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Growth of Varieties of Groups and Group Representations, and the Gel'fand–Kirillov Dimension

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During the past two decades, asymptotic methods of algebra have achieved substantial progress and found applications throughout mathematics (topology, functional analysis, probability theory, etc.). Central for this field is the concept of the growth of a finitely generated algebra (for associative algebras known also as the Gel'fand–Kirillov dimension), going back to Schwartz [Sch], Gel'fand-Kirillov [GK], and Milnor [M]. This concept has been intensively studied for groups, semigroups, associative algebras, and some other structures. In particular, it was discovered that any nontrivial identity that is satisfied in an algebraic structure has major influence on its growth. This suggests that there should be some connections between the growth of algebras and the theory of varieties (i.e., classes of algebras defined by identical relations).

In the present paper we introduce and study the concept of the *growth of a variety*. Although here we are mainly concerned with varieties of groups and group representations, it should be noted that this concept can

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be naturally extended to varieties of arbitrary algebraic structures, and thus may be of interest from the standpoint of universal algebra.

The paper consists of three sections. In Section 1 we introduce the notion of growth for varieties of group representations and propose two main problems to be studied. Section 2 is concerned with the growth of varieties of groups (or, equivalently, varieties of representations of group type). Finally, in Section 3 we investigate the growth of varieties of group representations of ring type.

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1. GROWTH OF VARIETIES: THE MAIN DEFINITIONS AND PROBLEMS

The considerations of this paper arose from the study of varieties of group representations. We recall briefly a few definitions, notation, and examples (for details see [PV] or [V2]). We assume that the reader is familiar with varieties of usual (that is, one-sorted) algebras.

Let K be an arbitrary but fixed field. Our objects of study are linear representations of groups over K. A representation ρ: $G \to \mathrm{Aut}_K V$ of a group G on a (left unitary) K-module V is often denoted by $\rho = (V, G)$; the result of the action of an element $g \in G$ on an element $v \in V$ is denoted by $v \cdot g$.

Let F be the free group of countable rank with free generators $x_1, x_2, \ldots$; KF be its group algebra over K; and $u = u(x_1, \ldots, x_n)$ be an element of KF. Suppose there is given a representation ρ: $G \to \mathrm{Aut}_K V$. We say that u is an *identity* (or a *law*) of ρ if for arbitrary $g_1, \ldots, g_n \in G$,

$$u(\rho(g_1), \ldots, \rho(g_n)) = 0$$

in $\mathrm{End}_K V$. A class of group representations is called a *variety* if it consists of all representations satisfying a certain set of identities. If $\mathscr{X}$ is a variety of group representations over K, then the set $\mathrm{Id}\,\mathscr{X}$ of all its identities is a two-sided ideal of KF, which is invariant under all endomorphisms of the group F. Such ideals are called *fully invariant* (or *verbal*). A standard argument shows that the map $\mathscr{X} \mapsto \mathrm{Id}\,\mathscr{X}$ is a bijection between varieties and verbal ideals of KF.

Let $\mathscr{X}$ be a variety and $I = \mathrm{Id}\,\mathscr{X}$ the corresponding verbal ideal. Regarding KF/I as an F-module, we obtain a representation

$$\mathrm{Fr}(\mathscr{X}) = (KF/I, F)$$

called the *free (cyclic) representation of countable rank of* $\mathscr{X}$. If $\rho = (V, G)$

is an arbitrary representation in $\mathscr{X}$, then every map sending the unit $1 + I$ of KF/I to V and the free generators x_i of F to elements of G can be uniquely extended to a homomorphism of representations $\mathrm{Fr}(\mathscr{X}) \to \rho$. Similarly, if F_n is a free group on n generators $x_1, \dots, x_n$ and $I_n = I \cap KF$, then the natural representation $\mathrm{Fr}_n(\mathscr{X}) = (KF_n/I_n, F_n)$ is the *free (cyclic) representation of rank n of* $\mathscr{X}$.

EXAMPLES. (1) A representation $\rho = (V, G)$ is called *stable of class n*, or simply *n-stable* (this terminology goes back to Kaloujnine and P. Hall), if there is a series of G-modules

$$0 = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = V$$

such that G acts trivially on each quotient A_{i+1}/A_i. A typical example of an n-stable representation is $ut_n(K) = (K^n, UT_n(K))$, where $UT_n(K)$ is the full unitriangular matrix group of degree n acting on K^n in the natural way. The class $\mathscr{S}^n$ of all n-stable representations is a variety because it is definable by a single identity $(x_1 - 1)(x_2 - 1) \cdots (x_n - 1)$. It is easy to see that $\mathrm{Id}(\mathscr{S}^n) = \Delta^n$, where Δ is the augmentation ideal of KF.

(2) A representation is *n-unipotent* if it satisfies the identity $(x - 1)^n$. The variety of all n-unipotent representations is denoted by $\mathscr{U}_n$. Evidently $\mathscr{S}^n \subseteq \mathscr{U}_n$. If K is a field then, by a classical theorem of Kolchin, every finite-dimensional unipotent representation is stable. In other words, for finite-dimensional representations over a field, "stable" and "unipotent" are the same. For infinite-dimensional representations, in general, this is not true.

To introduce the main concept of this paper, take an arbitrary variety $\mathscr{X}$ of group representations over a field K. *How fast is this $\mathscr{X}$ growing*? If $\mathscr{X}$ is locally finite-dimensional, a natural approach to this rather vague question could be the following. For every n, consider the free representation of rank n in $\mathscr{X}$:

$$\mathrm{Fr}_n(\mathscr{X}) = (KF_n/I_n, F_n).$$

Then the algebra KF_n/I_n is finite-dimensional; denoting

$$f_{\mathscr{X}}(n) = \dim_K(KF_n/I_n) \tag{1}$$

we obtain a function $f_{\mathscr{X}}(n)$. The growth of this function can be taken as a reasonable characterization of the growth of our variety. Such functions have been studied by several authors for various types of algebraic structures, starting with Higman's paper [Hi2] on varieties of groups.

However, if $\mathscr{X}$ is not locally finite-dimensional, the function (1) is not defined. In this case, instead of (1), we consider the function

$$g_{\mathscr{X}}(n) = \mathrm{GKdim}(KF_n/I_n) \tag{2}$$

where GKdim stands for the Gel'fand–Kirillov dimension. Recall that if A is a finitely generated unital algebra over a field K, then

$$\mathrm{GKdim}(A) = \varlimsup_m \frac{\ln d_V(m)}{\ln m} = \varlimsup_m \log_m(d_V(m)) \tag{3}$$

where V is any finite-dimensional generating subspace of A and

$$d_V(m) = \dim_K(K \cdot 1 + V + V^2 + \cdots + V^m).$$

The Gel'fand–Kirillov dimension reflects the rate of growth of the algebra and, in essence, is its "degree of polynomiality". More precisely, if A has polynomial growth of degree α then $\mathrm{GKdim}(A) = \alpha$, but if $\mathrm{GKdim}(A) = \alpha$ then α is the infimum of all real numbers ρ such that the growth of A is bounded by a polynomial of degree ρ. For details we refer the reader to [KL], a standard reference in the field.

Since for every n the number $g_{\mathscr{X}}(n)$ (if it exists) characterizes the growth of the free object of rank n of the variety $\mathscr{X}$, it is natural to call the function $g_{\mathscr{X}}\colon \mathbb{N} \to \mathbb{R}$ the *growth function of the variety* $\mathscr{X}$. Our initial question can now be stated more precisely: *what is the rate of growth of* $g_{\mathscr{X}}(n)$? The main idea is to take one more step and to introduce the following concept.

DEFINITION. If $\mathscr{X}$ is a variety of group representations, then its growth is

$$\begin{aligned} \mathrm{gr}(\mathscr{X}) &= \varlimsup_n \log_n(g_{\mathscr{X}}(n)) \\ &= \varlimsup_n \log_n(\mathrm{GKdim}(KF_n/I_n)). \end{aligned} \tag{4}$$

Notes. 1. This notion is clearly motivated by the definition of the Gel'fand–Kirillov dimension. Comparing (3) and (4), one can informally call $\mathrm{gr}(\mathscr{X})$ the "doubled Gel'fand–Kirillov dimension", or even "the growth of growths".

2. The growth $\mathrm{gr}(\mathscr{X})$ of a locally finite-dimensional variety $\mathscr{X}$ is not defined (since the GKdim of a finite-dimensional algebra is zero). But in this relatively simple case our construction is not needed, because for locally finite-dimensional varieties it is natural to use the function (1), rather than (2). We exclude these varieties from further consideration.

Once the basic concept of this paper is introduced, a host of questions cries out for exploration. How does the growth behave under various operations with varieties? Is there any relation between the growth of a variety and its axiomatic rank? its basis rank? Which abstract properties of a representation ρ actually determine the growth of var ρ?, and so on. Probably, the first two questions that should be solved in this direction are:

Problem 1. Characterize the varieties of group representations $\mathscr{X}$ such that $g_{\mathscr{X}}(n)$ is finite for all n. In other words, which varieties are locally finite-GK-dimensional?

Problem 2. Which real numbers can occur as $\mathrm{gr}(\mathscr{X})$ for a variety of group representations $\mathscr{X}$?

So far none of these problems has been solved in full generality. To explain some of the specific difficulties, consider for example Problem 2. It follows from results of Borho and Kraft [BK], Bergman [Bg], and Warfield [W] that the range of possible values of the Gel'fand–Kirillov dimension of an *individual algebra* is $\{0\} \cup \{1\} \cup [2, \infty]$. However, to determine the range of values of $\mathrm{gr}(\mathscr{X})$, one has to evaluate not the growth of a single algebra but the asymptotic behavior of a sequence of numbers that are defined asymptotically themselves. First of all, it is necessary to collect sufficient experimental material, i.e., calculate the growth of a number of concrete varieties.

The present paper is primarily concerned with two interesting particular cases: varieties of *group type* and varieties of *ring type* (see definitions below). Essentially they can be reduced to the study of varieties of groups and associative algebras, and are considered in Sections 2 and 3, respectively. In both cases Problems 1 and 2 are solved completely, and the results turn out to be strikingly different. Roughly speaking, varieties of group type are growing "very rapidly" and, unless certain rigid restrictions are satisfied, they have infinite growth. The key role here belongs to a well known theorem of Gromov [G]. On the other hand, varieties of ring type are growing "very slowly" and, if the characteristic of the ground field is 0, they all have growth 1.

This difference is in no way surprising. It is well known that the varieties of associative algebras over a field of characteristic 0 are much more "tame" objects than varieties of groups. There is a lot of evidence supporting this claim. For example, all varieties of algebras are finitely based (Kemer's theorem), while most of the varieties of groups are not. The results of the present paper provide additional evidence of this sort.

Concluding this introductory section, we note that the concepts introduced here can be naturally extended to algebras of arbitrary finite type.

Indeed, let $\mathfrak{B}$ be a variety of algebras with finitely many finitary operations. Take a finitely generated algebra A in $\mathfrak{B}$ and let $S = \{s_1, \dots, s_n\}$ be a finite generating set for A. For each natural number m, denote by $d_S(m)$ the cardinality of the set $D_S(m)$ of all elements of A that can be represented by a word of length at most m on S. For example, if $\mathfrak{B}$ is a variety of groups, then $d_S(m)$ is the number of distinct elements of A that can be expressed as a product of at most m elements $s_i^{\pm 1}$. (For arbitrary algebras, if some of the basic operations are nullary or unary, $d_S(m)$ may be infinite. But this obstacle can be overcome if one properly defines the length of a word—see for example [P]). The function $d_S(m)$ depends on the choice of the generating set S, but

$$\overline{\lim_{m}} \log_m(d_S(m)) = \mathrm{gr}(A)$$

is an invariant of A, which we will call the *growth* of A (although it would be more precise to call it the *rate of growth*). In particular, if A is a linear algebra over a field K, then by $d_S(m)$ one should of course denote not the cardinality of the set $D_S(m)$, but the dimension of its K-span, and then $\mathrm{gr}(A) = \mathrm{GKdim}(A)$. Now for every n let $\mathrm{Fr}_n(\mathfrak{B})$ be the free algebra of rank n in $\mathfrak{B}$, and let

$$g_{\mathfrak{B}}(n) = \mathrm{gr}(\mathrm{Fr}_n(\mathfrak{B})). \tag{5}$$

Then the *growth of the variety* $\mathfrak{B}$ can be defined similarly to (4):

$$\mathrm{gr}(\mathfrak{B}) = \overline{\lim_{n}} \log_n(g_{\mathfrak{B}}(n)). \tag{6}$$

These definitions will be used in the next two sections in the context of varieties of groups and associative algebras, respectively.

2. VARIETIES OF GROUP TYPE

Let $\mathfrak{B}$ be a variety of groups. Denote by $\omega\mathfrak{B}$ the class of all representations $\rho = (V, G)$ such that $G/\mathrm{Ker}\,\rho \in \mathfrak{B}$. This class is a variety because if $\mathfrak{B}$ is defined by group words $f_i \in F$, then $\omega\mathfrak{B}$ is defined by the $f_i - 1 \in KF$. It is easy to see that the map $\mathfrak{B} \mapsto \omega\mathfrak{B}$ is injective, and so there exists a natural embedding of the set of varieties of groups into the set of varieties of group representations over a given K. The varieties of group representations $\omega\mathfrak{B}$ are sometimes called *varieties of group type*.

In the present section Problems 1 and 2 are solved for varieties of group type. First we show that in this case they have purely group-theoretic content. Let $\mathscr{X} = \omega\mathfrak{B}$ be a variety of group type. If V is the verbal subgroup of F corresponding to $\mathfrak{B}$, then $F/V = G$ is the free group of countable rank of $\mathfrak{B}$ and $I = \mathrm{Id}\,\mathscr{X}$ is the kernel of the canonic epimorphism $KF \to K[F/V]$. Therefore the module of the relatively free representation $\mathrm{Fr}(\mathscr{X}) = (KF/I, F)$ is simply the group algebra KG. The same is true for the free objects of finite rank n: if $V_n = V \cap F_n$ and $I_n = I \cap KF_n$, then $KF_n/I_n \cong KG_n$, where $G_n = F_n/V_n$.

It is well known that $\mathrm{GKdim}(KH) = \mathrm{gr}(H)$ for every group H. By (2) and (5) we have

$$g_{\mathscr{X}}(n) = \mathrm{GKdim}(KF_n/I_n) = \mathrm{GKdim}(KG_n) = \mathrm{gr}(G_n) = g_{\mathfrak{V}}(n),$$

and so $\mathrm{gr}(\mathscr{X}) = \mathrm{gr}(\mathfrak{V})$. Thus, instead of varieties of group representations of the form $\omega\mathfrak{V}$, we may consider varieties of abstract groups. The following example shows that every natural number does occur as $\mathrm{gr}(\mathfrak{V})$ (and so as $\mathrm{gr}(\omega\mathfrak{V})$).

EXAMPLE. Let $\mathfrak{N}_c$ be the variety of nilpotent groups of class $\le c$. Our aim is to find $\mathrm{gr}(\mathfrak{N}_c)$. Set $\mathrm{Fr}_n(\mathfrak{N}_c) = G_n$, and find first the growth of this group. Take the lower central series

$$G_n = \gamma_1(G_n) \supset \gamma_2(G_n) \supset \cdots \supset \gamma_{c+1}(G_n) = 1$$

in G_n. Each factor $\gamma_i(G_n)/\gamma_{i+1}(G_n)$ is a free abelian group of some rank $r(i)$. By a theorem of Bass [Ba],

$$\mathrm{gr}(G_n) = \sum_{i=1}^{c} ir(i). \tag{7}$$

Further, the rank $r(i)$ is given by the classical Witt formula

$$r(i) = \frac{1}{i}\sum_{k|i} \mu(k)n^{i/k}$$

where the sum is taken over all divisors k of i and $\mu(k)$ is the Möbius function

$$\mu(k) = \begin{cases} (-1)^s & \text{if } k \text{ is a product of } s \text{ distinct primes } (s \ge 0), \\ 0 & \text{otherwise}. \end{cases}$$

Combining all these formulas, we obtain

$$\begin{aligned} \mathrm{gr}(G_n) &= \sum_{i=1}^{c} i\frac{1}{i}\sum_{k|i}\mu(k)n^{i/k} \\ &= \sum_{i=1}^{c}\sum_{k|i}\mu(k)n^{i/k} \\ &= n^c + \text{terms of lower degree in } n. \end{aligned}$$

Thus

$$\operatorname{gr}(\mathfrak{N}_c) = \overline{\lim_n} \log_n(n^c + \cdots) = c. \tag{8}$$

In particular, $\operatorname{gr}(\mathfrak{A}) = 1$, where $\mathfrak{A}$ is the variety of abelian groups.

Does there exist a variety of groups whose growth is a non-integer? In view of (8), one might guess that for some variety $\mathfrak{B}$ such that $\mathfrak{N}_c \subset \mathfrak{B} \subset \mathfrak{N}_{c+1}$ the growth of $\mathfrak{B}$ is strictly between c and $c+1$. But it is not the case: in fact, we will show that the growth of every variety of groups is an integer (provided it is defined and finite).

Before stating the main result of this section, we recall that a variety of groups is called *torsion-free* (or *pure*) if its free groups are torsion-free. If $\mathfrak{B}$ is an arbitrary variety, then the variety $\mathfrak{B}_0$ generated by all torsion-free groups from $\mathfrak{B}$ is torsion-free; it is called the *torsion-free part of* $\mathfrak{B}$.

If $\mathfrak{B}$ is a locally finite variety, then $g_{\mathfrak{B}}(n) = 0$ for every n and therefore the growth of $\mathfrak{B}$ cannot be defined by (6) (cf. Note 2 in the preceding section). For the sake of convenience of formulations, we formally define the growth of every locally finite variety to be equal to 0. Also, we will say that the growth of a variety $\mathfrak{B}$ is *infinite* if it is *not* finite. This means that either $g_{\mathfrak{B}}(n) = \infty$ for some n or $\overline{\lim} \log_n(g_{\mathfrak{B}}(n)) = \infty$.

The following theorem solves Problems 1 and 2 for varieties of groups (or varieties of representations of group type).

THEOREM 1. *Let $\mathfrak{B}$ be a variety of groups.*

(A) *All finitely generated groups in $\mathfrak{B}$ have finite (i.e., polynomial) growth if and only if $\mathfrak{B} \subseteq \mathfrak{N}_c\mathfrak{U}$ for some integer c and some locally finite variety $\mathfrak{U}$.*

(B) *If $\mathfrak{B}$ can be presented in the form $\mathfrak{B} = \mathfrak{N} \vee \mathfrak{U}$ with $\mathfrak{N}$ nilpotent and $\mathfrak{U}$ locally finite, then $\operatorname{gr}(\mathfrak{B})$ is equal to the nilpotency class of $\mathfrak{N}_0$ (and therefore is an integer). Otherwise, $\mathfrak{B}$ has infinite growth.*

The proof consists of several steps. First we prove (A). By a well known theorem of Gromov [G], the growth of a group G is polynomial if and only if G is nilpotent-by-finite. It follows that $\mathfrak{B}$ satisfies the condition of the theorem if and only if it is locally (nilpotent-by-finite). It remains to show that such a variety is nilpotent-by-(locally finite).

Let $\mathfrak{B}$ be a locally (nilpotent-by-finite) variety. Denote by $\mathfrak{N}$ the variety generated by all torsion-free nilpotent groups of $\mathfrak{B}$. We begin by showing that this $\mathfrak{N}$ is nilpotent. Let G be a free group of finite rank of $\mathfrak{N}$, then:

(i) G is polycyclic-by-finite;

(ii) G is residually (torsion-free nilpotent).

We claim that every group G satisfying (i) and (ii) is torsion-free nilpotent. Indeed, by (ii) there is a system of normal subgroups N_i of G such that G/N_i is torsion-free nilpotent and $\bigcap N_i = 1$. Clearly, if G/A and G/B are torsion-free nilpotent, then so is $G/(A \cap B)$. It follows that G has a strictly descending chain of normal subgroups

$$G \supset H_1 \supset H_2 \supset \cdots \tag{9}$$

such that each G/H_i is torsion-free nilpotent and $\bigcap H_i = 1$. Since the Hirsch numbers of the G/H_i are bounded by the Hirsch number of G, it follows that the chain (9) is finite, whence the claim follows.

Thus every free group of *finite rank* of $\mathfrak{N}$ is torsion-free nilpotent, that is, $\mathfrak{N}$ is a locally nilpotent variety whose free groups have no torsion. It follows from Zel'manov's theorem [Z] on the nilpotency of Engel Lie algebras over a field of characteristic 0 that such a variety must be nilpotent.

Now let c be the nilpotency class of $\mathfrak{N}$, and let G be the free group of rank $c + 1$ in $\mathfrak{V}$ freely generated by $x_1, x_2, \ldots, x_{c+1}$. By the assumption on $\mathfrak{V}$, this group is nilpotent-by-finite. Since a finitely generated nilpotent group is torsion-free-by-finite, there is a torsion-free nilpotent normal subgroup H of G with G/H finite. Therefore, for some e we have $x_1^e, x_2^e, \ldots, x_{c+1}^e \in H$ and

$$[x_1^e, x_2^e, \ldots, x_{c+1}^e] = 1$$

(because $H \in \mathfrak{N}$). This is a relation between free generators of G, and hence an identity of $\mathfrak{V}$. Since this identity determines the variety $\mathfrak{N}_c\mathfrak{B}_e$, we have

$$\mathfrak{V} \subseteq \mathfrak{N}_c\mathfrak{B}_e.$$

But then $\mathfrak{V} \subseteq \mathfrak{N}_c(\mathfrak{B}_e \cap \mathfrak{D})$. The variety $\mathfrak{B}_e \cap \mathfrak{D}$ is obviously locally finite, which completes the proof of (A).

To prove (B), we will need a number of auxiliary results. Some of them are of independent interest. We denote the torsion-free rank of an abelian group A by $r(A)$, and the Hirsch number of a polycyclic group G by $h(G)$.

LEMMA 1. *Let G be a finitely generated nilpotent group, T its torsion subgroup, and $H = G/T$. Then for any i,*

$$r(\gamma_i(G)/\gamma_{i+1}(G)) = r(\gamma_i(H)/\gamma_{i+1}(H)).$$

Proof. Set $\gamma_i(G) = G_i$ and $\gamma_i(H) = H_i$, so $H_i = G_iT/T$. By the isomorphism theorems,

$$\begin{aligned} H_i/H_{i+1} &= (G_iT/T)/(G_{i+1}T/T) \cong G_iT/G_{i+1}T \cong G_i/(G_i \cap G_{i+1}T) \\ &\cong (G_i/G_{i+1})/((G_i \cap G_{i+1}T)/G_{i+1}). \end{aligned}$$

Note further that $G_i \cap G_{i+1}T = (G_i \cap T)G_{i+1}$, so

$$(G_i \cap G_{i+1}T)/G_{i+1} = (G_i \cap T)G_{i+1}/G_{i+1} \cong (G_i \cap T)/(G_{i+1} \cap T).$$

Since T is periodic, this shows that H_i/H_{i+1} is isomorphic to a quotient of G_i/G_{i+1} modulo a periodic subgroup, and $r(G_i/G_{i+1}) = r(H_i/H_{i+1})$ follows. ∎

COROLLARY 1. *Let $\mathfrak{V}$ be a nilpotent variety. Then*

$$g_{\mathfrak{V}}(n) = g_{\mathfrak{V}_0}(n)$$

for every n, and so $\mathrm{gr}(\mathfrak{V}) = \mathrm{gr}(\mathfrak{V}_0)$.

Proof. Let G be the free group of rank n of $\mathfrak{V}$ and let T be its torsion part. Then G/T is the free group of rank n of $\mathfrak{V}_0$. By the Bass formula (7), the growth of a finitely generated nilpotent group is equal to $\Sigma ir(i)$, where $r(i)$ is the torsion-free rank of the ith quotient of the lower central series of this group. Together with Lemma 1, this implies that the groups G and G/T have the same growth, whence the claim follows. ∎

LEMMA 2. *Let G be a finitely generated torsion-free nilpotent group, and let p be a prime greater than the nilpotency class of G. Then the composition length of the group G/G^p is equal to $h(G)$, the Hirsch number of G.*

Proof. Our statement is equivalent to saying that if $h(G) = n$ then $|G/G^p| = p^n$. We argue by induction on $h(G)$. If $h(G) = 1$, then G is infinite cyclic and the claim is obvious. Let $h(G) = n$. Since G is a finitely generated torsion-free nilpotent group, there is an infinite cyclic normal subgroup H of G such that $G/H = \overline{G}$ is torsion-free. Then $h(\overline{G}) = n - 1$ and, by the induction hypothesis, $|\overline{G}/\overline{G}^p| = p^{n-1}$.

We prove that $H \cap G^p = H^p$. Take any $h \in H \cap G^p$, then $h = g_1^p \cdots g_k^p$ for some $g_i \in G$. Since p is greater than the nilpotency class of G, a product of pth powers must be again a pth power (see for example [H] or [Hi1]). Therefore $h = g^p$ for some $g \in G$ and, since G/H is torsion-free, we have $g \in H$ and so $h \in H^p$.

It follows that $G^pH/G^p \cong H/(H \cap G^p) = H/H^p$, which is a group of order p. Furthermore,

$$\overline{G}/\overline{G}^p \cong (G/H)/(G^pH/H) \cong G/G^pH \cong (G/G^p)/(G^pH/G^p).$$

Since $|G^pH/G^p| = p$ and $|\overline{G}/\overline{G}^p| = p^{n-1}$, we get $|G/G^p| = p^n$. ∎

LEMMA 3. *Let $\mathfrak{V}$ be a torsion-free nilpotent variety of class c. Then $h(\mathrm{Fr}_n(\mathfrak{V}))$ is a polynomial in n of degree precisely c.*

Proof. It follows from a theorem of Higman [Hi1] that

$$\mathfrak{V} = \mathrm{var}\big(\mathfrak{V} \cap \mathfrak{B}_p | p > c\big).$$

Therefore there exists $p > c$ such that the nilpotency class of $\mathfrak{W} = \mathfrak{V} \cap \mathfrak{B}_p$ is precisely c. Set $\mathrm{Fr}_n(\mathfrak{V}) = G_n$. By Lemma 2, $h(G_n)$ is equal to the composition length of $G_n/G_n^p = \mathrm{Fr}_n(\mathfrak{W})$. It remains to apply another result of Higman [Hi2] which states that if $\mathfrak{W}$ is a nilpotent variety of finite exponent and of class precisely c, then as a function of n the composition length of $\mathrm{Fr}_n(\mathfrak{W})$ is a polynomial of degree precisely c. ∎

COROLLARY 2. *Let $\mathfrak{V}$ be a torsion-free nilpotent variety, $G_n = \mathrm{Fr}_n(\mathfrak{V})$, and*

$$r(i) = r\big(\gamma_i(G_n)/\gamma_{i+1}(G_n)\big).$$

Then $r(i)$ is a polynomial in n of degree precisely i.

Proof. Since $A_n = G_n/\gamma_i(G_n)$ and $B_n = G_n/\gamma_{i+1}(G_n)$ are the rank n free groups of the varieties $\mathfrak{V} \cap \mathfrak{N}_{i-1}$ and $\mathfrak{V} \cap \mathfrak{N}_i$, it follows from Lemma 3 that $h(A_n)$ and $h(B_n)$ are polynomials in n of degrees $i - 1$ and i, respectively. Since

$$r(i) = h(B) - h(A),$$

$r(i)$ is a polynomial of degree i. ∎

PROPOSITION 1. *Let $\mathfrak{V}$ be a nilpotent variety and let c be the nilpotency class of $\mathfrak{V}_0$. Then $\mathrm{gr}(\mathfrak{V}) = c$.*

Proof. Let $\mathfrak{V}$ be a nilpotent variety. By Corollary 1 we may assume that $\mathfrak{V}$ is torsion-free. As usual, let c be the nilpotency class of $\mathfrak{V}$ and let $G_n = \mathrm{Fr}_n(\mathfrak{V})$. First we find the growth of G_n. Take the lower central series

$$G_n = \gamma_1(G_n) \supset \gamma_2(G_n) \supset \cdots \supset \gamma_{c+1}(G_n) = 1;$$

by choosing n large enough we may guarantee that the class of G is precisely c. From (7) we have

$$\mathrm{gr}(G_n) = \sum_{i=1}^{c} ir(i)$$

where $r(i) = r(\gamma_i(G_n)/\gamma_{i+1}(G_n))$. By Corollary 2, $r(i)$ is a polynomial of degree i in n, hence $\mathrm{gr}(G_n)$ is a polynomial of degree c: $\mathrm{gr}(G_n) = a_0 n^c + a_1 n^{c-1} + \cdots$ with $a_0 \neq 0$. Therefore

$$\begin{aligned}\mathrm{gr}(\mathfrak{V}) &= \overline{\lim_n} \log_n(g_{\mathfrak{V}}(n)) = \overline{\lim_n} \log_n(\mathrm{gr}(G_n)) \\ &= \overline{\lim_n} \log_n\left(a_0 n^c + a_1 n^{c-1} + \cdots\right) = c. \quad \blacksquare\end{aligned}$$

LEMMA 4. *For arbitrary varieties of groups* $\mathfrak{V}$ *and* $\mathfrak{W}$,

$$\mathrm{gr}(\mathfrak{V} \vee \mathfrak{W}) = \max\{\mathrm{gr}(\mathfrak{V}), \mathrm{gr}(\mathfrak{W})\}.$$

Proof. Clearly $\mathrm{gr}(\mathfrak{V} \vee \mathfrak{W}) \geq \max\{\mathrm{gr}(\mathfrak{V}), \mathrm{gr}(\mathfrak{W})\}$. To prove the reverse, note that $\mathrm{Fr}_n(\mathfrak{V} \vee \mathfrak{W}) = F_n/(V \cap W)$, where V and W are the corresponding verbal subgroups of F_n. Therefore

$$\mathrm{Fr}_n(\mathfrak{V} \vee \mathfrak{W}) \subseteq (F_n/V) \times (F_n/W) = \mathrm{Fr}_n(\mathfrak{V}) \times \mathrm{Fr}_n(\mathfrak{W}).$$

Since $\mathrm{gr}(A \times B) \leq \mathrm{gr}(A) + \mathrm{gr}(B)$ for any groups A and B, it follows that $g_{\mathfrak{V} \vee \mathfrak{W}}(n) \leq g_{\mathfrak{V}}(n) + g_{\mathfrak{W}}(n)$. But then

$$\begin{aligned}\mathrm{gr}(\mathfrak{V} \vee \mathfrak{W}) &= \overline{\lim} \log_n(g_{\mathfrak{V} \vee \mathfrak{W}}(n)) \\ &\leq \overline{\lim} \log_n(g_{\mathfrak{V}}(n) + g_{\mathfrak{W}}(n)) \\ &\leq \max\{\overline{\lim} \log_n(g_{\mathfrak{V}}(n), \overline{\lim} \log_n(g_{\mathfrak{W}}(n))\} \\ &= \max\{\mathrm{gr}(\mathfrak{V}), \mathrm{gr}(\mathfrak{W})\}. \quad \blacksquare\end{aligned}$$

Note that Lemma 4 is valid even if one of the varieties is of infinite growth, or of growth 0 (that is to say, it is locally finite).

LEMMA 5. *Every decomposable variety of groups has infinite growth, unless it is locally finite.*

Proof. Let $\mathfrak{V}$ be a decomposable variety, that is, $\mathfrak{V} = \mathfrak{X}\mathfrak{Y}$ for nontrivial varieties $\mathfrak{X}$ and $\mathfrak{Y}$. By (A), we may restrict ourselves to the case when $\mathfrak{V} \subseteq \mathfrak{N}_c\mathfrak{U}$ for some c and some locally finite variety $\mathfrak{U}$. If both $\mathfrak{X}$ and $\mathfrak{Y}$ are of nonzero exponent, then so is $\mathfrak{V}$ and, as a subvariety of $\mathfrak{N}_c\mathfrak{U}$, $\mathfrak{V}$ will be locally finite. Thus we may assume that either $\exp \mathfrak{X} = 0$ or $\exp \mathfrak{Y} = 0$. The latter is impossible, for otherwise $\mathfrak{V}$ would contain a subvariety of the form $\mathfrak{A}_p\mathfrak{A}$ which is not contained in any $\mathfrak{N}_c\mathfrak{U}$. Hence $\exp \mathfrak{X} = 0$, and so $\mathfrak{V}$ contains a subvariety of the form $\mathfrak{A}\mathfrak{A}_p$.

To complete the proof, it is enough to show that $\mathfrak{W} = \mathfrak{A}\mathfrak{A}_p$ is a variety of infinite growth. But this is easy, because

$$g_{\mathfrak{W}}(n) = (n-1)p^n + 1,$$

so $g_{\mathfrak{W}}(n)$ is an exponential function. Indeed, the free group $\mathrm{Fr}_n(\mathfrak{W})$ of rank n in $\mathfrak{W}$ is an extension of a free abelian group A of rank $r = (n-1)p^n + 1$ by an elementary abelian group of order p^n (by Schreier's formula). The growth of A is equal to r; the same is true for any finite extension of A. Thus $g_{\mathfrak{W}}(n) = \mathrm{gr}(\mathrm{Fr}_n(\mathfrak{W})) = r$. ∎

The rest of the proof is essentially based on results of Groves [Gr].

LEMMA 6. *Let $\mathfrak{V}$ be a nilpotent-by-(locally finite) variety, that is, $\mathfrak{V} \subseteq \mathfrak{N}_c\mathfrak{U}$ for some c and some locally finite variety $\mathfrak{U}$. If $\mathfrak{A}\mathfrak{A}_p \not\subseteq \mathfrak{V}$ for all primes p, then $\mathfrak{V} \subseteq \mathfrak{B}_n\mathfrak{A}^t$ for some natural numbers n and t.*

Proof. Consider first the case when $\mathfrak{V} \subseteq \mathfrak{A}\mathfrak{B}_m$ for some m. Repeating literally the proof of Lemma C in [Gr], we obtain that then $\mathfrak{V} \subseteq \mathfrak{B}_n\mathfrak{N}_t$ for some n and t. (The proof in [Gr] used the assumption that $\mathfrak{V}$ is contained in a product of finitely many varieties each of which is either soluble or Cross, but used it only to ensure that all finite exponent subvarieties of $\mathfrak{V}$ are locally finite. This also follows from our hypothesis.)

Since $\mathfrak{V}$ is nilpotent-by-(locally finite), for some m and s we have $\mathfrak{V} \subseteq \mathfrak{A}^s\mathfrak{B}_m$. Now we can proceed by induction on s. For $s = 1$, the result follows from the above. Suppose that $s > 1$. Then

$$\mathfrak{V} \subseteq \mathfrak{A}(\mathfrak{A}^{s-1}\mathfrak{B}_m \cap \mathfrak{V})$$

and, by the induction hypothesis, $\mathfrak{A}^{s-1}\mathfrak{B}_m \cap \mathfrak{V} \subseteq \mathfrak{B}_n\mathfrak{A}^t$ for some n and t. It follows that $\mathfrak{V} \subseteq \mathfrak{A}\mathfrak{B}_n\mathfrak{A}^t$, whence

$$\mathfrak{V} \subseteq (\mathfrak{A}\mathfrak{B}_n \cap \mathfrak{V})\mathfrak{A}^t.$$

Again by the induction hypothesis we have $\mathfrak{A}\mathfrak{B}_n \cap \mathfrak{V} \subseteq \mathfrak{B}_k\mathfrak{A}^r$ for some k and r, so that $\mathfrak{V} \subseteq \mathfrak{B}_k\mathfrak{A}^r\mathfrak{A}^t \subseteq \mathfrak{B}_k\mathfrak{A}^{r+t}$. ∎

Now we can prove Part (B) of Theorem 1. Let $\mathfrak{V}$ be a variety of finite growth. It follows from (A) that $\mathfrak{V} \subseteq \mathfrak{N}_c\mathfrak{U}$ for some c and a locally finite variety $\mathfrak{U}$. Moreover, by Lemma 5, $\mathfrak{V}$ may not contain subvarieties of the form $\mathfrak{A}\mathfrak{A}_p$. Therefore Lemma 6 implies that $\mathfrak{V} \subseteq \mathfrak{B}_n\mathfrak{A}^t$ for some natural numbers n and t.

Set $\mathfrak{W} = \mathfrak{V} \cap \mathfrak{A}^t$, and note that $\mathfrak{V} \subseteq \mathfrak{B}_n\mathfrak{W}$. Since $\mathfrak{W}$ is a soluble variety and $\mathfrak{A}\mathfrak{A}_p \not\subseteq \mathfrak{W}$ for all primes p, Theorem C(i) in [Gr] guarantees that $\mathfrak{W} \subseteq \mathfrak{B}_m \mathfrak{N}_d$ for some m and d. It follows that

$$\mathfrak{V} \subseteq \mathfrak{B}_n \mathfrak{B}_m \mathfrak{N}_d \subseteq \mathfrak{B}_{nm} \mathfrak{N}_d.$$

Since we also have $\mathfrak{V} \subseteq \mathfrak{N}_c \mathfrak{B}_k$ where $k = \exp \mathfrak{U}$, Lemma A from [Gr] implies that there exist a nilpotent variety $\mathfrak{N}$ and a locally finite variety $\mathfrak{U}'$ such that $\mathfrak{V} = \mathfrak{N} \vee \mathfrak{U}'$.

It remains to note that $\mathrm{gr}(\mathfrak{V}) = \mathrm{gr}(\mathfrak{N}) =$ nilpotency class of $\mathfrak{N}_0$ (Lemma 4 and Proposition 1).

Note. It follows from the proof of Theorem 1 that there is a gap in the growth of varieties of groups: the function $g_{\mathfrak{V}}$ grows either polynomially or at least exponentially. In other words, there are no varieties of "intermediate" growth.

3. VARIETIES OF RING TYPE

Following [V1], we first establish a connection between varieties of associative algebras over a field K and varieties of group representations over the same K. *For most of this section, it will be assumed that K is infinite*, but it is more convenient to start without that restriction. Let $K\langle\langle Z \rangle\rangle$ be the algebra (with 1) of formal power series over K in a countable set of noncommuting variables $Z = \{z_1, z_2, \ldots\}$. The elements $1 + z_i$ are invertible in $K\langle\langle Z \rangle\rangle$ and, according to Magnus [M], the map $x_i \mapsto 1 + z_i$ can be uniquely extended to a monomorphism of K-algebras $KF \to K\langle\langle Z \rangle\rangle$. In the following we will identify KF with its image in $K\langle\langle Z \rangle\rangle$, so that every $u \in KF$ can be uniquely written as a formal power series

$$u = u_{(0)} + u_{(1)} + \cdots + u_{(n)} + \cdots$$

where $u_{(n)} = \sum \lambda_{i_1 \cdots i_n} z_{i_1} \cdots z_{i_n}$ is a finite K-linear combination of monomials of degree n in the z_i. But then the free associative algebra $K\langle Z \rangle$ (without 1!) is naturally contained in KF, so we have an embedding of K-algebras

$$K\langle Z \rangle \subset KF \subset K\langle\langle Z \rangle\rangle.$$

The algebra $K\langle\langle Z \rangle\rangle$ has a natural filtration

$$K\langle\langle Z \rangle\rangle \supset A \supset A^2 \supset \cdots \supset \bigcap_{n=1}^{\infty} A^n = 0, \tag{10}$$

where A is the ideal of series without constant terms. For each subset $S \subseteq K\langle\langle Z\rangle\rangle$ let $\hat{S}$ denote its completion with respect to this filtration.

Now let $\mathscr{M}$ be a variety of (associative) algebras over K, and let $T = T(\mathscr{M})$ be the corresponding verbal ideal (or T-ideal) of $K\langle Z\rangle$. Set $\alpha T = \hat{T} \cap KF$. In other words,

$$\alpha T = \{u \in KF | \forall n: u_{(n)} \in T\}. \tag{11}$$

One can prove that αT is a verbal ideal of KF, so that it determines a variety of group representations which we denote by $\alpha\mathscr{M}$. The map $\mathscr{M} \mapsto \alpha\mathscr{M}$ yields the desired connection between the varieties of algebras and varieties of group representations over the field K.

This connection has a number of important properties, from which we recall two. First, if K is infinite, then the map α is injective. Second, if $\operatorname{char} K = 0$, then $\operatorname{Im}\alpha$ can be completely described: it consists of the so-called homogeneous Magnus varieties. For details we refer the reader to [V1] or [V2, Sect. 1.3].

From now on we assume that K is infinite.

In Section 2 we considered the injective map $\mathfrak{V} \mapsto \omega\mathfrak{V}$ from varieties of groups to varieties of group representations. Now we also have an injective map $\mathscr{M} \mapsto \alpha\mathscr{M}$ from varieties of associative algebras to varieties of group representations; it is natural to say that the varieties $\alpha\mathscr{M}$ are of *ring type*. Our aim here is to solve Problems 1 and 2 for such varieties. The answers turn out to be somewhat trivial.

THEOREM 2. *Let $\mathscr{X} = \alpha\mathscr{M}$ be an arbitrary variety of ring type.*[1] *Then*:
(a) *$g_{\mathscr{X}}(n)$ is finite for every n*;
(b) *if* $\operatorname{char} K = 0$ *then* $\operatorname{gr}(\mathscr{X}) = 1$.

Proof. The proof consists of two steps. First we prove that the analogues of (a) and (b) are valid for varieties of algebras. Second we show that the case of varieties of group representations of ring type can be reduced to that of varieties of algebras (although the reduction is not as straightforward as for varieties of group type and the map ω).

1. Let $\mathscr{M}$ be a nontrivial variety of algebras and let $T = T(\mathscr{M})$. Set $T_n = T \cap K\langle Z_n\rangle$, where $K\langle Z_n\rangle$ is the free associative algebra on $z_1, \dots, z_n$. Then $K\langle Z_n\rangle/T_n$ is the free algebra of rank n of $\mathscr{M}$.

The growth function $g_{\mathscr{M}}(n)$ and the growth $\operatorname{gr}(\mathscr{M})$ of the variety $\mathscr{M}$ are defined as described in Section 1:

$$g_{\mathscr{M}}(n) = \operatorname{GKdim}(K\langle Z_n\rangle/T_n), \qquad \operatorname{gr}(\mathscr{M}) = \overline{\lim_n} \log_n(g_{\mathscr{M}}(n)).$$

[1]As agreed in Section 1, we assume that $\mathscr{X}$ is not locally finite-dimensional.

Since $K\langle Z_n\rangle/T_n$ is a finitely generated PI-algebra, and the Gel'fand–Kirillov dimension of a finitely generated PI-algebra is finite [Be1, D], we have that

$$g_{\mathscr{M}}(n) \text{ is finite for all } n.$$

Moreover, it was recently proved by Berele [Be2] that if R is a PI-algebra over a field of characteristic 0 then there exists a linear function $f(n)$ such that the Gel'fand–Kirillov dimension of every n-generated subalgebra of R is at most $f(n)$. Thus $g_{\mathscr{M}}(n)$ is bounded by a linear function of n, and so

$$\text{if char } K = 0 \text{ then } \operatorname{gr}(\mathscr{M}) = 1.$$

(Note that $\operatorname{gr}(\mathscr{M})$ cannot be less than 1. Indeed, we may assume that $\mathscr{M}$ is not nilpotent; otherwise it is locally finite-dimensional and there is nothing to speak about. But then $\mathscr{M}$ contains the variety of all commutative algebras, whose growth is equal to 1.)

2. Now we return to varieties of group representations. Let $\mathscr{X} = \alpha\mathscr{M}$ be a variety of group representations of ring type. As usual, we write

$$T = T(\mathscr{M}), \qquad T_n = T \cap K\langle Z_n\rangle, \qquad I = \operatorname{Id}(\mathscr{X}), \qquad I_n = I \cap KF_n.$$

Consider the $\mathscr{X}$-free representation of countable rank $\operatorname{Fr}(\mathscr{X}) = (KF/I, F)$. Our next aim is to show that the algebra KF/I satisfies some multilinear identity of $\mathscr{M}$. Recall that $I = \hat{T} \cap KF$, where $\hat{T}$ is the completion of T with respect to the filtration (10). Clearly

$$\hat{T} = \bigcap_{n=1}^{\infty} (A^n + T).$$

Furthermore, $A/(A^n + T) \in \mathscr{M}$ for any n, and so

$$A/\hat{T} \subseteq \prod A/(A^n + T) \in \mathscr{M}.$$

Since the ground field K is infinite, some nontrivial multilinear identity $f(z_1, \dots, z_m)$ must hold in $\mathscr{M}$. Set

$$f([u_1, v_1], [u_2, v_2], \dots, [u_m, v_m]) = g(u_1, v_1, u_2, \dots, u_m, v_m)$$

where $[u, v] = uv - vu$; then $g(u_1, v_1, u_2, \dots, u_m, v_m)$ is also a nontrivial multilinear identity of $\mathscr{M}$. In addition, each of the

$$g(1, v_1, u_2, \dots, v_m), g(u_1, 1, u_2, \dots, v_m), \dots, g(u_1, v_1, \dots, u_m, 1)$$

is the zero polynomial. Since g is an identity of $A/\hat{T}$, it follows from the latter property that it is an identity of the algebra

$$K\langle\langle Z\rangle\rangle/\hat{T} = K\cdot 1 \oplus \left(A/\hat{T}\right)$$

and therefore also of the subalgebra

$$(KF + \hat{T})/\hat{T} \cong KF/(\hat{T} \cap KF) = KF/I,$$

as required.

Thus KF/I is a PI-algebra. Therefore $g_{\mathscr{X}}(n) = \mathrm{GKdim}(KF_n/I_n)$ is finite for any n. Moreover, by the result we quoted from [Be2], there is a linear function $f(n) = kn$ such that the Gel'fand–Kirillov dimension of every n-generated subalgebra of KF/I is at most kn. Since KF_n/I_n is a subalgebra of KF/I generated by $2n$ elements,

$$g_{\mathscr{X}}(n) = \mathrm{GKdim}(KF_n/I_n) \le 2kn,$$

whence $\mathrm{gr}(\mathscr{X}) = 1$. ∎

In conclusion we recall that our work was motivated by the following question: for a given variety $\mathscr{X}$, what is the rate of growth of the function $g_{\mathscr{X}}\colon \mathbb{N} \to \mathbb{R}$, where $g_{\mathscr{X}}(n)$ is the rate of growth of the rank n free object in $\mathscr{X}$? For this purpose, in (6) we introduced an asymptotic invariant, $\mathrm{gr}(\mathscr{X})$, of the variety $\mathscr{X}$. This invariant is convenient if the function $g_{\mathscr{X}}$ is growing polynomially. If it is not the case, there are other ways to estimate the rate of growth of $g_{\mathscr{X}}$. Some types of a superpolynomial growth (of the order function of a locally finite variety of groups) were discussed by Higman in [Hi2], and one can follow his example.

On the other hand, the growth of varieties of associative algebras (over fields of characteristic 0) need not be classed simply as linear, but could be differentiated according to the slope of that linear function. Explicitly, in this context we could change to the definition

$$\mathrm{gr}(\mathscr{M}) = \overline{\lim_{n}} \frac{g_{\mathscr{M}}(n)}{n}. \tag{12}$$

For example, it is known that if $\mathscr{M}_k$ is the variety generated by the algebra of $k \times k$ matrices, then $\mathrm{GKdim}(\mathrm{Fr}_n(\mathscr{M}_k)) = k^2(n-1) + 1$. In terms of (12), this gives that $\mathrm{gr}(\mathscr{M}_k) = k^2$. Another example: if $\mathscr{T}_k$ is the variety generated by the algebra of triangular $k \times k$ matrices, then $\mathrm{gr}(\mathscr{T}_k) = k$ (personal communication of Berele). It would be interesting to solve an analogue of Problem 2 in this situation.

Note that so far we do not have examples of a variety (of any sort) whose growth is a non-integer.

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