Finite Groups with Trivial Multiplicator and Large Deficiency

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Abstract. Generalizing examples of Swan and Wiegold, this note shows how to construct more finite groups (including some perfect groups) whose Schur multiplicator is trivial but whose abelianized deficiency is arbitrarily large.

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1. Introduction

Given a finite presentation for a group in terms of generators and defining relations, one obtains the deficiency of that presentation by subtracting the number of generators from the number of relations. The deficiency of a finitely presentable group G is the minimum of the deficiencies of its (finite) presentations; we write it as $\operatorname{def} G$. If G is finite, then $\operatorname{def} G \geqslant 0$.

B. H. Neumann [11] asked whether the deficiency of a finite group has to be zero whenever the Schur multiplicator of the group is trivial. The first examples to show that the answer is negative were made by Swan [12]: he gave an infinite set of finite groups whose multiplicators are trivial but whose deficiencies admit no upper bound. Later Wiegold [15] produced a different construction to the same end (and, as was reported in [15], Neumann immediately added a slight modification to reduce the number of generators).

Our aim here is to generalize both constructions. This will yield new examples which can be tailored to various purposes. In particular, it will be seen that there exist finite perfect groups with trivial multiplicator but arbitrarily large deficiency. As is the nature of generalizations, the process will direct our attention to various features of the two constructions, separating similarities from differences.

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When a finite group G is written as F/R with F free of finite rank, conjugation in F yields a G-module structure on R/R'. The number obtained by subtracting the rank of F from the minimum of the cardinalities of the G-module generating sets of R/R' has been called the abelianized deficiency of G (p. 149 in Wiegold [15]). In symbols, we write this as

$$\mathsf{abdef}\, G = d_G(R/R') - d(F).$$

(It was proved by Gruenberg in [7], and again as part (ii) of Corollary 7.9 in [8], that $d_G(R/R') - d(F)$ does not depend on the choices involved in writing G as F/R, so in this context we do not have to take the minimum of all possibilities.) As $\mathsf{def}\, G = d_F(R) - d(F)$ for some choice of F/R and of course $d_F(R) \geqslant d_G(R/R')$, it follows that $\mathsf{def}\, G \geqslant \mathsf{abdef}\, G$. It is easy to adapt the arguments of Swan [12] and Wiegold [15] to show that in their examples there is no upper bound on the abelianized deficiencies either. The same will be true of the generalizations.

We shall write the Schur multiplicator of a finite group G as $\mathsf{M}(G)$. The connection between deficiency and the Schur multiplicator may be viewed as follows. In the situation considered above, the quotient $R/[R, F]$ is the direct sum of $\mathsf{M}(G)$ with a free abelian group of rank $d(F)$. On the other hand, $R/[R, F]$ is a G-homomorphic image of R/R', and the action of G on $R/[R, F]$ is trivial, so $d_G(R/R') \geqslant d(R/[R, F])$. Thus

$$\mathsf{def}\, G \geqslant \mathsf{abdef}\, G \geqslant d(\mathsf{M}(G)).$$

(In different notation, this was (3) in Wamsley's survey [14]).

Before the fundamental paper [12] of Swan and the developments that flowed from it, it was not known that $d_G(R/R') - d(F)$ is an invariant of G, and in any case there was no method for calculating this number, so the only lower estimate of $\mathsf{def}\, G$ came from the inequality $\mathsf{def}\, G \geqslant d(\mathsf{M}(G))$. A group G is called efficient if $\mathsf{def}\, G = d(\mathsf{M}(G))$ (beware: for infinite G, the definition is different). In the light of the present discussion, G cannot be efficient unless the minimum number of generators of R/R' can be read off the largest G-trivial quotient $R/[R, F]$ of this module. In general, modules with this property are fairly rare, and so one can expect that inefficient groups are far more common than efficient ones. It is not the examples of Swan and Wiegold, and the generalizations to be presented here, that are the rare exceptions, but the groups that have been found to be efficient.

As usual in such pseudo-statistical speculations, other views are also quite plausible. While modules with this property are rare in general, they may not be so rare among the modules which arise as R/R', and they may be even less rare among the R/R' that arise with $\mathsf{M}(F/R) = 0$. Or: rarity is determined by the examples known at any one time, and by the skills available to the beholder. I, for one, would not know where to look for an efficient finite group that is not already in the literature, but would be happy to attempt to 'make to measure' any number of further examples with $\mathsf{abdef}\, G > d(\mathsf{M}(G))$.

One cannot close such a discussion without mentioning that the existence of a finite G with $\operatorname{def} G > \operatorname{abdef} G$ is still an open problem and looks as intractable as ever. (This was Question 6 in Wamsley's survey [14].) Recall that Gruenberg proved that if a finite group G is written as F/R and if T is a normal subgroup of F contained in R with R/T finite or soluble, then $d_F(R/T) \leqslant d_G(R/R')$ (Propositions 2.8 and 7.10 in [8]; note that in the second line of p. 47 'finite soluable' should be replaced by 'soluble'). Thus in attempting to prove $\operatorname{def} G > \operatorname{abdef} G$ for some particular G, it is a waste of time to investigate the F/T with R/T finite or soluble.

2. The Constructions

Both of the constructions that we generalize are based on a nontrivial module of order 7 for the group of order 3. All the examples are semidirect products $Q \ltimes N$ where Q is the group of order 3 (or, in the Wiegold–Neumann case, the cyclic group of order 6) and N ranges through certain groups made from copies of the module. Swan's N are simply the direct sums of copies of the module. Wiegold first takes a free product of two copies of the module, and his N are the quotients of that free product over suitable terms of the lower central series.

Perhaps the most important common property of the two constructions is that the orders of the group and module are coprime, so the orders of the semidirect factors will also be coprime. The action of Q on N (used in defining $Q \ltimes N$) yields an action of Q on $\mathsf{M}(N)$ (see the next section). Let $\mathsf{M}(N)^Q$ denote the subgroup of of $\mathsf{M}(N)$ consisting of the elements fixed by Q. The key to ensuring that $\mathsf{M}(Q \ltimes N) = 0$ is the following.

Lemma 1. *If the orders of Q and N are coprime, then*

$$\mathsf{M}(Q \ltimes N) = \mathsf{M}(Q) \oplus \mathsf{M}(N)^Q.$$

The relevant special case of this lemma is explicitly mentioned in Swan [12] and justified by reference to the spectral sequence of the extension $N \to Q \ltimes N \to Q$. In its present generality, it can be read off as a special case from Theorem 2 of Tahara [13], who did not use spectral sequences. A proof without cohomology will be given in the next section. The reader who prefers the spectral sequence proof may still find some interest in the implicit group theoretic interpretation provided by our argument for some ingredients of the spectral sequence. (Wiegold chose to avoid this lemma in favour of a simple and direct but somewhat *ad hoc* argument.)

Both Swan and Wiegold use a simple consequence of the Reidemeister–Schreier Theorem, namely that if H is a subgroup of a group G then

$$1 + \operatorname{def} G \geqslant (1 + \operatorname{def} H)\big/|G : H|,$$

to ensure that the groups G they construct have large deficiency. The lower estimate they use for def H is $d(\mathsf{M}(H))$, so in effect they are using that

$$1+\operatorname{def} G \geqslant (1+d(\mathsf{M}(H)))/|G:H|.$$

One can equally easily deduce from the foregoing that

$$1+\operatorname{abdef} G \geqslant (1+\operatorname{abdef} H)/|G:H| \geqslant (1+d(\mathsf{M}(H)))/|G:H|,$$

so in fact both constructions yield groups of arbitrarily large *abelianized* deficiency. We shall also rely on this argument.

In the generalizations we are about to describe, the task is to choose first a group Q and then infinitely many groups N on which Q acts, in such a way that $(|N|,|Q|)=1$, $\mathsf{M}(N)^Q=0$, and the set of the numbers $d(\mathsf{M}(N))$ is not bounded above. By the discussion so far, we know that we shall have $\mathsf{M}(Q \ltimes N)=\mathsf{M}(Q)$ with the set of the numbers $\operatorname{abdef}(Q \ltimes N)$ unbounded. In particular, if also $\mathsf{M}(Q)=0$, then our semidirect products will have trivial multiplicator and unbounded abelianized deficiency.

For the generalization of Swan's construction, the critical property of Q is to have an element that is not conjugate to its inverse. (Thus for this purpose the group of order 2 and $SL(2,5)$ are bad, but all larger finite cyclic groups and $SL(2,7)$ are good.) Such a Q always has an irreducible complex character, χ say, that is different from its complex conjugate $\overline{\chi}$ (see V.13.7a in Huppert [10]). Then χ^2 does not involve the trivial character (because the scalar product of χ^2 with the trivial character is also the scalar product of χ with $\overline{\chi}$). Equivalently, for the simple module U that affords χ we have $(U \otimes U)^Q=0$. Next, choose any prime p congruent to 1 mod $|Q|$. Over the field of p elements, which we write as $\mathbb{F}_p$, the representation theory of Q is the same as over the complex field, so there is an simple $\mathbb{F}_pQ$-module V such that $(V \otimes V)^Q=0$. For each positive integer k, let N_k be the direct sum of k copies of V. Of course then also $(N_k \otimes N_k)^Q=0$, so we can appeal to the completely elementary V.25.4 in [10] for the conclusion that the Sylow p-subgroup of $\mathsf{M}(Q \ltimes N_k)$ is trivial: in view of Lemma 1, this yields that $\mathsf{M}(N_k)^Q=0$ and $\mathsf{M}(Q \ltimes N_k)=\mathsf{M}(Q)$. Alternatively, we can follow (p. 186 of) Beyl and Tappe [1] and deduce $\mathsf{M}(N_k)^Q=0$ from $\mathsf{M}(N_k) \cong N_k \wedge N_k \leqslant N_k \otimes N_k$. Since N_k is elementary abelian and $d(N_k)=k \dim V \geqslant k$, we know that $d(\mathsf{M}(N_k)) \geqslant \frac{1}{2}k(k-1)$, so the $d(\mathsf{M}(N_k))$ have no upper bound.

It may be worth noting here that the way Neumann made 2-generator examples from Wiegold's 3-generator groups can also be used to make 2-generator examples from Swan's many-generator groups. Choose a group Q with $\mathsf{M}(Q)=0$ and a module V of characteristic p, as above. For each positive integer k that is prime to $p|Q|$, set $Q_k=C_k \times Q$ (where C_k is a cyclic group of order k) and let N_k be the Q_k-module induced from V (so $Q_k \ltimes N_k$ is the twisted wreath product of V by Q_k, the action of Q on V providing the twisting). Then $d(Q_k \ltimes N_k)=\min\{2, d(Q)\}$ and $\mathsf{M}(Q_k \ltimes N_k)=0$ for all k, while the abelianized deficiencies tend to infinity.

(Note that the subgroups $Q \ltimes N_k$ form a subsequence in the corresponding sequence of generalized Swan examples.) If Q is perfect and $p > 3$, one can replace the C_k by the $SL(2, q)$ with q running through the primes larger than $|Q|$ that are congruent neither to 1 nor to -1 modulo p, and the same conclusions will still hold, while the $Q_k \ltimes N_k$ will remain perfect. (To see that $d(Q_k \ltimes N_k) \leqslant \min\{2, d(Q)\}$, argue first that Q_k can be generated by $\min\{2, d(Q)\}$ elements, then write Q_k as F/R with F a free group of this rank and appeal to a theorem of Gaschütz [6] for the fact that N_k, like every one-generator Q_k-module of characteristic p, is a quotient of $R/R'R^p$ where R^p stands for the subgroup $\langle r^p \mid r \in R \rangle$.)

For the generalization of the Wiegold–Neumann construction, the critical property of Q is to have a nontrivial central element, z say, which does not generate all of Q. (Thus here all groups of prime order and all nonabelian simple groups are bad, but cyclic groups of composite order and perfect groups like $SL(2, q)$ for odd q, $q \neq 3$, are good.) Let R be any subgroup of Q such that $R < \langle R, z \rangle < Q$, and let p be a prime which does not divide $|Q|$ and is congruent to 1 modulo $|Rz|$ (that is, modulo the order of the element Rz in the factor group $\langle R, z \rangle / R$). Choose a 1-dimensional $\mathbb{F}_p\langle R, z \rangle$-module, U say, on which R acts trivially and z acts by a scalar of multiplicative order $|Rz|$. Then z acts as that scalar also on the $\mathbb{F}_p Q$-module V induced from U, and V can be written as a direct sum of $|Q : \langle R, z \rangle|$ subspaces, each of dimension 1, that are permuted by Q. It follows that the free product of $|Q : \langle R, z \rangle|$ groups of order p, which we shall call P, admits an action by Q which is such that V is Q-isomorphic to P/P'. Write the lower central series of P as $P = P_1 > \cdots > P_i > \cdots$; for each positive integer k such that $k + 1$ is not divisible by $|Rz|$, set $N_k = P/P_{k+1}$. From Theorem 2.6 of Haebich [9] we know that $\mathsf{M}(N_k)$ is isomorphic to P_{k+1}/P_{k+2}. It follows that P/P_{k+2} is a Schur covering group of N_k. The action of Q on P yields an action on N_k, and we form the semidirect product $G = Q \ltimes N_k$ with reference to this. The action of Q on P also yields actions on the covering group P/P_{k+2} and on the copy P_{k+1}/P_{k+2} of $\mathsf{M}(N_k)$ in that covering group. These actions match in the way that is necessary to ensure that when $\mathsf{M}(N_k)$ is viewed as a Q-module in the sense of Lemma 1, it is Q-isomorphic to P_{k+1}/P_{k+2} (see the last two paragraphs of the next section). It is easy to calculate, following Wiegold [15], that P_{k+1}/P_{k+2} is an elementary abelian p-group and z has nontrivial powering action on P_{k+1}/P_{k+2}: thus $\mathsf{M}(N_k)^Q = 0$ as required. Further, Wiegold asserted in [15] that the rank of P_{k+1}/P_{k+2} tends to infinity with k, and this yields that our set of the $d(\mathsf{M}(N_k))$ has no upper bound.

However, I must admit that I have not been able to locate or devise a proof for the proposition that the rank of P_{k+1}/P_{k+2} tends to infinity. (I am grateful for a last-minute suggestion that it may be possible to deduce it from Gaglione [5].) To this extent, the justification of the construction given above is incomplete.

One way to sidestep this difficulty is by exploiting the freedom to choose p. By Dirichlet's Theorem, there are infinitely many primes satisfying the present requirements. Write P_k and N_k as $P_{k,p}$ and $N_{k,p}$ to indicate that our groups depend on this choice as well. If $p > k+1$, then $N_{k+1,p}$ is a regular p-group (because its nilpotency class is small) and so it has exponent p (because it is generated by elements of order p). It follows that in this case $N_{k+1,p}$ is a free group of the variety $\mathfrak{B}_p \cap \mathfrak{N}_{k+1}$ of groups of exponent p and class at most $k+1$, and $P_{k+1,p}/P_{k+2,p}$ is the last term of its lower central series. Thus the rank of $P_{k+1,p}/P_{k+2,p}$ is given by Witt's Formula, from which one can see that this rank is independent of p and does tend to infinity with k. For each k such that $k+1$ is not divisible by $|Rz|$, choose N_k as one of the $N_{k,p}$ with $p > k+1$: then the sequence of groups $Q \ltimes N_k$ will have all the properties we claimed.

In the generalized Wiegold–Neumann construction, V had to be a monomial module in order that action on it should lift to action on a free product P of cyclic groups with $P/P' \cong V$. The other important point was that some element of Q act on V as a nontrivial scalar, so one can deduce that this element, and therefore also Q, acts fixed-point-free on certain lower central factors of P. If instead we choose P as a free group of $\mathfrak{B}_p \cap \mathfrak{N}_{p-1}$, then V does not have to be monomial: then any action of the p'-group Q on P/P' can be lifted to an action on P. Witt's Formula is just a special case of the character formula of Brandt [2], which enables one to compute the action on the lower central factors of these P. Using this, it may be possible to verify that the action of Q is fixed-point-free even when no individual element of Q is known to act without fixed points. Of course these P are finite and so no single one of them will yield an infinite family of examples.

3. A Proof of Lemma 1

The aim of this section is to present an elementary proof of Lemma 1.

In preparation, we establish two simple propositions. They concern the (right) action of a finite group Q on a finitely generated abelian group A. It will be convenient for the moment to write A additively. Let A^Q denote the subgroup consisting of the fixed points of Q in A, and A_0 the torsion subgroup of A. Write $[A, Q]$ for the subgroup of A generated by the elements $a(x-1)$ with $a \in A$ and $x \in Q$.

(a) *If the index* $|A : A^Q|$ *is prime to* $|Q|$, *then* $[A, Q] \cap A^Q = 0$.

(b) *If* $|A_0|$ *is prime to* $|Q|$, *then* $[A, Q] \cap A_0 = [A_0, Q]$.

For the proof, let η denote the element $\sum_{y \in Q}(1-y)$ of $\mathbb{Z}Q$, and note that $(x-1)\eta = |Q|(x-1)$ whenever $x \in Q$: thus

$$a\eta = \begin{cases} 0 & \text{if } a \in A^Q, \\ |Q|a & \text{if } a \in [A, Q]. \end{cases}$$

It follows that $[A, Q] \cap A^Q$ has exponent dividing $|Q|$. If $|A : A^Q|$ is prime to $|Q|$, then $A = A^Q + |Q|A$ and hence $[A, Q] = [|Q|A, Q] = |Q|[A, Q]$. A finitely generated abelian group that is divisible by $|Q|$ has no nonzero element of order dividing $|Q|$, so in this case $[A, Q] \cap A^Q = 0$. On the other hand, if $|A_0|$ is prime to $|Q|$ then there is a positive integer k such that $|Q|^k \equiv 1 \pmod{|A_0|}$, and then $a \in [A, Q] \cap A_0$ implies that $a = |Q|^k a = a\eta^k \in A_0\eta \leqslant [A_0, Q]$. This completes the proof of (a) and (b).

We are now ready to start on the proof of Lemma 1.

Let π denote the set of the prime divisors of $|Q|$, and π' the set of all other primes. Then $\mathsf{M}(Q)$ is a π-group and $\mathsf{M}(N)$ is a π'-group, so to prove Lemma 1 it will be sufficient to show that $\mathsf{M}(Q \ltimes N)$ has a subgroup isomorphic to $\mathsf{M}(N)^Q$ with quotient isomorphic to $\mathsf{M}(Q)$.

The definition of multiplicator that we use is this: given a finite group G, write it as F/R with F free of finite rank, and set $\mathsf{M}(G) = (F' \cap R)/[F, R]$. (Accordingly, in this section we use multiplicative rather than additive language even where the multiplicator is concerned.) As the discussion on pp. 29–31 of Beyl and Tappe [1] explains, there is in fact a functor M from finite groups to abelian groups; so in particular there is, for each G, a distinguished homomorphism

$$\operatorname{Aut} G \to \operatorname{Aut} \mathsf{M}(G), \quad \alpha \mapsto \mathsf{M}(\alpha).$$

Composition with this homomorphism converts any action on G into an action on $\mathsf{M}(G)$. Given a semidirect product $Q \ltimes N$ formed with respect to some action of Q on N, it is in this sense that we have an action of Q on $\mathsf{M}(N)$. In terms of normal subgroups in a free group, this comes to the following.

Write $Q \ltimes N$ as F/S with F free of finite rank, and let R/S correspond to the normal subgroup N of $Q \ltimes N$. Then $F/R = Q$ and R is also a free group of finite rank, so we have

$$\begin{aligned} \mathsf{M}(Q \ltimes N) &= (F' \cap S)/[F, S], \\ \mathsf{M}(Q) &= (F' \cap R)/[F, R], \\ \mathsf{M}(N) &= (R' \cap S)/[R, S]. \end{aligned}$$

Moreover, the relevant action of Q on $\mathsf{M}(N)$ is that which comes from conjugation in F. To justify this last claim, recall that if $\alpha \in \operatorname{Aut} N$ then $\mathsf{M}(\alpha)$ is defined by choosing any endomorphism, ε say, of R such that $(rS)\alpha = (r\varepsilon)S$ for all r in R, and setting $\mathsf{M}(\alpha)\colon r'[R, S] \mapsto (r'\varepsilon)[R, S]$ for each r' in $R' \cap S$. (The last sentence of Lemma 3.1 in [1] shows that the $\mathsf{M}(\alpha)$ so defined depends only on α and not on the choice of ε.) The α that are relevant to our claim come from conjugation in F; the same conjugation can also be used to define the corresponding ε, and then our claim follows. Note also that, by Maschke's Theorem, $\mathsf{M}(N)^Q$ is isomorphic to the largest Q-trivial quotient of $\mathsf{M}(N)$, so in the present terms

$$\mathsf{M}(N)^Q \cong (R' \cap S)/[R, S][F, R' \cap S].$$

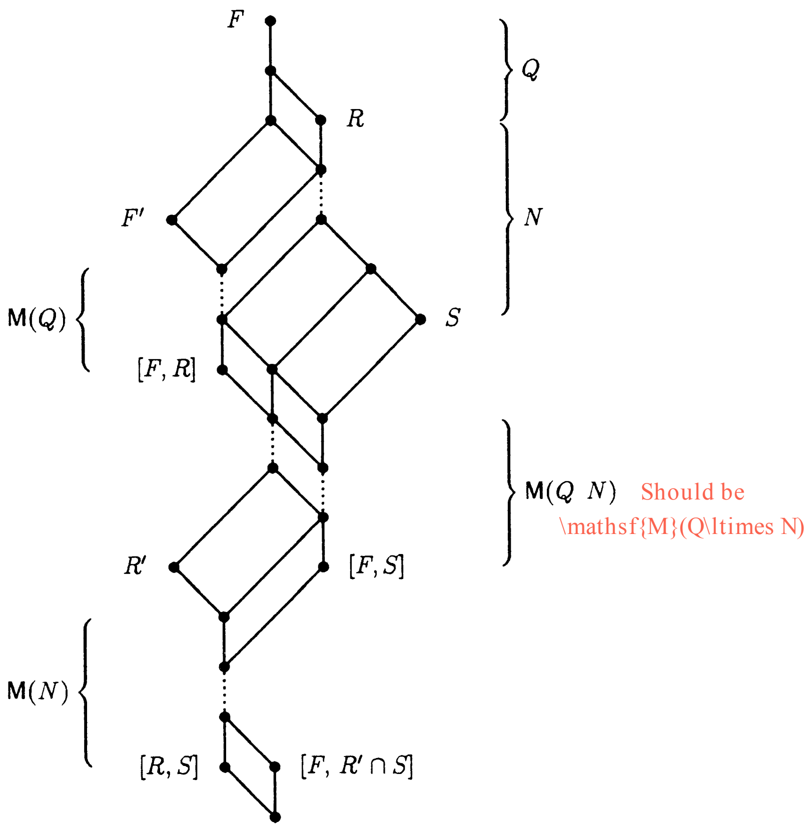
A modular lattice generated by two chains is always distributive: thus in the normal subgroup lattice of F, the sublattice generated by the two chains

$$F \geqslant R \geqslant S \geqslant [F, S] \geqslant [F,\ R' \cap S] \quad \text{and} \quad F' \geqslant [F, R] \geqslant R' \geqslant [R, S]$$

is distributive. The figure shows the Hasse diagram of the distributive lattice $\mathcal{L}$ defined on these nine generators by the displayed inclusions and

$$F \geqslant F', \quad R \geqslant [F, R] \geqslant [F, S], \quad R' \geqslant [F,\ R' \cap S]$$

as defining relations: its verification is an elementary lattice theory exercise.



The sublattice in the normal subgroup lattice of F is a homomorphic image of this $\mathcal{L}$. Dotted lines have been used to denote edges which are present in $\mathcal{L}$ but, as we shall see, always collapse in F (in the sense that the two endpoints of the edge are equal in F). Note that once those collapses are established, the planned proof of

Lemma 1 will be complete, with $(R' \cap S)[F, S]/[F, S]$ as the required subgroup in $\mathsf{M}(Q \ltimes N)$.

By one of the elementary isomorphism theorems,

$$(F' \cap R)/([F' \cap [F, R]S) \cong (F' \cap R)S/[F, R]S.$$

The left hand side is a quotient of the π-group $\mathsf{M}(Q)$ and the right hand side is a section of the π'-group R/S, so both must have order 1. This proves the first pair of collapses. (It was noted in [1], at the top of p. 31, that the functorial nature of M very directly yields that $\mathsf{M}(Q)$ is not only a quotient but even a module direct summand of $\mathsf{M}(Q \ltimes N)$, and no coprimality was used there. However, the translation of that argument into the present setting would involve a more complicated lattice.)

The proposition (a) established at the beginning of this section may now be applied with $A = R'[F, S]$: since $R'S/R'[F, S] \leqslant A^Q$, the coprimality condition is satisfied. Should be A=R/R'[F,S]
The conclusion gives that $[F, R] \cap R'S \leqslant R'[F, S]$, and the second pair of collapses follows. Next, apply (b) with $A = S/[R, S]$: then $A_0 = (R' \cap S)/[R, S]$, so the coprimality condition holds. The conclusion gives that $R' \cap [F, S] \leqslant [R, S][F, R' \cap S]$, the last collapse we wanted to show. This completes the proof of Lemma 1.

We conclude this section by sketching two justifications for a claim we used implicitly in the discussion of the generalized Wiegold–Neumann construction, namely that if some action on a group N lifts to an action on a covering group of N, then by restriction to the copy of $\mathsf{M}(N)$ in that covering group we get the same action as by composition with the homomorphism $\operatorname{Aut} N \to \operatorname{Aut} \mathsf{M}(N)$ provided by the functor M. Given a covering group of N, this time write that as F/S (where F is free of finite rank), with R/S as the copy of $\mathsf{M}(N)$ in F/S: then

$$F/R = N, \quad F' \cap S = [F, R], \quad (F' \cap R)S = R,$$

so that there is a natural isomorphism

$$\mathsf{M}(N) = (F' \cap R)/[F, R] \cong R/S.$$

Let α be an automorphism of F/R that lifts to an automorphism of F/S, say, to α^*, and let us use our freedom to define $\mathsf{M}(\alpha)$ in terms of an endomorphism ε of F that lifts α because it lifts α^*. Of course then ε and α^* agree on R/S, so the action of $\mathsf{M}(\alpha)$ on $(F' \cap R)/[F, R]$ and the action of α^* on R/S are intertwined by the natural isomorphism $F' \cap R)/[F, R] \cong R/S$ above. This is just the rigourous form of what we had to establish.

The claim we have just proved could not serve as a general description on how to convert an action on N to an action on $\mathsf{M}(N)$ (for example, if N is elementary abelian of order 2^3, the natural action of $\operatorname{Aut} N$ does not lift to any Schur covering group of N). However, it is possible to give such a description, in terms of covering groups rather than free groups, by reference to the familiar result that any two covering groups of of a finite group N are isoclinic. Strictly speaking, what one proves there is that, given Replace \mathsf{N}
any two surjective maps $\sigma_i : N_i \to N$ with $\ker \sigma_i \leqslant N_i{}' \cap \mathsf{Z}(\mathsf{N})$ and $\ker \sigma_i \cong \mathsf{M}(N)$, by N_i

there is a unique isomorphism $N_1' \to N_2'$ with a certain property. If σ_1 is fixed and σ_2 ranges through the $\sigma_1\alpha$ with $\alpha \in \operatorname{Aut} N$, then to each α we get a uniquely determined automorphism, α_1 say, of N_1'. This α_1 maps $\ker\sigma_1$ onto itself, and the restriction of α_1 to that kernel 'is' $\mathsf{M}(\alpha)$. (In fact, $\alpha \mapsto \alpha_1$ defines a homomorphism $\operatorname{Aut} N \to \operatorname{Aut} N_1'$.) The second justification promised lies in checking that if α lifts to an automorphism α^* of N_1 then the restriction of α^* to N_1' also has the property which characterizes α_1, so that restriction must be α_1.

4. Some Explicit Examples

In the perfect group $SL(2,7)$, the elements of order 7 are not conjugate to their inverses; over any $\mathbb{F}_p$ with $p \equiv 1 \bmod 168$, this group has an simple module V of dimension 3 such that $V \otimes V$ contains no fixed point other than 0. In fact, Section 232 of Burnside [3] (see particularly its last sentence) explicitly describes such a representation over any field containing a square root of -7, so we can even choose $p = 11$. Of course the centre is nontrivial, the multiplicator is trivial, and the deficiency is zero (see for example Campbell and Robertson [4]), and therefore this group can play the role of Q in both constructions. For the generalized Wiegold–Neumann construction, $SL(2,5)$ presents a smaller perfect alternative in the role of Q. The $Q \ltimes N$ obtained with these choices seem to be the first perfect finite groups proved to have trivial multiplicator but positive deficiency.

To provide a better understanding of the constructions and to illustrate further aspects of the available methods, we explore here some of the smallest perfect examples that we can make and determine the exact abelianized deficiency of these groups. The latter calculations will be based on the following result, whose proof is deferred to the next section and in which $\mathbb{F}_p$ stands for the trivial Q-module of p elements.

Lemma 2. *If Q is a finite p'-group acting on a finite p-group N, then*

$$\mathsf{abdef}(Q \ltimes N) = \max\{\,\mathsf{abdef}\,Q,\ d_Q(\mathbb{F}_p \oplus \mathsf{M}(N)) - 1\,\}.$$

Take the generalized Swan construction first, with $G_k = Q \ltimes N_k$ where $Q = SL(2,7)$ and N_k is the direct sum of k copies of the Q-module V of order 11^3 mentioned above, so

$$|G_k| = 336 \times 11^{3k}.$$

Though we do not need it, for orientation we note that it can be seen using the results of Gaschütz [6] that

$$d(G_k) = 1 + \lceil k/3 \rceil$$

where by $\lceil x \rceil$ we denote the unique integer such that $x \leqslant \lceil x \rceil < x + 1$. We shall prove that

$$\operatorname{abdef} G_k = \left\lceil \tfrac{1}{6}k(k+1) \right\rceil - 1. \tag{1}$$

In particular, $\operatorname{abdef} G_1 = \operatorname{abdef} G_2 = 0$ but $\operatorname{abdef} G_3 = 1$.

Towards the proof of (1), we already know that $\operatorname{abdef} Q = 0$ (because even $\operatorname{def} Q = 0$). By the general rules of multilinear algebra,

$$\mathsf{M}(N_k) = N_k \wedge N_k = (V^{\oplus k}) \wedge (V^{\oplus k}) = (V \wedge V)^{\oplus k} \oplus (V \otimes V)^{\oplus \frac{1}{2}k(k-1)}.$$

As for every 3-dimensional simple module, the exterior square $V \wedge V$ of V is also 3-dimensional and simple. Any simple module of prime dimension on which the derived group acts nontrivially is in fact absolutely simple, and this applies to both V and $V \wedge V$. For this particular V, it is also easy to see that the direct complement of $V \wedge V$ in $V \otimes V$ is absolutely simple. Call that 6-dimensional module W: then

$$\mathsf{M}(N_k) = (V \wedge V)^{\oplus \frac{1}{2}k(k+1)} \oplus W^{\oplus \frac{1}{2}k(k-1)}.$$

The way for counting the minimum number of generators of a semisimple module can be found in Lemma 7.12 of [8]; using that and the fact that the multiplicity of an absolutely simple module of dimension d in the largest semisimple quotient of the regular module is d, we obtain that

$$d_Q(\mathbb{F}_{11} \oplus \mathsf{M}(N_k)) = \left\lceil \tfrac{1}{6}k(k+1) \right\rceil,$$

and so (1) follows from Lemma 2.

Next we turn to the generalized Wiegold–Neumann construction, this time writing $H_k = Q \ltimes N_k$, still with $Q = SL(2,7)$. There is then only one nontrivial element in $\mathsf{Z}(\mathrm{Q})$, and that must be our z. The largest subgroup R that does not contain z is of order 21, and then the index of $\langle R, z \rangle$ is 8; the smallest p we can choose is 5, and the smallest permitted value of k is 2. Now N_4 is the free group of rank 8 in the variety $\mathfrak{B}_5 \cap \mathfrak{N}_4$ which consists of all groups of exponent 5 and nilpotency class at most 4, so the ranks of the first four lower central factors of P can be calculated by Witt's Formula: they are 8, $\frac{1}{2}(8^2 - 8)$, $\frac{1}{3}(8^3 - 8)$, and $\frac{1}{4}(8^4 - 8^2)$. We only need the first three to deduce that

$$|H_2| = 336 \times 5^{36} \quad \text{and} \quad |\mathsf{M}(N_2)| = 5^{168},$$

with $\mathsf{M}(N_2)$ elementary abelian.

Notice that 168 is also the dimension of the unique largest $\mathbb{F}_5 Q$-module which can be generated by a single element and on which z acts as the scalar -1. There is no reason to expect that $\mathsf{M}(N_2)$ is that particular module, but calculation shows that it is, and hence

$$\operatorname{abdef} H_2 = 0$$

follows by Lemma 2. This coincidence was first established by a direct computer calculation that one could not hope to perform by hand. I am greatly indebted to Dr M. F. Newman for this, for his other contributions that will be mentioned below, and for many illuminating discussions on related matters.

There is another argument which justifies this conclusion and which does not rely on a machine; it can be sketched as follows. Groups of exponent p and nilpotency class at most $p-1$ are the same, in the strongest possible sense, as Lie algebras over $\mathbb{F}_p$ that are nilpotent of class at most $p-1$. In particular, N_3 is just the rank 8 free Lie algebra in the variety of the Lie algebras over $\mathbb{F}_5$ that are nilpotent of class at most 3. Call this Lie algebra $L(8, \mathbb{F}_5, 3)$, or briefly L. Also in this sense, $\mathsf{M}(N_2)$ is L^3, and both are $\mathbb{F}_5 Q$-modules. The connection between the natural actions of $GL(8,5)$ on L/L^2 and on L^3 was described by Brandt in [2]. That connection can be restricted to the action of Q, it survives the 'extension of scalars' which replaces $\mathbb{F}_5$ by its algebraic closure, and also the 'reduction of constants' which establishes the connection between the representations of Q over that algebraic closure and the representations of Q over the complex field $\mathbb{C}$. Thus the connection between the actions of Q on L/L^2 and on L^3 does not change if we replace L by $L(8, \mathbb{C}, 3)$. Let χ denote the complex character corresponding to the action of Q on P/P', that is, the character afforded by the complex L/L^2: by Brandt's Formula, the character afforded by L^3 is then given by the rule that its value at an element x of Q is $\frac{1}{3}(\chi(x)-\chi(x^3))$. Here χ is the character of Q induced from the character of $\langle R, z\rangle$ which is 1 on R and -1 on the rest of $\langle R, z\rangle$, so it is easy to calculate, and therefore so is $L_3(\chi)$. On the other hand, the character corresponding to the unique largest module which can be generated by a single element and on which z acts as the scalar -1 has value 168 at the identity element, -168 at the central involution, and 0 at every other element of Q. It remains to observe that this character and $L_3(\chi)$ coincide, and the argument is complete.

The next example in this sequence is H_4. A slight extension of the argument above yields first that

$$|H_4| = 336 \times 5^{1212} \quad \text{and} \quad |\mathsf{M}(N_4)| = 5^{6552}.$$

Further, it justifies that $d_Q(\mathsf{M}(N_4))$ is the minimum number of generators of the $\mathbb{C}Q$-module that affords the character $L_5(\chi)$ whose value at x is $\frac{1}{5}(\chi(x)-\chi(x^5))$. Here χ is the same as above. Using the character table of Q and the orthogonality relations, an easy hand calculation shows that $L_5(\chi)$ is the sum of

158 copies of each of two irreducible characters of degree 4,
234 copies of each of two irreducible characters of degree 6, and
310 copies of an irreducible character of degree 8.

It follows that $d_Q(\mathsf{M}(N_4)) = 40$ and so, by Lemma 2,

$$\text{abdef } H_4 = 39.$$

To attempt to confirm this by the kind of direct machine calculations that were used above would be stretching the resources available to us at this time.

For a related example with positive deficiency which is not as far out of reach as this, we exploit the observations of the last paragraph of Section 2: one can imitate the construction of H_2 with P/P' the other 8-dimensional simple $\mathbb{F}_5 Q$-module on which the central involution acts nontrivially, even though that module is not monomial. This new version of H_2 also has order 336×5^{36}, but now the role of χ goes to the faithful simple character, ψ say, of degree 8, and $L_3(\psi)$ turns out to be the sum of

3 copies of each of two irreducible characters of degree 4,
6 copies of each of two irreducible characters of degree 6, and
9 copies of an irreducible character of degree 8.

The conclusion is that the abelianized deficiency of this version of H_2 is 1. This group is, of course, much larger than G_3, which remains the smallest perfect group we know with trivial multiplicator and positive deficiency.

The second smallest is also built on this last pattern, with $Q = SL(2,5)$, $p = 11$, and P/P' the direct sum of two isomorphic 2-dimensional simple modules. (There are two such simple modules, but they are interchanged by an automorphism of Q, so the isomorphism type of $Q \ltimes N_2$ does not depend on which one we use). The resulting $Q \ltimes N_2$ has order 120×11^{10}, and its abelianized deficiency is 2.

The last two examples were discovered by Dr M. F. Newman and first justified by machine computations. I am grateful for his permission to include them here.

5. The Proof of Lemma 2

Set $G = Q \ltimes N$. In the notation already used in the proof of Lemma 1, abdef $G = d_G(S/S') - d(F)$, so the problem is to determine $d_G(S/S')$. By the theorem of Gruenberg [7] (see Theorem 7.8 in [8]), $d_G(S/S') = \max_q d_G(S/S'S^q)$ where q ranges through the prime divisors of $|G|$ and S^q stands for the subgroup $\langle s^q \mid s \in S \rangle$.

For simplicity, put $d = d(F)$ and assume that $d \geqslant 2$. It follows from a fundamental result of Gaschütz [6] (see Lecture 2 in [8]) that there is an exact sequence

$$0 \to S/S'S^q \to (\mathbb{F}_q G)^{\oplus d} \to \mathbb{F}_q G \to \mathbb{F}_q \to 0 \tag{2}$$

of G-homomorphisms (where, as usual in this context, $\mathbb{F}_q$ stands both for the field of q elements and for the trivial G-module of q elements). In view of Schanuel's Lemma and the Krull-Schmidt Theorem (used as in the deduction of Corollary 2.5 in [8]), the existence of such a sequence may in fact be taken as the definition of $S/S'S^q$. Similarly, $R/R'R^q$ is characterized by the existence of an exact sequence

$$0 \to R/R'R^q \to (\mathbb{F}_q Q)^{\oplus d} \to \mathbb{F}_q Q \to \mathbb{F}_q \to 0. \tag{3}$$

Consider first the case $q \neq p$. Given that N is a q'-group, every $\mathbb{F}_q G$-module W can be written canonically as $W^N \oplus [W, N]$, with

$$d_G(W) = \max\{d_G(W^N),\ d_G([W, N])\}.$$

In this manner, (2) splits into two sequences, and the first of those is

$$0 \to (S/S'S^q)^N \to (\mathbb{F}_q Q)^{\oplus d} \to \mathbb{F}_q Q \to \mathbb{F}_q \to 0$$

because, in the relevant sense, $(\mathbb{F}_q G)^N = \mathbb{F}_q Q$. Comparing this sequence with (3), we conclude that $(S/S'S^q)^N \cong R/R'R^q$. The second sequence resulting from (2) is

$$0 \to [S/S'S^q, N] \to [\mathbb{F}_q G, N]^{\oplus d} \to [\mathbb{F}_q G, N] \to 0 \to 0.$$

Since $[\mathbb{F}_q G, N]$ is a direct summand of $\mathbb{F}_q G$, it is projective, and therefore this sequence splits; hence by the Krull-Schmidt Theorem $[S/S'S^q, N] \cong [\mathbb{F}_q G, N]^{\oplus(d-1)}$. It follows that $d_G(S/S'S^q) = \max\{d_Q(R/R'R^q),\ d-1\}$, whence one readily sees that, with q ranging over the prime divisors of $|Q|$,

$$\max_q\,(d_G(S/S'S^q) - d) = \max_q\,(d_Q(R/R'R^q) - d) = \mathsf{abdef}\, Q.$$

It remains to consider $d_G(S/S'S^p)$. This is the same as the minimum number of generators of the largest semisimple quotient of $S/S'S^p$. On that quotient, N acts trivially (because every normal p-subgroup acts trivially on every simple module of characteristic p); conversely, the largest quotient on which N acts trivially is semisimple (by Maschke's Theorem). Thus the largest semisimple quotient of $S/S'S^p$ is $S/[R, S]S^p$, and $d_G(S/S'S^p) = d_Q(S/[R, S]S^p)$.

The torsion subgroup of the finitely generated abelian group $S/[R, S]$ is $(R' \cap S)/[R, S]$, isomorphic to the p-group $\mathsf{M}(N)$. It follows that (in additive terminology) $S/[R, S]S^p$ is a direct sum of two summands, namely of $S/(R' \cap S)S^p$ and of the Frattini factor group of $\mathsf{M}(N)$. Since $|Q|$ is prime to p, there is such a direct decomposition of $S/[R, S]S^p$ as Q-module, and so

$$d_Q(S/[R, S]S^p) = d_Q\big(S/(R' \cap S)S^p \oplus \mathsf{M}(N)\big).$$

The proof of Lemma 2 will therefore be complete if we show that

$$S/(R' \cap S)S^p \cong \mathbb{F}_p \oplus (\mathbb{F}_p Q)^{\oplus(d-1)}.$$

Here the right hand side is familiar, for by the easy (coprime) case of the Gaschütz theory quoted above we know that $R/R'R^p \cong \mathbb{F}_p \oplus (\mathbb{F}_p Q)^{\oplus(d-1)}$: thus what we need is that $S/(R' \cap S)S^p \cong R/R'R^p$. As $S/(R' \cap S)$ and $R'S/R'$ are Q-isomorphic, so are their largest exponent-p quotients: $S/(R' \cap S)S^p \cong R'S/R'R^p$. We have reduced our target to $R'S/R'R^p \cong R/R'R^p$, and this much can be seen as follows.

If A is any finitely generated $\mathbb{Z}$-free $\mathbb{Z}Q$-module and B is a maximal submodule of p-power index, then

$$A/pA \cong A/B \oplus B/pA \cong pA/pB \oplus B/pA \cong B/pB.$$

Repeating this argument shows that $A/pA \cong B/pB$ whenever B is a submodule of p-power index. This may then be applied with $A = R/R'$ and $B = R'S/R'$ because, being a quotient of N, $R/R'S$ has p-power order. The proof of Lemma 2 is now complete.

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