_{j-1}.
\] (8.1)

In these terms, we may re-state Lemma 8.2 as follows.

**Theorem 8.4.** If \( p > 2 \) and \( p \nmid n \), then each nonprojective indecomposable direct summand of \( L^n \) is isomorphic to a \( D^i \otimes S^{n-2i} \) with \( 0 \leq n - 2i \leq p - 2 \), and the Krull–Schmidt multiplicity of such a \( D^i \otimes S^{n-2i} \) in \( L^n \) is
\[
\sum_{j=1}^{[n/2]} c_{ij}^{(n)} (\dim L^n_j - \dim L^n_{j-1}).
\]

In view of Witt’s formulas (2.6), once the \( c_{ij}^{(n)} \) are available, the practicality of this version cannot be in doubt.

**Proof of (8.1).** So far, we have always thought in terms of a fixed \( p \) while \( n \) ranged over the positive integers. As (8.1) is completely independent of \( p \), in this proof we may fix \( n \) instead and choose \( p \) as we please: assume that \( p > n \). Then all elements of \( S_n \) are \( p \)-elements, \( c_{ij}^{(n)} = d_{ij}^{(n)} = \delta_{ij} \) (the Kronecker delta), and \( \beta^{(n-i,j)} = \chi^{(n-i,j)} \). One can readily see that a submodule of \( L^n \) isomorphic to \( D^i \otimes S^{n-2i} \) intersects \( L^n_m \) in a one-dimensional space when \( j \leq m \leq n - j \) and avoids it otherwise. It follows that the multiplicity of \( D^i \otimes S^{n-2i} \) in \( L^n \) is \( \dim L^n_j - \dim L^n_{j-1} \); on the other hand, by Lemma 8.2 and what we have just seen, this multiplicity is \( \frac{1}{n} \sum_{d|n} \mu(d) \chi^{(n-j,i)}(a^{n/d}) \). This proves (8.1).

To see how easy it is to access the \( c_{ij}^{(n)} \), we conclude this discussion with a paraphrase of the recipe of James [16] for calculating the decomposition numbers \( d_{ij}^{(n)} \). Say that \( a \) contains \( b \) to base \( p \) if \( a \) and \( b \) are nonnegative integers such that \( \lceil \log_p a \rceil > \lceil \log_p b \rceil \) and, for each nonzero digit of \( b \) in base \( p \) arithmetic, the corresponding digit of \( a \) is the same. (Thus if \( a \) has \( k + 1 \) nonzero digits when written to base \( p \), then there exist precisely \( 2^k \) numbers \( b \) that are contained in \( a \) to base \( p \), namely the numbers obtained from \( a \) by omitting the leading digit and changing some (or all or none) of the other \( k \) nonzero digits of \( a \) to zero.) In these terms, the rule may be put quite simply:
\[
d_{ij}^{(n)} = \begin{cases} 
1 & \text{if } n - 2i + 1 \text{ contains } j - i \text{ to base } p, \\
0 & \text{otherwise}.
\end{cases}
\] (8.2)

It is a remarkable consequence of this rule that deleting the first row and first column of the matrix \( d^{(n)} \) yields the matrix \( d^{(n-2)} \). As \( d^{(n)} \) is unitriangular, it follows also that deleting the first row and first column of \( c^{(n)} \).
yields $c^{(n-2)}$. Conversely, if we construct the $d^{(n)}$ and $c^{(n)}$ for $n = 1, 2, \ldots$ in turn, at each new value of $n$ we need to add only one new row and one new column to the $d^{(n-2)}$ and $c^{(n-2)}$ that we already have.

The $GL(2, p)$-module structure of the $L^n$ with $p > 3$ and $p | n$ remains an entirely open problem. Small examples show that obvious analogues of the results and methods discussed here will not work.

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