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Module structure of the free Lie ring on three generators

By

L. G. Kovács and Ralph Stöhr

Abstract. Let L^n denote the homogeneous component of degree n in the free Lie ring on three generators, viewed as a module for the symmetric group S_3 of all permutations of those generators. This paper gives a Krull-Schmidt Theorem for the L^n: if $n > 1$ and L^n is written as a direct sum of indecomposable submodules, then the summands come from four isomorphism classes, and explicit formulas for the number of summands from each isomorphism class show that these multiplicities are independent of the decomposition chosen.

A similar result for the free Lie ring on two generators was implicit in a recent paper of R. M. Bryant and the second author. That work, and its continuation on free Lie algebras of prime rank p over fields of characteristic p, provide the critical tools here. The proof also makes use of the identification of the isomorphism types of $\mathbb{Z}$-free indecomposable $\mathbb{Z}S_3$-modules due to M. P. Lee. (There are, in all, ten such isomorphism types, and in general there is no Krull-Schmidt Theorem for their direct sums.)

1. Introduction and statement of result. R. M. Bryant and the second author proved (as part (ii) of the corollary in [2]) that the free Lie ring on two generators, viewed as a module for the symmetric group S_2 which permutes the two free generators, has no nonzero direct summand on which this group acts trivially. More precisely, it can be seen from Theorem 1 of [2] and from the formula displayed on p. 284 of [2] that, as $\mathbb{Z}S_2$-module, the nth homogeneous component of this Lie ring is the direct sum of

$$\frac{1}{2n}\sum_{d|n}\mu(d)2^{n/d} + \frac{1}{2n}\sum_{d|n,\ 2|d}\mu(d)2^{n/d}$$

copies of the first homogeneous component and

$$-\frac{1}{n}\sum_{d|n,\ 2|d}\mu(d)2^{n/d}$$

copies of the second. (These sums are taken over all, or over all even, divisors d of n; if a range of summation is empty then the sum is read as 0; and μ is the Möbius function.)

The aim of this note is to provide a similar result for the free Lie ring L on three generators, viewed as a module for the group S_3 of all permutations of those generators. The

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integral representation theory of S_3 is nothing like as trivial as that of S_2, but it is well understood from the work of Lee [6]. We show here that there is a Krull-Schmidt Theorem for the homogeneous components L^n of L: if $n > 1$ and L^n is written as a direct sum of indecomposable submodules, then these submodules come from four isomorphism classes, and explicit formulas (that are independent of the decomposition) can be given for the number of summands from each isomorphism class. (For comparison, we recall that there are, in all, ten isomorphism types of $\mathbb{Z}$-free indecomposable $\mathbb{Z}S_3$-modules, and different direct sums formed from them can be isomorphic.)

The first homogeneous component L^1 is, of course, indecomposable, but it is not isomorphic to a direct summand of any other L^n: this is the reason for the exclusion $n > 1$ which will apply throughout. One of the four relevant indecomposables is L^2, and another is the regular module $\mathbb{Z}S_3$. The last two are the unique submodule U of $\mathbb{Z}$-rank 2 in L^1 and the unique factormodule V of L^1 with $\mathbb{Z}$-rank 2. (These are unique on the usual understanding that in the context of integral representations we consider only $\mathbb{Z}$-free modules and restrict attention to submodules that yield $\mathbb{Z}$-free quotients.)

Some more *ad hoc* notation will make it easier to state the full result. If X is a module and Y is an indecomposable module such that, in any unrefinable direct decomposition of X, the number of summands isomorphic to Y is independent of the decomposition, we call that number the (Krull-Schmidt) multiplicity of Y in X and write it as $[X \div Y]$. Let $\delta(n)$ be 1 if n is a power of 2 and 0 otherwise.

Theorem. *If $n > 1$ and L^n is written as a direct sum of indecomposable submodules, then each summand is isomorphic to one of L^2, $\mathbb{Z}S_3$, U and V. Moreover,*

$$\begin{aligned}
[L^n \div L^2] &= -\frac{1}{n}\sum_{d|n,\,2|d} \mu(d)3^{n/d} - \frac{\delta(n)}{n},\\
[L^n \div \mathbb{Z}S_3] &= \frac{1}{6n}\sum_{d|n} \mu(d)3^{n/d} + \frac{1}{3n}\sum_{d|n,\,3|d} \mu(d)3^{n/d} - \frac{[L^n \div L^2]}{2},\\
[L^n \div U] &= \sum_{d} [L^{n/d} \div L^2]
\end{aligned}$$

with the last sum being over the divisors d of n that are powers of 3 different from 1, *while*

$$[L^n \div V] = -\frac{1}{n}\sum_{d|n,\,3|d} \mu(d)3^{n/d} - [L^n \div U].$$

We see no prospect of any similar result for free Lie rings on more than three generators.

2. Proof of the Theorem. Let $\{x, y, z\}$ be a free generating set of L. Let a be any permutation of order 3 in S_3, and b the transposition in S_3 which fixes x and swaps y with z. Denote by $\langle a\rangle$ and $\langle b\rangle$ the subgroups of S_3 generated by a and by b, respectively.

Step 1 is that L^n as $\mathbb{Z}\langle b\rangle$-module has no nonzero direct summand on which b acts trivially. To see this, consider the subset $\mathscr{F}$ of L consisting of all left-normed Lie monomials of the form $[[\dots[y,x],\dots],x]$ and $[[\dots[z,x],\dots],x]$. Note that $\mathscr{F}$ is permuted by b, and that the action of b on $\mathscr{F}$ is free. Let J be the Lie subalgebra of L generated by $\mathscr{F}$. It is well known in the context of Lazard elimination (see for example Theorem 0.6 in Reutenauer [7]) that $\mathscr{F}$ is a free generating set of J, and that J is an ideal in L with quotient of $\mathbb{Z}$-rank 1. The first homogeneous component J^1 of J (in its own grading as free Lie algebra on $\mathscr{F}$) is then a free $\mathbb{Z}\langle b\rangle$-module of infinite rank, and $J = (L^1 \cap J^1) \oplus \bigoplus_{m \geqq 2} L^m$. If J has a nonzero $\mathbb{Z}\langle b\rangle$-module direct summand on which b acts trivially, then it also has such a direct summand with $\mathbb{Z}$-rank 1; this small direct summand lies in the Lie subring generated by some finitely generated $\mathbb{Z}\langle b\rangle$-module direct summand of J^1, and is of course a direct summand in that subring as well. This possibility is ruled out by the first statement of Corollary 2 in [3] read with $p = 2$ (which for this case is a direct extension of part (ii) of the Corollary of [2]), so the proof of step 1 is complete.

Step 2 is that $L^n/3L^n$ as $(\mathbb{Z}/3\mathbb{Z})\langle a\rangle$-module has no 1-dimensional direct summand. This follows from Theorem 1 of [3] read with $p = 3$.

Step 3 is to inspect the list of $\mathbb{Z}$-free indecomposable $\mathbb{Z}S_3$-modules in Lee [6], or on p. 752 of Curtis and Reiner [4], and check that all but four of them are ruled out (as in Section 7 of [5]) by these tests. The remaining four isomorphism types of indecomposables are those described above. It is also easy to see that for any finite direct sum of indecomposables from these four isomorphism classes, the multiplicities can be recovered from the modules obtained on tensoring with $\mathbb{C}$ and from tensoring with $\mathbb{Z}/3Z$. Since in those contexts there is a Krull-Schmidt Theorem, it follows that for such sums there is one also over $\mathbb{Z}$. The details of this step are left to the reader.

Step 4 is to prove the third multiplicity formula of the theorem. Tensoring L with $\mathbb{Z}/3\mathbb{Z}$, one obtains a free Lie algebra to which one can apply the results of [3] read with $p = 3$. Three of our indecomposable $\mathbb{Z}S_3$-modules remain indecomposable and pairwise non-isomorphic, while the regular module becomes the direct sum of two nonisomorphic indecomposables modules, namely of $L^1/3L^1$ and $L^2/3L^2$. This proves that $[L^n/3L^n \div U/3U] = [L^n \div U]$ and $[L^n/3L^n \div L^1/3L^1] = [L^n \div \mathbb{Z}S_3]$. The consequence that we need from the rather technical central results of [3] was given as Theorem 5.3 in [5]; it yields that $[L^n/3L^n \div U/3U]$ is the sum of the $[L^{n/d}/3L^{n/d} \div L^1/3L^1]$ where d ranges over the divisors d of n that are powers of 3 different from 1. Thus $[L^n \div U]$ is the sum of the $[L^{n/d} \div \mathbb{Z}S_3]$, and the third formula of the theorem is proved.

Step 5 is to work out what can be said about the multiplicities we seek by using the character λ_n of S_3 afforded by $L^n \otimes \mathbb{C}$. We shall take 1, a and b as representatives of the conjugacy classes of elements in S_3. The orthogonality relations give that in $L^n \otimes \mathbb{C}$ the multiplicity of the trivial 1-dimensional $\mathbb{C}S_3$-module is $\frac{1}{6}[\lambda_n(1) + 2\lambda_n(a) + 3\lambda_n(b)]$, the multiplicity of the nontrivial 1-dimensional $\mathbb{C}S_3$-module is $\frac{1}{6}[\lambda_n(1) + 2\lambda_n(a) - 3\lambda_n(b)]$, and the the multiplicity of the 2-dimensional simple $\mathbb{C}S_3$-module is $\frac{1}{6}[2\lambda_n(1) - 2\lambda_n(a)]$. On the other hand, $L^2 \otimes \mathbb{C}$ is the sum of a nontrivial 1-dimensional and a 2-dimensional simple, $\mathbb{C}S_3$ is the sum of the two different 1-dimensionals and two copies of the 2-dimensional simple, while both $U \otimes \mathbb{C}$ and $V \otimes \mathbb{C}$ are isomorphic to the 2-dimensional simple module. It follows that

$$
\begin{aligned}
&[L^n \div \mathbb{Z}S_3] = \tfrac{1}{6}[\lambda_n(1) + 2\lambda_n(a) + 3\lambda_n(b)],\\
&[L^n \div L^2] + [L^n \div \mathbb{Z}S_3] = \tfrac{1}{6}[\lambda_n(1) + 2\lambda_n(a) - 3\lambda_n(b)],\\
&[L^n \div L^2] + 2[L^n \div \mathbb{Z}S_3] + [L^n \div U] + [L^n \div V] = \tfrac{1}{6}[2\lambda_n(1) - 2\lambda_n(a)],
\end{aligned}
$$

whence

$$[L^n \div L^2] = -\lambda_n(b),$$
$$[L^n \div \mathbb{Z}S_3] = \tfrac{1}{6}\lambda_n(1) + \tfrac{1}{3}\lambda_n(a) - \tfrac{1}{2}[L^n \div L^2],$$
$$[L^n \div V] = -\lambda_n(a) - [L^n \div U].$$

The final step 6 is to calculate the values of λ_n from the permutation character λ_1 afforded by $L^1 \otimes \mathbb{C}$. For $g = 1, a, b$, the character formula of Brandt [1] says that

$$\lambda_n(g) = \frac{1}{n}\sum_{d|n}\mu(d)\lambda_1(g^d)^{n/d}.$$

As $\lambda_1(1) = 3$, $\lambda_1(a) = 0$, $\lambda_1(b) = 1$, this comes to

$$\lambda_n(1) = \frac{1}{n}\sum_{d|n}\mu(d)3^{n/d},$$
$$\lambda_n(a) = \frac{1}{n}\sum_{d|n,\,3|d}\mu(d)3^{n/d},$$
$$\lambda_n(b) = \frac{1}{n}\sum_{d|n,\,2|d}\mu(d)3^{n/d} + \frac{\delta(n)}{n},$$

provided we use the elementary fact that

$$\sum_{d|n,\,2\nmid d}\mu(d) = \delta(n).$$

This completes the proof of the theorem. $\square$

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References

[1] ANGELINE J. BRANDT, The free Lie ring and Lie representations of the full linear group. Trans. Amer. Math. Soc. **56,** 528–536 (1944).
[2] R. M. BRYANT and R. STÖHR, Fixed points of automorphisms of free Lie algebras. Arch. Math. **67,** 281–289 (1996).
[3] R. M. BRYANT and RALPH STÖHR, On the module structure of free Lie algebras. Trans. Amer. Math. Soc., to appear.
[4] CHARLES W. CURTIS and IRVING REINER, Methods of representation theory, Vol I. New York 1981.
[5] L. G. KOVÁCS and RALPH STÖHR, Lie powers of the natural module for GL(2). J. Algebra (to appear).
[6] MYRNA PIKE LEE, Integral representations of dihedral groups of order $2p$. Trans. Amer. Math. Soc. **110,** 213–231 (1964).
[7] CRISTOPHE REUTENAUER, Free Lie algebras. Oxford 1993.

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Anschriften der Autoren:

L. G. Kovács
Australian National University
Canberra ACT 0200
Australia

Ralph Stöhr
UMIST, PO Box 88
Manchester M60 1QD
England