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On the Number of Conjugacy Classes of a Finite Group

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INTRODUCTION

We are concerned here with determining general properties of $k(G)$, the number of conjugacy classes of a finite group G. In the first section, we prove that any subgroup, G, of the symmetric group S_d satisfies $k(G) \leqslant 5^{d-1}$. We reduce the proof of a proposed improvement of this to $k(G) \leqslant 2^{d-1}$ to the case that G is "almost simple", using the O'Nan-Scott theorem (see [1] or [12], for example). We also give examples to show that there is lower bound of the form c^d ($c > 1$, a constant) for the maximum number of conjugacy classes of a transitive subgroup of the symmetric group S_d (a lower bound of this nature being essentially obvious if we allow intransitive subgroups).

In the second section, we improve the above bound in the case that G is a *solvable* subgroup of the symmetric group S_d to $k(G) \leqslant (\sqrt{3})^{d-1}$ for $d > 2$.

In the third section, we turn our attention to linear groups. We prove that if G is a finite solvable subgroup of $GL(n, \mathbb{C})$, then $k(H) \leqslant 3^{n-1}$ for every subgroup H of $G/F(G)$. This has an application to the so-called "$k(GV)$-problem". We prove:

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THEOREM. *Let G be a finite solvable group with $|F(G)| = p^r$ for some prime p, some positive integer r. Then $k(G) \leqslant 3^{r-1}p^r$. (Note that we do* not *assume that $G/F(G)$ and $F(G)$ have coprime orders here).*

This has an application to Brauer's $k(B)$ problem in the case of solvable groups.

COROLLARY. *Let G be a finite solvable group, p be a prime, B be a p-block of G of positive defect d. Then $k(B) \leqslant 3^{d-1}p^d$. In particular, if $p > 7$, then $k(B) < p^{(3d-1)/2}$.*

In the final section, we consider analogues of the results of Section 3 for insoluble groups. We prove, using the classification of the finite simple groups:

THEOREM. *There is a constant c such that whenever G is a finite subgroup of $GL(n, \mathbb{C})$ and H is a subgroup of $G/F(G)$, then $k(H) \leqslant c^{n-1}$.*

This has as consequences:

THEOREM. There is a constant c (the same as in the theorem above) such that whenever p is a prime and G is a p-solvable group with $|F^*(G)| = p^r > 1$ (or equivalently, $O_{p'}(G) = \{1_G\}$ and $|O_p(G)| = p^r > 1$), we have $k(G) \leqslant c^{r-1}p^r$.

THEOREM. *There is a constant c (the same as above, and independent of the prime p and the defect d) such that whenever p is a prime, G is a p-solvable group, and B is a p-block of G of positive defect d, then $k(B) \leqslant c^{d-1}p^d$.*

Some Assumed Results

We collect here some results which will be used frequently throughout the paper, and will sometimes be quoted without explicit reference.

A1 (see P. X. Gallagher, [7]). Let G be a finite group, H be a subgroup of G. Then $[G : H]^{-1}\, k(H) \leqslant k(G) \leqslant [G : H]\, k(H)$.

A2 (P. X. Gallagher, [7]). Let G be a finite group, N be a normal subgroup of G, χ be an irreducible character of N, $I(\chi)$ be its inertial subgroup. Then the number of irreducible characters of G which lie over (G-conjugates of) χ is at most $k(I(\chi)/N)$.

(A3) (See P. X. Gallagher [7] or H. Nagao [14]). Let G be a finite group, N be a normal subgroup of G. Then $k(G) \leqslant k(N)\, k(G/N)$.

(A4) (Dixon, Winter, see Isaacs [10]). Let G be a p-solvable linear group of degree n (over a field of characteristic p). Then $[G : O_p(G)]_p \leqslant p^{4n/3}$, and if p is odd, then $[G : O_p(G)]_p \leqslant p^{np/(p-1)^2}$.

A5. Let G be a p-constrained group with $O_{p'}(G)=\{1_G\}$. Then

$$k(G)\leqslant |G|_p \qquad \text{if } p^3 \text{ does not divide } |G|.$$
$$k(G)\leqslant |G|_p^2/p^2 \qquad \text{if } p^3 \text{ divides } |G|.$$

(This follows from Brauer-Feit [4], since G has only one p-block by a minor variant of a theorem of P. Fong).

A6. Let G be a solvable completely reducible subgroup of $GL(n, q)$ for some prime q. Let $r\geqslant n+1$ be a prime. Then G is r-closed, except possibly when r is a Fermat prime and $O_{r,2}(G)>O_r(G)$.

In any case, if $r\geqslant n+2$, then G is r-closed (possibly $q=r$ here).

(This is well-known, and follows from the Hall–Higman theorem, after "lifting" G to a subgroup of $GL(n, \mathbb{C})$, then reducing (mod r), extending scalars to F, the algebraic closure of $GF(r)$, and considering $G/O_r(G)$ as a completely reducible subgroup of $GL(n, F)$ on the direct sum of the composition factors of the module obtained).

A7. Let G be a finite subgroup of $GL(2, \mathbb{C})$. Then either G has an Abelian normal subgroup of index at most 2, or $Z(G)$ has even order and $G/Z(G)$ is isomorphic to one of

$$A_4, S_4, \qquad \text{or} \qquad A_5$$

(see part A of [6], for example).

1. On Permutation Groups

In this section, we give a general bound for the number of conjugacy classes of a permutation group of degree d. The constant we obtain in the first theorem is obtained by using a theorem of Praeger and Saxl ([15]). Theorems of L. Babai ([2] and [3]) have the consequence that for sufficiently large d, the order of a *primitive* permutation group of degree d (not containing A_d) is at most 2^{d-1}. It will become apparent to the reader from the proof below that if it could be established that all primitive permutation groups of degree d have at most 2^{d-1} conjugacy classes, then the same inequality would follow for all permutation groups.

The results of Babai and Praeger–Saxl are independent of the classification of finite simple groups, so that Theorem 1.2 below is also independent of the classification.

It is known that there is a constant A such $\pi(d)$, the number of partitions of d (also the number of conjugacy classes of the symmetric group S_d), is always smaller than $\exp(A\sqrt{d})$. However, we present an elementary proof of a weaker upper bound for the number of conjugacy classes of the symmetric group S_d.

LEMMA 1.1. *For every integer $d \geqslant 2$, $k(S_d) \leqslant 2^{d-1}$ and $k(A_d) \leqslant 2^{d-1}$.*

Proof. This is true by inspection when $d < 5$. We prove by induction that for $d \geqslant 5$, $k(S_d) < 2^{d-2}$, and then both claims of the lemma will follow by virtue of A1. Again, the stated inequality is true by inspection for $5 \leqslant d < 10$, so we assume that $d \geqslant 10$, and the result has been established for smaller values of d (greater than 4).

Any permutation of $\{1, 2, ..., d\}$ except for a d-cycle has its shortest cycle of length $d/2$ or less when expressed as a product of disjoint cycles. Every permutation whose shortest cycle is an a-cycle is conjugate within S_d to a permutation whose shortest cycle is $(12 \cdots a)$. Let G_a denote the elementwise stabilizer of $\{1, 2, ..., a\}$ in S_d, which is isomorphic to S_{d-a} when $a < d$. Then, by induction, when $d - a \geqslant 5$, (so certainly when $a \leqslant d/2$), there are fewer than $2^{(d-a)-2}$ G_a-conjugacy classes of permutations whose shortest cycle is $(12 \cdots a)$.

Letting q denote the integer part of $d/2$, it follows that there at most $1 + 2^{d-(2+q)}[1 + 2 + \cdots + 2^{q-1}]$ conjugacy classes of S_d, and this is certainly less than 2^{d-2}.

We can now prove:

THEOREM 1.2. *Let G be any subgroup of the symmetric group S_d. Then G has at most 5^{d-1} conjugacy classes.*

Proof. The result is vacuous when $d = 1$, so we assume that $d > 1$, and that the result has been established for smaller values of d.

We may assume that G is transitive, for if Ω_1 is one orbit of G on $\{1, 2, ..., d\}$ of length $d_1 < d$, let $\Omega_2 = \{1, 2, ..., d\} - \Omega_1$, $d_2 = d - d_1$, $K_i = C_G(\Omega_i)$ for each i.

Then $k(G) \leqslant k(K_1)\, k(G/K_1)$, and K_1 is faithfully represented as a group of permutations of Ω_2, G/K_1 is faithfully represented as a group of permutations of Ω_1, so that by hypothesis $k(G) \leqslant 5^{d_1 - 1} 5^{d_2 - 1} = 5^{d-2}$.

We may assume that G is primitive, for let $H = G_1$, and suppose that $H \max L < G$. Let $a = [L : H]$, $b = [G : L]$, and let $C = \mathrm{core}_G(L)$. Then G/C is isomorphic to a subgroup of the symmetric group S_b, so that $k(G/C) \leqslant 5^{b-1}$. Also, C has b orbits on $\{1, 2, ..., d\}$, each of length a, and the argument used above shows that $k(C) \leqslant (5^{a-1})^b$. Hence $k(G) \leqslant k(C)\, k(G/C) \leqslant 5^{d-1}$.

By Lemma 1.1, we may assume that G is neither S_d nor A_d. But then by the aforementioned theorem of Praeger and Saxl, we even have $|G| \leqslant 4^d$. Now $5^{d-1} > 4^d$ for $d > 7$, while for $3 \leqslant d \leqslant 7$, $|G| < d!/2 < 5^{d-1}$. Hence we certainly have $k(G) \leqslant 5^{d-1}$ in this case, and the proof is complete.

Remarks. There is no constant c such that for every d, every permutation group of degree d has *order* at most c^d, as the symmetric group S_d (with d sufficiently large compared to any specified c) illustrates. Furthermore, as soon as we stray away from primitive permutation groups, wreath product constructions allow examples other than the full symmetric and alternating groups violating such a bound.

When d is divisible by 4, the symmetric group S_d has a subgroup with $5^{d/4}$ conjugacy classes (a direct product of $d/4$ copies of S_4), so that there is a *lower* bound of the form c^d for any general estimate of the maximum number of conjugacy classes of a permutation group of degree d.

Even if we restrict our attention to transitive subgroups of S_d, such a lower bound still exists:

LEMMA 1.3. *Let $d = 3^{n+1}$, where n is a non-negative integer. Then a Sylow 3-subgroup of the symmetric group S_d has more than γ^d conjugacy classes, where $\gamma = 3^{1/6}$.*

Proof. By A1, we have $k(G \text{ wr } C_p) \geqslant k(G)^p/p$ for any finite group G. An easy induction argument shows that $k(C_p \text{ wr } C_p \text{ wr } C_p \cdots \text{wr } C_p) \geqslant p^{1/(p-1)} \cdot \beta^{p^{n-1}}$, where $\log_p(\beta) = (p-2)/(p-1)$, and there are $n+1$ occurrences of C_p in the iterated wreath product. The lemma (and analogous results for other primes) follow easily.

Remark. We illustrate that, with the aid of the O'Nan–Scott theorem, it is possible to reduce the proof of the claim that every permutation group G of degree d has at most 2^{d-1} conjugacy classes to the case that $T \leqslant G \leqslant \operatorname{Aut}(T)$ for some non-Abelian simple group T.

The reduction to the case that G is primitive follows the lines used in Theorem 1.2. By the O'Nan–Scott theorem there are three cases (other than the "almost simple" one) to consider:

(a) *G of affine type.* Here, $d = p^n$ for some prime p, G has a minimal normal subgroup M of order p^n with $C_G(M) = M$.

In this case, a Sylow p-subgroup of G has order dividing $d^{(n+1)/2}$ and, by A5, $k(G) \leqslant d^{n+1}$. We only need to consider cases in which $d^{n+1} > 2^{d-1}$, or $(n+1)\log_2(d) > (d-1)$.

If p is odd, this forces $p = 3$, $n \leqslant 2$, or $p = 5$, $n = 1$. Both cases where $n = 1$ yield $k(G) \leqslant d < 2^{d-1}$. If $p = 3$ and $n = 2$, then $k(G) \leqslant 27k(S)$ where S is a Sylow 2-subgroup of G. But S is isomorphic to a subgroup of a semidihedral group of order 16, so $k(S) \leqslant 8$, $k(G) \leqslant 216 < 2^{d-1} = 256$.

If $p = 2$, this forces $n^2 + n > 2^n - 1$, $n < 5$. By inspection, we only need to consider $n=3$ and $n=4$. It is easy to eliminate the case $n=3$, since every subgroup of $GL(3,2)$ has 7 or fewer conjugacy classes, so that $k(G) \leqslant 56 < 2^7 = 2^{d-1}$.The case $n=4$ is almost as easy: Let $X = G/M$, regarded as a subgroup of $GL(4,2)$. Let P be a maximal parabolic of $GL(4,2)$, U be its unipotent radical. Then $[X : X \cap P] \leqslant 15$, and $k(X \cap P) \leqslant k(X \cap P \cap U)$ $k(X \cap P/X \cap P \cap U) \leqslant 56$, since U has order 8, and $(X \cap P)/(X \cap P \cap U)$ is isomorphic to a subgroup of $GL(3,2)$. Thus $k(X) \leqslant 15 \times 56 < 2^{10}$. Hence $k(G) < 2^{14}$, and we wished to prove that $k(G) \leqslant 2^{d-1} = 2^{15}$.

(b) *G of product type.* Here, $d = a^b$ for certain integers a and b (both greater than 1), G is a subgroup of $H \operatorname{wr} S_b$ (in product action), where H is a primitive subgroup of S_a. By induction, we may assume that every subgroup of H has at most 2^{a-1} conjugacy classes, so that every subgroup of the base group B has at most 2^{ab-b} conjugacy classes. By induction, as $G/G \cap B$ is isomorphic to a subgroup of S_b, we may assume that $k(G/G \cap B) \leqslant 2^{b-1}$, and $k(G \cap B) \leqslant 2^{ab-b}$, so $k(G) \leqslant 2^{ab-1}$, and certainly $k(G) \leqslant 2^{d-1}$.

(c) *G of diagonal type.* Here, M, the socle of G, is isomorphic to a direct product of $r > 1$ copies of a non-Abelian simple group T, and $d = |T|^{r-1}$. Furthermore, G/M is isomorphic to a subgroup of $[\operatorname{Out} T] \times S_r$. By induction, we may assume that

$$k(G) \leqslant k(T)^r \, |\operatorname{Out} T| \, 2^{r-1}.$$

To dispose of this case, it suffices to prove that

$$r[\log_2(k(T)) + 1] + \log_2(|\operatorname{Out}(T)|) \leqslant |T|^{r-1}.$$

But T can be generated by fewer than $\log_2(|T|)$ elements, and no non-identity element of $\operatorname{Aut}(T)$ can fix all of those generators, so it follows that $|\operatorname{Aut}(T)| < |T|^{\log_2(|T|)}$. Also, it is certainly true that $k(T) < |T|$.

Hence it's enough to prove that $r + (r-1)x + x^2 \leqslant 2^{(r-1)x}$, where $x = \log_2(|T|)$. Since T has order at least 60, we certainly have $x > 5$.

It follows easily that the desired inequality is valid for $r > 2$. When $r = 2$, it is easy to check that $2 + y + y^2 \leqslant 2^y$ for any real number $y \geqslant 5$.

2. On Solvable Permutation Groups

In this section, we show that Theorem 1.2 can be considerably sharpened for solvable subgroups of the symmetric group S_d. This is not surprising since J. D. Dixon (see part A of [6], for example) has shown that every solvable subgroup of S_d has *order* at most $24^{(d-1)/3}$, which is already smaller than 5^{d-1}.

In Theorem 2.2 below, we will prove that if $d > 2$, then every solvable subgroup of S_d has at most $(\sqrt{3})^{d-1}$ conjugacy classes. Before we do this, we note a slightly more careful version of A2, which will be useful to us.

Let N be a normal subgroup of the finite group G, and suppose that *every* subgroup of G/N has at most t conjugacy classes. Let χ be an irreducible character of N. Then the number of irreducible characters of G which lie over (a G-conjugate of) χ is at most $k(I(\chi)/N)$, where $I(\chi)$ is the inertial subgroup of χ, so is at most t. Since the number of G-orbits on the irreducible characters of N is the number of G-conjugacy classes of N, we see that

A2′. $k(G) \leqslant t \times \#(G\text{-conjugacy classes of } N)$.

In the context we are presently considering this is useful, since if G/N is a subgroup of S_d, then of course so are all its subgroups.

Before we can prove Theorem 2.2, we need to carefully analyse the primitive case.

LEMMA 2.1. *Let Y be a solvable primitive permutation group of degree a, and let X be a subnormal subgroup of Y.*

Then:

(i) *If* $a > 4$, $k(X) \leqslant \sqrt{2}^a$.

(ii) *If a is prime, or* $a = 8$, $k(X) \leqslant a$.

(iii) *If* $a = 9$, $k(X) \leqslant 11$.

(iv) *If* $a > 2$, $k(X) \leqslant \sqrt{3}^{a-1}$.

Proof. Part (iv) follows immediately from part (i) if $a > 4$, and is true by inspection for $a = 3$ or 4.

Let Y be a primitive solvable permutation group of degree $a = p^n$, where p is prime, and $a > 4$.

Then $Y = TV$, where V is minimal normal in Y, of order a, $V = F(Y)$, T acts irreducibly on V, and $T \cap V$ is trivial. Let X be a subnormal subgroup of Y, $U = X \cap V$. Then $U = F(X)$. We may assume that a is *not* prime (and even that $|U|$ is not prime), since $k(X) \leqslant |U|$ if $|U|$ is prime (by inspection of the holomorph of C_p), and since $\sqrt{2}^a > a$ when $a > 4$. Hence we assume that $n > 1$, and that U has order greater than p.

In that case, X/U is a faithful p-solvable group of linear transformations of U, and $O_p(X/U)$ is trivial, so that $[X:U]_p \leqslant |U|^{4/3}$ by A4.

Let W be a Hall p'-subgroup of X. Then $k(WU) \leqslant |U|^2/4$, by A5, and $k(X) \leqslant [X : WU]\, k(WU)$ by A1, so that:

A8: $k(X) \leqslant \frac{1}{4}|U|^{10/3}$.

Certainly, then, $k(X) \leqslant \frac{1}{4}a^{10/3}$.

We wish to prove that $k(X) \leqslant (\sqrt{2})^a$. This will be true as long as $(a+4)/2 \geqslant 10\log_2(a)/3$. When $a \geqslant 32$, this inequality is satisfied. Since a is a prime power, but is not prime, and since $a > 4$, we only need to consider $a \in \{8, 9, 16, 25, 27\}$.

Case $a = 8$. In this case, there are only two possibilities for T. Either T is cyclic of order 7, or T is a Frobenius group of order 21. Since X is subnormal in Y, we have $X = VT$, or $X = VO_7(T)$, or else $X \leqslant V$. In any case, $k(X) \leqslant 8$ by direct inspection.

Case $a = 25$. In this case, T is isomorphic to a subgroup of $GL(2, 5)$, and T is a $5'$-group by A4. Then $k(X) \leqslant |U| \leqslant 25$, by A5, so that $k(X) \leqslant \sqrt{2}^{25}$.

Case $a = 27$. Now $\sqrt{2}^{27} = 8192\sqrt{2} > 11,300$. If $|U| \leqslant 9$, then by A8, $k(X) \leqslant 9^{10/3}/4 < 9^3 = 729$. Hence we may suppose that $V \leqslant X$. If 13 divides $[X:V]$, then X/V is 13-closed by A6. In that case $|F(X/V)| \leqslant 26$, on consideration of the centralizers of elements of order 13 in $GL(3, 3)$, and setting $M = (X/V)/O_2(X/V)$ we see that $k(M) \leqslant 13$, so $k(X/V) \leqslant 26$, $k(X) \leqslant 26 \cdot 27 < 729$.

If 13 does not divide $[X:V]$, then $[X:V]_{3'} \leqslant 32$, and $[X:V]_3 \leqslant 9$ by A4, so that $|X| \leqslant 27 \cdot 32 \cdot 9 < 9000$, and hence certainly $k(X) < 11,300$.

Case $a = 9$. If 3 does not divide $[X:U]$, then $k(X) \leqslant 9$ by A5. If 3 divides $[X:U]$, then as X is subnormal in Y, we must have $X \cong V \rtimes SL(2, 3)$ or $X \cong V \rtimes GL(2, 3)$. In these cases, we may check that $k(X) = 10, 11$ respectively.

Case $a = 16$. In this case, $(\sqrt{2})^a = 256$. If $|U| \leqslant 8$, then $k(X) \leqslant \frac{1}{4}8^{10/3}$ $(= 256)$ by A8. Hence we may assume that $V < X$.

In that case, consideration of the structure of $GL(4, 2)$ shows that $|F(X/V)| \in \{3, 5, 9, 15\}$. (If 7 divides $|T|$, then T is 7-closed by A6, but then $O_7(T)$ has a non-trivial fixed-point on V, contrary to the fact that T acts faithfully and irreducibly on V).

If $F(X/V)$ has order 3, then $[X:V] \leqslant 6$, $|X| \leqslant 96$. If $F(X/V)$ has order 5, then $k(X/V) \leqslant 5$, so $k(X) \leqslant 80$. If $F(X/V)$ has order 9, then X/V is 3-closed and hence $k(X/V) \leqslant 9$ by A5, so $k(X) \leqslant 144$.

If $F(X/V)$ has order 15, set $L = X/V$, $S = L/O_3(L)$. Then $C_S(O_5(S)) \leqslant O_5(S)$, so that $k(S) \leqslant 5$, $k(L) \leqslant 15$, and $k(X) \leqslant 240$.

In all cases, then, $k(X) \leqslant 256$, as required to complete the proof of the Lemma.

Now we can prove:

THEOREM 2.2. *Let d be a positive integer, G be a solvable subgroup of the symmetric group S_d. Suppose that G has a orbits of length 2 on $\{1, 2, ..., d\}$, b orbits not of length 2. Then $k(G) \leqslant 2^a \cdot (\sqrt{3})^{d-2a-b}$. In particular, if $d > 2$, then $k(G) \leqslant (\sqrt{3})^{d-1}$.*

Proof. Choose a counterexample with d minimal. As in earlier arguments, G is transitive. By inspection, $d > 5$. Let $H = G_1$. We first claim that if $H < L < G$, then either $[G:L] = 2$ or $[L:H] = 2$.

For let $[G:L] = f$, $[L:H] = e$, and suppose that both e and f are greater than 2. Let $C = \text{core}_G(L)$. Then $H < CH \leqslant L$. Let $h = [CH:H] \leqslant e$. Then C has ef/h orbits on $\{1, 2, ..., d\}$, each of length h. If $h > 2$, then by hypothesis, $k(C) \leqslant [(\sqrt{3})^{h-1}]^{ef/h} \leqslant (\sqrt{3})^{d-f}$. Also, G/C is isomorphic to a subgroup of the symmetric group S_f, so that by hypothesis $k(G/C) \leqslant (\sqrt{3})^{f-1}$. Hence $k(G) \leqslant (\sqrt{3})^{d-1}$, contrary to the choice of G and d. Thus $h = 2$, so that $L > CH$. Hence $d = [G:L][L:CH][CH:H] \geqslant 4f$.

But now C is an elementary Abelian 2-group, so that $k(C) \leqslant 2^{d/2}$, while $k(G/C) \leqslant (\sqrt{3})^{f-1} \leqslant (\sqrt{3})^{(1/4\,d)-1}$. Hence $k(G) \leqslant (\sqrt{2})^d \cdot (\sqrt{3})^{(1/4\,d)-1}$. But we are assuming that $k(G) > (\sqrt{3})^{d-1}$, and this is easily seen to be a contradiction, as $2^{1/2} \cdot (\sqrt{3})^{1/4} < \sqrt{3}$.

Now we have four possibilities to consider:

(i) H max G.

(ii) H max K max G with $[G:K] = 2$.

(iii) H max K max G with $[K:H] = 2$.

(iv) H max K max L max G, with $[K:H] = [G:L] = 2$.

Case (i) is disposed of by Lemma 2.1.

Case (ii). Let $[K:H] = a$ (a prime power). Let $C = \text{core}_K(H)$. We claim that $a < 5$. For suppose otherwise. Then for any x in $G - K$, $C \cap C^x = \{1_G\}$, as $\text{core}_G(H)$ is trivial. Now K/C and K/C^x are both isomorphic to primitive permutation groups of degree a, and CC^x is normal in K, so by Lemma 2.1, we have $k(K/C) \leqslant (\sqrt{2})^a$ and $k(CC^x/C^x) \leqslant (\sqrt{2})^a$. Hence $k(K) \leqslant k(K/C)\, k(C/C \cap C^x) \leqslant 2^a$.

Now $k(G) \leqslant 2k(K) \leqslant 2^{a+1}$. But we are assuming that $k(G) > (\sqrt{3})^{2a-1}$, so we have $2\sqrt{3} > (3/2)^a$, which is false for $a \geqslant 4$.

Since we know that $d = 2a > 5$, we have $a = 3$ or $a = 4$. But if $a = 3$, then $|G| \leqslant 2\,|K/C|^2 \leqslant 72$, while we are assuming that G has more than $\sqrt{3}^5$ conjugacy classes, so at least 16 conjugacy classes. Now S_3 wr C_2 has 9 conjugacy classes, so that $|G| \leqslant 36$.

Since G has at least 16 conjugacy classes, this is easily seen to force G

to have order precisely 36 (for if G has order 18 or 24, then G is forced to be Abelian by elementary considerations). Then K has order 18, and C has order 3. Thus CC^x is a normal Sylow 3-subgroup of G. Since it is easy to see that G can have no Abelian normal subgroup of order greater than 9, $F(G) = CC^x$, and $k(G) \leqslant 9$ by A5.

If $a = 4$, then in the argument above, $k(K/C) \leqslant 5$ by inspection, and $k(C/C \cap C^x) \leqslant 5$ also, so that $k(G) \leqslant 50$. We are assuming that $k(G) \geqslant (\sqrt{3})^{2a-1}$, so $k(G) \geqslant 47$. This forces $k(K/C) = k(CC^x/C^x) = 5$, so since K/C and K/C^x are isomorphic to primitive permutation groups of degree 4, we see that $K/C \cong C \cong S_4$, and that $G > K$. Hence G is isomorphic to the full wreath product S_4 wr C_2, which only has 20 conjugacy classes, contrary to assumption.

Case (iii). Let $[G : K] = a$, again a prime power, $C = \mathrm{core}_G(K)$. We claim first that $a \leqslant 9$. For suppose that $a \geqslant 5$. Then C is an elementary Abelian 2-group of order at most 2^a, and G/C is isomorphic to a primitive permutation group of degree a, so by Lemma 2.1, $k(G) \leqslant (\sqrt{2})^{3a}$.

We are assuming that $k(G) > (\sqrt{3})^{2a-1}$, so that $(\sqrt{2})^{3a} > (\sqrt{3})^{2a-1}$ and hence $(9/8)^a < 3$. Since a is a prime power, this forces $a \leqslant 9$, as claimed.

If $a = 9$, then we have the sharper estimate $k(G/C) \leqslant 11$, so that $k(G) \leqslant 11 \cdot 2^9$, whereas we are assuming that $k(G) > (\sqrt{3})^{2a-1} = 3^9/\sqrt{3}$, which is easily seen to be a contradiction.

If $a = 7$ or 8, then $k(G/C) \leqslant a$, so $k(G) \leqslant a \cdot 2^a$, and in both cases, $(\sqrt{3})^{2a-1} > a2^a$, contrary to assumption.

If $a = 5$, then as G/C is isomorphic to a primitive solvable permutation group of degree 5, every subgroup of G/C has 5 or fewer conjugacy classes, so by A2′ we have the estimate $k(G) \leqslant 5$ #(G-conjugacy classes of C). Now C has order 32 or less, and G contains an element of order 5 which is a product of two 5-cycles, so has a centralizer of order at most 10. Hence there are at most eight G-conjugacy classes of C, and $k(G) \leqslant 40$. But we are assuming that $k(G) > (\sqrt{3})^9$, a contradiction.

If $a = 4$, then $k(G) > (\sqrt{3})^7$, so $k(G) \geqslant 47$. Now G/C is isomorphic to a primitive permutation group of degree 4, so G/C is isomorphic to S_4 or A_4. We must have $|C| = 16$, otherwise $k(G) \leqslant 40$. Now G contains an element x of order 3 which is a product of two 3-cycles, so that $C_C(x)$ has order (at most) 4. Hence there are at most eight G-conjugacy classes of C. But, using A2′, we have the sharper estimate $k(G) \leqslant 5$ #(G-conjugacy classes of C), since every subgroup of G/C has at most 5 conjugacy classes. This yields $k(G) \leqslant 40$, a contradiction.

If $a = 3$, then $k(G) > \sqrt{3}^5$, so $k(G) \geqslant 16$. Now G is isomorphic to a subgroup (of order divisible by 3) of the wreath product C_2 wr S_3, so has order dividing 48. If G has order 48, then $G \cong C_2$ wr $S_3 \cong C_2 \times S_4$, so G has 10 conjugacy classes, a contradiction. If G has order 24, then G is easily seen to be Abelian, since G has 16 or more conjugacy classes, a contradiction, as G is a transitive permutation group of degree 6.

This concludes the elimination of case (iii).

Case (iv). Let $[L:K]=a$ (a prime power). We claim that $a<5$. For suppose otherwise. Let $C=\mathrm{core}_L(H)$, $D=\mathrm{core}_L(K)$. For any x in $G-L$, $C\cap C^x=\{1_G\}$. Then L/D is isomorphic to a primitive permutation group of degree a, and D/C is an elementary Abelian 2-group of order at most 2^a. By Lemma 2.1, $k(L/D)\leqslant(\sqrt{2})^a$, so that $k(L/C)\leqslant(\sqrt{2})^{3a}$. Now set $M=CC^x$. Then $k(M D^x/D^x)\leqslant(\sqrt{2})^a$ by Lemma 2.1, so $k(M/M\cap D^x)\leqslant(\sqrt{2})^a$. Hence $k(M/C^x)\leqslant k(M/M\cap D^x)\,k(M\cap D^x/C^x)\leqslant(\sqrt{2})^{3a}$, so that $k(C)\leqslant(\sqrt{2})^{3a}$, $k(L)\leqslant 2^{3a}$, and $k(G)\leqslant 2^{3a+1}$.

We are assuming that $k(G)>(\sqrt{3})^{4a-1}$, so that $9^a<(2\sqrt{3})\cdot 8^a$. This forces $a\leqslant 9$ once more, as a is a prime power.

If $a=9$, we have $k(L/D)\leqslant 11$ and $k(M/M\cap D^x)\leqslant 11$, so we obtain the sharper estimate $k(L)\leqslant 11^2 2^{18}$, $k(G)\leqslant 11^2 2^{19}$. But we are assuming that $k(G)>(\sqrt{3})^{4a-1}$. This is a contradiction, because $11^4\cdot 2^{38}<2^{52}<3^{35}$.

If $a=8$, we have $k(L/D)\leqslant 8$ and $k(M/M\cap D^x)\leqslant 8$, so we obtain the sharper estimate $k(L)\leqslant 8^2 2^{16}$, $k(G)\leqslant 2^{23}$. But we are assuming that $k(G)>(\sqrt{3})^{4a-1}$. This is a contradiction, because $2^{46}<2\cdot 3^{30}<3^{31}$.

If $a=7$, we have $k(L/D)\leqslant 7$ and $k(M/M\cap D^x)\leqslant 7$, so we obtain the sharper estimate $k(L)\leqslant 7^2 2^{14}$, $k(G)\leqslant 7^2 2^{15}$. But we are assuming that $k(G)>(\sqrt{3})^{4a-1}$. This is a contradiction, because $49^2 2^{30}<2^{42}<3^{27}$.

If $a=5$, then, as in an earlier argument, there are at most eight L/C conjugacy classes of D/C, and $k(L/C)\leqslant 40$. Similarly, $k(M/C^x)\leqslant 40$, so that $k(L)\leqslant 1600$, $k(G)\leqslant 3200$. This is a contradiction, since we are assuming that $k(G)>(\sqrt{3})^{4a-1}=3^9\cdot\sqrt{3}$.

Thus $a\leqslant 4$, as claimed. If $a=2$, then G is a 2-group of order dividing 128. We are assuming that G has more than $(\sqrt{3})^{4a-1}$ conjugacy classes, so that $k(G)\geqslant 47$. If $|G|<128$, then G is easily seen to be Abelian, which is a contradiction, as G is a permutation group of degree 8. If G has order 128, then G is isomorphic to C_2 wr C_2 wr C_2, so that $k(G)=20$, a contradiction.

If $a=3$, then we are assuming that $k(G)>(\sqrt{3})^{11}>413$. But L/C is isomorphic to a subgroup of C_2 wr S_3, from which it follows that $k(L/C)\leqslant 10$ and $k(M/C^x)\leqslant 10$, so that $k(L)\leqslant 100$, $k(G)\leqslant 200$, a contradiction.

If $a=4$, then we are assuming that $k(G)>(\sqrt{3})^{15}>3600$. The argument used in case (iii) shows that $k(L/C)\leqslant 40$, and a similar argument shows that $k(M/C^x)\leqslant 40$, so that $k(L)\leqslant 1600$, $k(G)\leqslant 3200$, the final contradiction.

Remark. Let d be any integer greater than 1. Then there are non-negative integers x and y such that $d=3x+2y$ (and these may be chosen so that $y < 3$). Then the symmetric group S_d has an Abelian subgroup A of order $3^x 2^y$ which has x orbits of length 3 on $\{1, 2, ..., d\}$, y orbits of length 2. Hence for every $d > 1$, there is a permutation group of degree d realising the bound of Theorem 1.2.

3. On the $k(GV)$ Problem

The $k(GV)$ problem is to prove that whenever G is a group of linear transformations of the finite-dimensional vector space V (over a finite field of characteristic coprime to the order of G), then $k(GV) \leqslant |V|$. This has been done independently when G has odd order by Gluck ([18]) and Knörr (unpublished). It has also been done when G is nilpotent (or, more generally, supersolvable), by Knörr ([11]). In general, though, the best existing bound is not much better than $|V|^2$. If we drop the assumption of coprimeness, but assume instead that G is completely reducible in its action on V, then the inequality $k(GV) \leqslant |V|$ does not hold in general.

One reason for interest in the $k(GV)$ problem is that H. Nagao ([14]) proved that an affirmative solution would imply that whenever B is a p-block of defect d of the finite p-solvable group H, then $k(B) \leqslant p^d$.

R. Gow ([9]) proved that if $|V| = p^d$ for some prime p, then $k(GV) \leqslant f(d)\,|V|$, where f is some fixed function on the natural numbers.

Gow's idea is as follows: "lift" G to a subgroup of $GL(d, \mathbb{C})$ and use Jordan's theorem, which asserts the existence of an Abelian normal subgroup, A, of G, of index bounded in terms of d alone. Then we have $k(GV) \leqslant k(G/A)\, k(AV)$, and $k(AV) \leqslant |V|$ is easily proved.

When G is solvable, L. Dornhoff (see part A of [6], for example) has shown that under the above circumstance G has an Abelian normal subgroup A with $[G:A] \leqslant 2^{(4d-3)/3} \cdot 3^{(10d-3)/9}$, so that when G is solvable, $f(d)$ may be replaced by $2^{(4d-3)/3} \cdot 3^{(10d-3)/9}$, as noted in [9].

In a similar fashion, we have $k(GV) \leqslant k(G/F(G)) \cdot k(F(G)\, V)$, and by Knörr's result, we know that $k(F(G) \cdot V) \leqslant |V|$.

Our main result in this section is:

Theorem 3.1. *Let G be a finite solvable group, p be a prime. Suppose that $|F(G)| = p^r$, where r is a positive integer, Then $k(G) = 3^{r-1} p^r$.*

Proof. Let $U = \Phi(G)$. Then $F(G/U) = F(G)/U$, and $k(G) \leqslant k(U)\, k(G/U) \leqslant |U|\, k(G/U)$, so it suffices to consider the case that $U = \{1_G\}$, and hence that $G^* = G/F(G)$ is a completely reducible group of linear transformations of the elementary Abelian p-group $F(G)$. Let $F_2(G)$ be the full pre-image of $F(G^*)$ in G. Then $F_2(G) = XF(G)$, where X is a nilpotent p'-group acting faithfully on $F(G)$, so that $k(F_2(G)) \leqslant p^r$ by

Knörr's result. Now $k(G) \leqslant k(F_2(G))\, k(G/F_2(G)) \leqslant p^r k(G^*/F(G^*))$. By the Fong-Swan theorem (see [6, part B], for example), G^* is isomorphic to a subgroup of $GL(r, \mathbb{C})$, so it is more than sufficient to prove:

THEOREM 3.2. *Let G be a finite solvable subgroup of $GL(r, \mathbb{C})$. Then $k(H/H \cap F(G)) \leqslant 3^{r-1}$ for every subgroup H of G.*

Proof. Choose G and r to violate the theorem with r minimal. Let V be the $\mathbb{C}G$-module affording the given representation of G.

Suppose that V is reducible, say $V = U \oplus W$. Let $K = C_G(U)$, $L = C_G(W)$, A be the full pre-image in G of $F(G/K)$, B be the full pre-image in G of $F(G/L)$, $s = \dim(U)$, $t = \dim(W)$. Then $F(G) = A \cap B$. Let H be any subgroup of G. We need to prove that $k(H/H \cap F(G)) \leqslant 3^{r-1}$.

Now $k(H/H \cap A \cap B) \leqslant k(H/H \cap A)\, k(H \cap A/H \cap A \cap B) = k(HA/A)$ $k((H \cap A)\, B/B) \leqslant 3^{s-1} \cdot 3^{t-1}$, contrary to the choice of G. Hence V is irreducible.

Next we claim that V is primitive. For suppose that $V \cong \mathrm{ind}_M^G(U)$ for some proper subgroup M of G, some $\mathbb{C}M$-module U. Let $s = [G : M]$, $t = \dim(U)$, $X = \mathrm{core}_G(M)$, H be any subgroup of G.

Then $k(H/H \cap F(G)) \leqslant k(H/H \cap F(X)) \leqslant k(H/H \cap X)\, k(H \cap X/H \cap F(X))$. Now $H/H \cap X \cong HX/X$, which is isomorphic to a permutation group of degree s, so that $k(H/H \cap X) < (\sqrt{3})^s$.

Now $\mathrm{Res}_X^G(V)$ is a direct sum of at least s irreducible $\mathbb{C}X$-modules (counting multiplicities), so an argument like that used to show that G is irreducible shows that $k(H \cap X/H \cap F(X)) \leqslant 3^{s(t-1)}$. Since $s > 1$, we obtain $k(H/H \cap F(G)) < 3^{st-1}$. Thus G is primitive.

We note here that the validity (or otherwise) of the conclusion of the theorem depends only on the isomorphism type of $G/F(G)$.

We next claim that every non-central normal subgroup of G is irreducible. For let N be such a non-central normal subgroup of G. Then if N is not irreducible, (by standard Clifford theory) there is a finite group G^* with $G^*/Z(G^*) \cong G/Z(G)$ such that $V = U \otimes W$ as $\mathbb{C}G^*$-module, where both $\dim(U)$ and $\dim(W)$ are greater than 1. But G^* may be chosen so that $U \oplus W$ is a faithful $\mathbb{C}G^*$-module (and has dimension at most r, with equality if and only if U and W both have dimension 2).

By the choice of G and r (and, if $r = 4$, using the argument used to show G irreducible), the conclusion of the theorem is valid for G^*, a contradiction, as $G^*/F(G^*) \cong G/F(G)$.

Now, adjoining scalars or replacing G by a supplement to $Z(G)$ as necessary, we may assume that G is unimodular, and that $Z(G) \leqslant \Phi(G)$.

There is a prime p such that $O_p(G)$ is non-central in G. Hence $O_p(G)$ is irreducible, so $C_G(O_p(G)) \leqslant Z(G) \leqslant O_p(G)$ (using unimodularity), and $F(G)$ is a p-group.

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Let $Z = Z(G)$, M be a normal subgroup of G such that M/Z is a minimal normal subgroup of G/Z. Then $M \leqslant F(G)$, and M is irreducible. Hence $C_G(M) \leqslant Z \leqslant M$. But it is well-known (see part A of [6], for example), that $O_p(G) = ZE$ where E is an extra-special p-group, and by a result of P. Hall, we have $E = (E \cap M)\, C_E(E \cap M) \leqslant M$, so that $M = O_p(G)$ and $G/O_p(G)$ is an irreducible group of linear transformations of the elementary Abelian p-group $O_p(G)/Z$.

Let $r = p^m$. Then E has order p^{2m+1}, and G/Z is isomorphic to a primitive *permutation* group of degree p^{2m}. Also, by the Fong-Swan theorem, G/ZE is isomorphic to a subgroup of $GL(2m, \mathbb{C})$.

We claim that $2m \geqslant p^m$ (which forces $p = 2$, $m \leqslant 2$). If not, then the conclusion of the theorem is valid for $G^* = G/O_p(G)$, regarded as a subgroup of $GL(2m, \mathbb{C})$. Let H be any subgroup of G. To derive a contradiction from our current assumptions, we need to show that $k(H/H \cap ZE) \leqslant 3^{r-1}$. Since $H/H \cap ZE \cong HZE/ZE$, we may as well assume that $ZE \leqslant H$.

Let $F_2(G)$ denote the full pre-image of $F(G^*)$ in G. Then $k(H/H \cap F_2(G)) = k(HF_2(G)/F_2(G)) \leqslant 3^{2m-1}$ (since the conclusion of the theorem is valid for G^*).

Now we have $k(H/F(G)) \leqslant k(H/H \cap F_2(G))\, k(H \cap F_2(G)/F(G)) \leqslant 3^{2m-1} k(H \cap F_2(G)/F(G))$.

Now $H \cap F_2(G) = O_p(G)\, Y$, where Y is a nilpotent p'-group which acts faithfully on $O_p(G)/Z$, so by Knörr's result, $k(H \cap F_2(G)/Z) \leqslant p^{2m}$. In particular, $k(Y) < p^{2m}$.

Hence we see that $k(H/F(G)) < 3^{2m-1} p^{2m}$. We needed to prove that $k(H/F(G)) \leqslant 3^{r-1}$, where $r = p^m$. This will be so if $2m - 1 + 2m \log_3(p) \leqslant p^{m-1}$.

The only cases where this inequality is not satisfied (when $p^m > 2m$, as we are currently assuming) are when $p = 3$, $m = 1$ and $p = 2$, $m = 3$. If $p = 3$ and $m = 1$, then $G/F(G)$ is isomorphic to a subgroup of $SL(2, 3)$, and all subgroups of $SL(2, 3)$ have 7 or fewer conjugacy classes, whereas $3^{r-1} = 9$ in this case.

If $p = 2$ and $m = 3$, then $3^{r-1} = 3^7 = 2187$. We derive a contradiction by showing that no subgroup of $G/F(G)$ has 2187 or more conjugacy classes. For $G^* = G/F(G)$ is isomorphic to an irreducible subgroup of $Sp(6, 2)$, so its only possible prime divisors are 2, 3, 5 and 7.

If 7 divides $|G^*|$, then G^* is 7-closed by A6. It follows easily that $F(G^*)$ has order dividing 63 and that $[G^* : F(G^*)] \leqslant 6$, so that G^* has order less than 400. Thus we may assume that 7 does not divide $|G^*|$.

If 5 divides $|G^*|$, then G^* is 5-closed, for otherwise $F(G^*)$ is a 3-group. But a Sylow 3-subgroup of G^* is isomorphic to a subgroup of $C_3 \text{ wr } C_3$, so none of its subgroups admits an automorphism of order 5, a contradiction. But now let T^* be a Sylow 5-subgroup of G^*. Then T^* has a non-trivial fixed-point in its action on $F(G)/Z$ (which has order 64), contrary to the fact that G^* acts irreducibly on $F(G)/Z$ and T^* is normal in G^*.

Thus we may assume that G^* is a $\{2, 3\}$-group and that $F(G^*)$ is a 3-group.

If $F(G^*)$ is elementary Abelian of order 27, then every proper subgroup of G^* has order less than 2187, while $k(G^*) \leqslant 729$ by A5.

In any other case, $F(G^*)$ is generated by at most two elements, and a Sylow 2-subgroup of G^* has order dividing 16, so G^* has order less than 2187, the required contradiction.

Thus $p=2$, and $m=1$ or 2. But $m=1$ yields $r=2$, and $G/F(G)$ isomorphic to C_3 or S_3, so every subgroup of $G/F(G)$ has 3 or fewer conjugacy classes, contrary to the choice of G.

If $m=2$, then $3^{r-1}=27$. In this case, $G^* = G/F(G)$ (which is isomorphic to a subgroup of $Sp(4, 2)$) is 5-closed by A6. Choose $H \leqslant G^*$.

If 5 divides the order of G^*, then (upon consideration of the centralizer of an element of order 5 in $GL(4, 2)$), $|F(G^*)|$ divides 15, G^* has order dividing 60. If 5 does not divide the order of H in this case, then H has order at most 12. If 5 does divide the order of H, then $k(H/O_3(H)) \leqslant 5$, so that $k(H) \leqslant 15$.

If G^* is a $\{2, 3\}$-group, then G^* has a Sylow 3-subgroup of order at most 9, which is $F(G^*)$. If 9 divides $|H|$, then $F(H) = F(G^*)$, so $k(H) \leqslant 9$ by A5. Otherwise, since $H/O_3(H)$ is isomorphic to a subgroup of a semi-dihedral group of order 16, we have $k(H/O_3(H)) \leqslant 8$, $k(H) \leqslant 24$.

This completes the proofs of Theorems 3.2 and 3.1.

Remark. The last result has proved something a little stronger. For when $V = O_p(G)$ is elementary Abelian of order p^r and G/V is a finite solvable completely reducible group of linear transformations of V, A2′ yields that $k(G) \leqslant 3^{r-1}$ $\#$(G-conjugacy classes of $F_2(G)$), since *every* subgroup of $G/F_2(G)$ has at most 3^{r-1} conjugacy classes.

By arguments similar to those of Gow [9], we also obtain:

COROLLARY 3.3. *Let G be a finite solvable group, p be a prime, B be a p-block of G of positive defect d. Then $k(B) \leqslant 3^{d-1}p^d$. In particular, if $p > 7$, then $k(B) < p^{(3d-1)/2}$.*

4. ON THE $k(GV)$ PROBLEM IN THE INSOLUBLE CASE

In this section, we imitate the arguments of the previous section to prove:

THEOREM 4.1. *There is a fixed constant c (independent of the prime p, and of the integer r) such that whenever G is a finite p-solvable group with $|F^*(G)| = p^r > 1$, then $k(G) \leqslant c^{r-1}p^r$.*

We may argue as before to reduce to the case that $G/F(G)$ is a faithful completely reducible group of linear transformations of the elementary Abelian p-group $F^*(G)$. Again by the Fong–Swan theorem, $G^* = G/F(G)$ is isomorphic to a subgroup of $GL(r, \mathbb{C})$. Also, $k(F_2(G)) \leqslant p^r$ by Knörr's result (though it may be the case here that $F_2(G) = F(G)$), so it suffices to prove that $k(G^*/F(G^*)) \leqslant c^{r-1}$ for some fixed c. Thus it will more than suffice to prove:

THEOREM 4.2. *There is a constant c (independent of the integer r) such that whenever G is a finite subgroup of $GL(r, \mathbb{C})$, every subgroup of $G/F(G)$ has at most c^{r-1} conjugacy classes.*

We do not attempt to suggest an optimal choice for c (indeed, some of our estimates below are extremely generous) but we first show that, if c is taken large enough, the proof of this last theorem may be reduced to the case that G is a primitive linear group which has a unique component (that is, quasi-simple subnormal subgroup), M say, with $C_G(M) = Z(G)$.

More precisely, we prove:

LEMMA 4.3. *Let c_0 be any real number greater than 64. Suppose that for some integer r there is a finite subgroup L of $GL(r, \mathbb{C})$ such that some subgroup of $L/F(L)$ has more than c_0^{r-1} conjugacy classes.*

Then there is an integer d, and there is a primitive finite subgroup G of $GL(d, \mathbb{C})$ such that:

(i) *G has a unique component, M say.*

(ii) *$C_G(M) = Z(G)$.*

(iii) *$G/Z(G)$ has a subgroup with more than c_0^{d-1} conjugacy classes.*

To prove this, we first require:

LEMMA 4.4. *We have $q^m/\log_{16}(q^m) \geqslant 2m + 4$ for every prime q and every positive integer m. Consequently, $q^{2m^2+4m} \leqslant 16^d$, where $d = q^m$, for every prime q and every positive integer m.*

Proof. For a fixed positive integer m, $x^m/\log_{16}(x^m)$ increases for $x > e^{1/m}$. When $m = 1$, it suffices to check the inequality for $q = 2$ and $q = 3$, and the inequality is easily seen to be valid in those cases. When $m > 1$, it suffices to check the inequality for $q = 2$. Hence we need to show that $2^m/\log_{16}(2) \geqslant 2m^2 + 4m$ for $m \geqslant 2$. When $m = 2$, this inequality is satisfied since $\log_{16}(2) = \frac{1}{4}$. But when $x \geqslant 2$, $2(x+1)^2 + 4(x+1) < 2(2x^2 + 4x)$, from which the desired inequality easily follows.

Proof of Lemma 4.3. We fix a real number $c_0 \geqslant 64$. Suppose that L as in the hypotheses of the lemma exists. Then we may choose an integer d

which is minimal subject to: there is a finite subgroup G of $GL(d, \mathbb{C})$ such that $G/F(G)$ has subgroup with more than c_0^{d-1} conjugacy classes. From A7, $d>2$. Hence $c_0^{d-1} \geqslant 16^d$.

As in the proof of Theorem 3.2, (this time making use of Theorem 1.2), we may argue that G is primitive. Also, as in that proof, we may deduce that G is absolutely tensor indecomposable, that every non-central normal subgroup of G is irreducible, and we may assume that G is unimodular, and that $Z(G) \leqslant \Phi(G)$.

We claim that $F(G)=Z(G)$. For otherwise, there is a prime q such that $O_q(G)$ is irreducible, and as in the earlier proof, $d=q^m$ for some integer m, $O_q(G)=Z(G)\,E$, where E is extra-special of order q^{2m+1}, $C_G(O_q(G)) \leqslant O_q(G)$, and $G/O_q(G)$ is isomorphic to a subgroup of $Sp(2m, q)$. We derive a contradiction by showing that every subgroup of $G/O_q(G)$ has at most c_0^{d-1} conjugacy classes.

Let H be a subgroup of G containing $O_q(G)$. Then $H/Z(G)$ is a q-constrained group, with no non-identity normal q'-subgroup, whose Sylow q-subgroup has order dividing q^{m^2+2m}.

By A5, $k(H/Z(G)) \leqslant q^{2m^2+4m} \leqslant 16^d$ (the last inequality holding by virtue of Lemma 4.4) $\leqslant c_0^{d-1}$ so that certainly $k(H/O_q(G)) \leqslant c_0^{d-1}$, the desired contradiction.

Next, we claim that the given representation of G is not tensor induced (as a projective representation) from a projective representation of any proper subgroup of G.

For suppose that the given representation is so induced (say from a subgroup H of index $t>1$) when viewed as a projective representation of G. Let V be the underlying $\mathbb{C}G$-module.

Let $X=\mathrm{core}_G(H)$, and let L be any subgroup of G containing $Z(G)$. Then $k(L/Z(G)) \leqslant k(L/Z(X)) \leqslant k(L/L \cap X)\, k(L \cap X/Z(X))$. Now $k(L/L \cap X) = k(LX/X) \leqslant 5^{t-1}$, as LX/X is isomorphic to a subgroup of the symmetric group S_t.

Hence we are done if $X \leqslant Z(G)$. Suppose that $X > Z(X)$. Then X is irreducible, and $V=U_1 \otimes \cdots \otimes U_t$, where $\dim(U_i)=d^{1/t}$ for each i, each U_i affording an irreducible projective representation of X.

Now we may choose a central extension, X^*, of X, minimal such that each U_i has the structure of a $\mathbb{C}X^*$-module. Then $U_1 \oplus \cdots \oplus U_t$ is a faithful $\mathbb{C}X^*$-module, of dimension at most d, but is reducible, so we may conclude, as in earlier arguments, that every subgroup of $X^*/F(X^*)$ has at most c_0^{d-t} conjugacy classes. Of course, $X^*/F(X^*) \cong X/F(X)$.

But $F(X)=Z(X)$ since $F(G)=Z(G)$. Hence $k(L \cap X/Z(X)) \leqslant c_0^{d-t}$ and $k(L/Z(X)) \leqslant c_0^{d-1}$. This contradiction shows that V is not tensor induced in the above sense.

By Kovács [13], G has a unique component, so now all parts of the lemma have been proved.

Remark. For the purposes of our Theorem 4.1, we only need to bound $k(G/F(G))$ by c^{r-1}, but once the stronger version has been proved, we will be able to deduce as before that $k(G) \leqslant c^{r-1}$ #(G-conjugacy classes of $F_2(G)$) when $G/F(G)$ acts as a completely reducible group of linear transformations of $F(G)$.

The following result must be well known, and follows by arguments like those of Section 1.

LEMMA 4.5. *Let X be a nilpotent subgroup of the symmetric group S_t. Then $|X| \leqslant 2^{t-1}$.*

Now we return to the proof of Theorem 4.2: invoking Lemma 4.3, it suffices to prove that if G is a finite primitive subgroup of $GL(r, \mathbb{C})$, with a component M such that $C_G(M) = Z(G)$, then we find a constant $c > 64$, independent of G and r, such that every subgroup of $G/Z(G)$ has at most c^{r-1} conjugacy classes.

By the classification of the finite simple groups, there are three cases to consider:

Case (i): *$M/Z(M)$ Sporadic, or the Tits Group ${}^2F_4(2)'$.* We can dispense with this case by choosing c sufficiently large, since there are only finitely many possibilities for $G/Z(G)$.

Case (ii): *$M/Z(M)$ an Alternating Group.* In this case, if $M/Z(M)$ is isomorphic to the alternating group A_t, then if $t > 7$, we see that $G/Z(G)$ is isomorphic to A_t or S_t, (so that all of its subgroups have 5^{t-1} or fewer conjugacy classes by Theorem 1.2), and that $Z(M)$ has order dividing 2. Also, $r > 2$ by A7.

By choosing c large enough, we can dispense with the exceptional cases, so we suppose that $t > 7$. Then M has an elementary Abelian 3-subgroup of order $3^{(t-2)/3}$ or greater, so by a theorem of Blichfeldt (see part A of [6], for example) $3^{(t-2)/3} < 6^{r-1}$, and so certainly $t - 1 < 6r - 5$. Hence in this case, every subgroup of $G/Z(G)$ has $[\sqrt{5^{13}}]^{r-1}$ or fewer conjugacy classes.

Case (iii): *$M/Z(M)$ a Group of Lie Type.* Let l be the characteristic of M, and let Q be a Sylow l-subgroup of M. Set $U = Q/Q \cap Z(M)$. Then inspection of [5] shows that $[M : Z(M)] < |U|^3$ (this observation is really due to E. Artin).

Now Q has an Abelian normal subgroup (A say), containing $Q \cap Z(M)$, such that Q/A is isomorphic to a (nilpotent) subgroup of the symmetric group S_r. By Lemma 4.5, $[Q : A] \leqslant 2^{r-1}$. Since G is primitive, it follows from Blichfeldt's theorem that $[A : A \cap Z(M)] < 6^{r-1}$. Hence $|U| < 12^{r-1}$, $[M : Z(M)] < (1728)^{r-1}$.

Now $G/MZ(G)$ is isomorphic to a subgroup of $\mathrm{Out}(M/Z(M))$, and $|\mathrm{Out}(M/Z(M))| \leqslant |U|$ (from [5], again).

Hence $[G : Z(G)] < (12^4)^{r-1}$, so certainly all subgroup of $G/Z(G)$ have fewer than $(12^4)^{r-1}$ conjugacy classes, which suffices to complete the proof.

We have, again arguing as in Gow [9]:

THEOREM 4.6. *There is a fixed constant c, independent of the prime p and the defect d, such that whenever p is a prime and B is a p-block of positive defect d of a finite p-solvable group G, then $k(B) \leqslant c^{d-1}p^d$.*

We remark that there is no direct analogue of Theorem 4.2 for finite general linear groups, even if we allow the constant c to depend on the size of the field. For when n is even, $GL(n, q)$ has an Abelian subgroup of order $q^{n^2/4}$. We suggest however:

Conjecture. Let q be a prime power. Then there is a constant $c = c(q)$, depending on q (but independent of n), such that whenever G is a subgroup of $GL(n, q)$, and H is a completely reducible subgroup of G, then $k(H/H \cap F(G)) \leqslant c^{n-1}$.

In fact, it suffices to consider the case that $G = GL(n, q)$, for letting $S(G)$ denote the subgroup of scalar matrices within G, and Z denote $Z(GL(n, q))$, $H/H \cap F(G)$ is an epimorphic image of $H/H \cap S(G)$, which is in turn isomorphic to HZ/Z. Of course, unless both $n = 2$ and $q < 4$, $Z = F(GL(n, q))$ and by taking $c > 3$ these exceptional cases can be ignored.

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