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A Problem of Grätzer and Wehrung on Groups

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Abstract. Given a group G, is there a group $\overline{G}$ which contains G as a proper subgroup and is such that every nontrivial normal subgroup N of G can be uniquely represented in the form $\overline{N} \cap G$ with $\overline{N}$ a normal subgroup of $\overline{G}$? The answer is negative for elementary abelian G of prime-square order, but is positive for all G if one allows two such representations when $N = 1$.

Keywords: normal subgroup lattices

In response to a question raised in [2], a forthcoming paper [3] by Grätzer and Wehrung proves that every lattice L with more than one element has a proper congruence-preserving extension K (that is, a proper extension K such that every congruence of L has exactly one extension to K). The latter paper raises the analogous question about groups in the following form.

"Let us say that a variety **V** of groups has the *Normal Subgroup Preserving Extension Property* (NSPEP, for short), if every group G in **V** with more than one element has a proper supergroup $\overline{G}$ in **V** with the following property: every normal subgroup N in G can be uniquely represented in the form $\overline{N} \cap G$, where $\overline{N}$ is a normal subgroup of $\overline{G}$.

"Not every group variety **V** has NSPEP, for instance, the variety **A** of Abelian groups does not have NSPEP.

"Problem. Does the variety **G** of all groups have NSPEP? Find all group varieties having NSPEP?"

We show here that the answer is: *no nontrivial variety of groups has NSPEP.* The reason lies in the simple fact that if the normal subgroup lattice of a group H is isomorphic to that of an elementary abelian group of prime-square order, then H is isomorphic to that group. (Indeed, if A, B, C are any three proper nontrivial normal subgroups of H, then C is disjoint from, and is therefore centralized by, both A and B. The product AB being H, it follows that C is central. For the same reason, so are A and B; therefore, H is abelian. The rest of the proof is obvious. For a more general result, see Curzio [1] or [4, Theorem 9.1.11].) If **V** is a nontrivial variety of groups, then it contains some elementary abelian group G of prime-square order; a matching $\overline{G}$ would have to be isomorphic to G, so G could not be a proper subgroup in it.

However, the variety of all groups has what one might call the *Almost Normal Subgroup Preserving Extension Property* (ANSPEP), namely the following. *Given any group G, there is a group $\overline{G}$ which contains G as a proper subgroup and is such that every nontrivial normal subgroup N of G can be uniquely represented in the form $\overline{N} \cap G$ with $\overline{N}$ a normal subgroup of $\overline{G}$, while the trivial normal subgroup of G has precisely two such representations.* To see this, take any nonabelian simple group S and let $\overline{G}$ be the (restricted standard) wreath product S wr G. Then $\overline{G}$ is the semidirect product of a top group and a normal subgroup called the base group; the latter is the direct product of a conjugacy class of subgroups known as the coordinate subgroups, and it is precisely the normalizer of each of these coordinate subgroups. The top group is isomorphic to G and we shall simply identify it with G. Denote the base group by B and note that the coordinate subgroups are isomorphic to S. The key point for now is that *each nontrivial normal subgroup K of $\overline{G}$ contains B.* (Proof. If K met B trivially, it would centralize it, but elements outside B cannot even normalize a coordinate subgroup. If $1 \neq b \in B \cap K$, write b as a product of elements from the coordinate subgroups, according to the direct decomposition of B, and choose a coordinate subgroup C which is nontrivially involved in this expression of b. Since the centre of S is trivial, there is an element c in C that does not commute with the C-component of b, and then the commutator $[b, c]$ is a nontrivial element of $K \cap C$. Since S is simple, we must have $K \cap C = C$. As B is the product of the conjugates of C, in fact $K \geq B$.) Thus, if N is a nontrivial normal subgroup of G, the one and only one $\overline{N}$ such that $\overline{N} \cap G = N$ is given by $\overline{N} = NB$, while if $N = 1$ then $\overline{N}$ can (and must) be taken as 1 or B.

This construction cannot be used to show that any other variety of groups has ANSPEP. Indeed, it seems unlikely that any other variety would have this property. On the other hand, both definitions make sense not only for varieties of groups, but also for other classes of groups. It is easy to see that the class of simple groups has NSPEP, as every simple group can be embedded in a strictly larger simple group. The class of finite simple groups also has NSPEP, but the class of all finite groups has only ANSPEP. The above argument also shows that the class of finitely generated groups has ANSPEP.

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