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ABSTRACT

We give a description, in terms of twisted wreath products, of all finite groups G with a core-free maximal subgroup H such that the socle of G is a non-abelian minimal normal subgroup of G, and is complemented in G by H. Moreover, we deal with the problem of finding all subgroups G* of these G such that firstly, their socle coincides with the socle of G and is also minimal normal, and secondly, $G^* \cap H$ is a (necessarily core-free) maximal subgroup of G*.

A finite group G is called *primitive* if it has a maximal subgroup H such that $\mathbf{Core}_G(H) = 1$; here $\mathbf{Core}_G(H) = \bigcap_{g \in G} H^g$, the unique largest normal subgroup of G contained in H. Of course, G is primitive in this sense if and only if it admits a faithful representation as a primitive permutation group; however, our interest in the present paper is in abstract groups and their structure: we will not be concerned with the various permutation representations that primitive groups may have. A partial description of the structure of (finite — this assumption will be made of all groups in this note) primitive groups has been given by

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BAER [2]: *If* H *is a maximal subgroup of a group* G *with* $\mathrm{Core}_G(H) = 1$ *then precisely one of the following three statements holds:*

(I) $S(G) = C_G(S(G)) \cdot\!\trianglelefteq G$ — *in this case* $G = HS(G)$ *and* $H \cap S(G) = 1$;

(II) $S(G) \cdot\!\trianglelefteq G$ *and* $C_G(S(G)) = 1$;

(III) $S(G) = M_1 \times M_2$ *where* $M_i \cdot\!\trianglelefteq G$ *and* $C_G(M_i) = M_{3-i}$ $(i = 1,2)$ — *here* $G = HM_i$, $H \cap M_i = 1$ *and* $M_i \cong_H S(H) = H \cap S(G) \cdot\!\trianglelefteq H$ $(i = 1,2)$.

Conversely, if one of the following three conditions holds, then G *is a primitive group* (*and satisfies* (I), (II), *or* (III), *respectively*):

(I') G *is a semidirect product of a group* H *with a faithful irreducible* H-*module* S *over some prime field*;

(II') G *has a non-abelian minimal normal subgroup* S *such that* $C_G(S) = 1$;

(III') G *is the semidirect product of a group* H *possessing a unique minimal normal subgroup* S, *where* S *is non-abelian, with a copy of* S.

(We have used the following notation: $M \cdot\!\trianglelefteq G$ means that M is a minimal normal subgroup of G; and $S(G) = \langle M \mid M \cdot\!\trianglelefteq G\rangle$, the socle of G. We will also write $H <\cdot\, G$ to express the fact that H is a maximal subgroup of G.)

A primitive group G satisfying condition (I), (II), or (III) will be called of type (I), (II), or (III), respectively — this will be indicated by writing $G \in \mathcal{P}_{I}$, $G \in \mathcal{P}_{II}$, $G \in \mathcal{P}_{III}$. For G in $\mathcal{P}_I$ or $\mathcal{P}_{III}$ there is always a minimal normal subgroup complementing some core-free maximal subgroup; in fact, this is true for all minimal normal subgroups and all core-free maximal subgroups. The purpose of this paper is to give a precise description of all those primitive groups of type II where a core-free maximal subgroup is complemented by a non-trivial normal subgroup (which, by the above result, is necessarily minimal): these are the groups admitting a representation as a primitive permutation group with a single non-abelian regular normal subgroup. (They are also the groups which were missing in the original version of the O'Nan-Scott Theorem as given in [4]; a correct version appeared in [1]). Following Lafuente, we will denote the class of all such groups by $\mathcal{P}_{II}'$.

A first step towards our goal is the observation that — as a consequence of the Schreier conjecture — groups in $\mathcal{P}_{\mathrm{II}}'$ cannot have a simple socle, which is due to

ASCHBACHER-SCOTT [1]: *$\mathcal{P}_{\mathrm{II}}'$ does not contain any almost simple groups.*

(Recall that an *almost simple group* is [a group isomorphic to] a subgroup of the automorphism group of a non-abelian simple group containing all inner automorphisms.)

In view of this lemma, the socle of a group in $\mathcal{P}_{\mathrm{II}}'$ is always a direct product of at least two copies of a non-abelian simple group; since it is a minimal normal subgroup, the simple direct factors are permuted transitively (via conjugation) by the elements of any (core-free maximal) subgroup complementing the socle. Such groups are always (non-trivial) twisted wreath products:

BERCOV [3], LAFUENTE [7]: *Let* G *be a group and* $H, E \leq G$. *Assume that the normal closure* E^G *of* E *in* G *is complemented in* G *by* H, *and is a direct product of the* H-*conjugates* E^t *of* E, *where* t *ranges through a right transversal* T *of* $\mathbf{N}_H(E)$ *in* H:

$$G = H(E^G), \; H \cap (E^G) = 1, \; \text{and} \; E^G = \times_{t \in T} E^t \; (= E^T).$$

Set $N = \mathbf{N}_H(E)$. *Then*

$$G \cong E \wr_N H,$$

the twisted wreath product of E *and* H (*with respect to the action of* N *on* E *given by conjugation*); *indeed, a suitable isomorphism maps* E^G *to the base group and* H *to its canonical complement.*

It will be convenient to employ the following notation for twisted wreath products.

Let E and H be groups, suppose that $\lambda \in \mathbf{Hom}(N,\mathbf{Aut}(E))$ for some subgroup N of H, and consider any right transversal T of N in H containing 1.

Put

$$E^T = \times_{t\in T} E^t, \text{ where } E^t \cong E \text{ for all } t \in T;$$

we will, in fact, always write a (distinguished) isomorphism from E to E^t by

$$e \longmapsto e^t \text{ for all } e \in E$$

and require that for $t = 1$ this is the identity.

An action of H on E^T (that is, a certain $\lambda^* \in \mathbf{Hom}(H,\mathbf{Aut}(E^T))$ may now be defined by

$$(e^t)^{(h^{\lambda^*})} = (e^{(n^{\lambda})})^{t'} \text{ for all } e \in E,\ h \in H \text{ and } t \in T,$$

where $n \in N$ and $t' \in T$ are related to h and t by $th = nt'$. (Often the homomorphisms λ and λ^* will be omitted from our formulas, especially when the action is understood from the context; then the above formula reads $e^{th} = e^{nt'}$.)

The *twisted wreath product of* E *and* H (with respect to the action λ of N on E) is the semidirect product $H(E^T)$ formed with respect to the action λ^* of H on E^T. It will be denoted by $E \wr_{(T,\lambda)} H$ (or by $E \wr_N H$, if the action of N on E is understood: note that the resulting group $E \wr_{(T,\lambda)} H$ does not really depend on the specific choice of a transversal T).

A (sufficient, yet not necessary) criterion for the base group E^T of such a twisted wreath product G ($= E \wr_{(T,\lambda)} H = E \wr_{(T,\lambda)} H$) with E non-abelian simple to be the only minimal normal subgroup of G is derived from the following fact:

$$C_G(E^T) \le \mathbf{Core}_H(N^{\#})E^T,$$

where $N^{\#} = \{\, n \in N \mid n \text{ induces an inner automorphism in } E\}$.

Consequently,

$$\mathbf{Core}_H(N^{\#}) = 1 \textit{ implies that } G \in \mathcal{P}_{\mathrm{II}}.$$

We conclude this introductory section by fixing some more notation and terminology.

For any group E, $E^{(n)}$ ($n \in \mathbb{N}$) denotes the direct product $E \times \ldots \times E$ of n copies of E. The *canonical diagonal subgroup of* $E^{(n)}$ (isomorphic to E) is the group

$$\Delta(E^{(n)}) = \{ (e,\ldots,e) \mid e \in E \}.$$

If $E^{(n)}$ is given as E^T as above (the isomorphisms from E^1 to E^t again being written as $e \longmapsto e^t$ for all $e \in E$), the same term will refer to the subgroup

$$\Delta(E^T) = \{ \prod_{t\in T} e^t \mid e \in E \}.$$

§1 Primitive groups with non-abelian regular normal subgroups

In view of Baer's Theorem, a regular normal subgroup of a primitive group is necessarily minimal. Therefore, from the Bercov-Lafuente Proposition we get that any such group G is isomorphic to a twisted wreath product of a non-abelian simple group E with a group H. The following result gives a primitivity criterion for such twisted wreath products.

(We would like to alert the reader that there is a misprint in the statement of this result given in [5]. Furthermore, we would like to draw attention to a different proof of this theorem, which will be given in [6] as part of a more general investigation. Here we present a direct proof.)

1.1 THEOREM. *Let* $G = E \wr_{(T,\lambda)} H$, *where* T *denotes a right transversal of the subgroup* N *of* H *and* λ *is a homomorphism from* N *into* $\mathbf{Aut}(E)$ *for some non-abelian simple group* E.

Put $C = \ker(\lambda) = C_N(E)$ *and* $D = S(N \bmod C)$.

(a) *The following three statements are equivalent in pairs:*

(i) $H <\cdot\, G$.

(ii) $D/C \cong_N E$, *and if for some subgroup* M *of* H, $\lambda' : M \longrightarrow \mathbf{Aut}(E)$ *extends* λ *(i.e.,* $N \le M$ *and* $\lambda'|_N = \lambda$*), then* $\lambda' = \lambda$ *(i.e.,* $N = M$*).*

(iii) $D/C \cong_N E$, $N = N_H(C) \cap N_H(D)$, *and* $D \leq X$ *whenever* $C < X \leq H$ *with* $N \leq N_H(X)$.

(b) *Assume conditions (i-iii) from (a) and suppose that* $\mathbf{Core}_G(H) = 1$. *Then either* $G \in \mathcal{P}_{II}$ *and* $\mathbf{Core}_H(D) = 1$, *or else* $G \in \mathcal{P}_{III}$ *and* $\mathbf{Core}_H(D) = S(H)$.

Proof. (a) (i) $\Longrightarrow$ (ii): Consider a proper subgroup K of H containing N. In T there exist transversals R, S of, respectively, N, K in K, H. Then

$$K(E^K) = K(E^R) = E \wr_{(R,\lambda)} K.$$

Now, if K is not maximal in $E \wr_{(R,\lambda)} K$, say $K < K^* < E \wr_{(R,\lambda)} K$, we can find a non-trivial K-invariant proper subgroup B of E^R, namely, $B = K^* \cap E^R$. In this case $1 \neq B^H = B^{KS} = B^S = \times_{s \in S} B^s < E^T$ and thus $H < H(B^H) < E \wr_{(T,\lambda)} H$. This contradiction against (i) shows:

(*) $K <\!\cdot\, E \wr_{(R,\lambda)} K$ whenever $N \leq K \leq H$ (R a right transversal of N in K).

Application of (*) with $K = N$ yields maximality of N in NE ($= NE^1$). Hence NE/C is a primitive group with core-free maximal subgroup N/C complemented by the simple minimal normal subgroup EC/C ($\cong E$). The Aschbacher-Scott Lemma gives that $NE/C \in \mathcal{P}_{III}$, and then Baer's Theorem proves the first claim in (ii):

$$D/C \cong_N E.$$

In order to verify the second claim in (ii) we assume that $\lambda' : M \longrightarrow \mathbf{Aut}(E)$ extends λ, where $N < M \leq H$. From $R \neq \{1\}$ for a right transversal R of N in M we obtain that

$$B = \{ \prod_{r \in R} e^{(r^{-1})^{\lambda'} r} \mid e \in E \}$$

is a non-trivial proper subgroup of E^R: it is obviously isomorphic to E. Hence it will suffice to establish the M-invariance of B, for then $M < MB < M(E^R) = M(E^M)$ — a contradiction against (*), completing the proof that (i) implies (ii). Now, for all $r \in R$ and all $m \in M$ there exist $n = n(r,m) \in N$ and $r' = r'(r,m) \in R$ such that $rm = nr'$; and then for all $e \in E$,

$$(e^{(r^{-1})^{\lambda'} r})^m = (e^{(r^{-1})^{\lambda'}})^{(nr')} = e^{(r^{-1})^{\lambda'} n^{\lambda} r'} = e^{((r^{-1}nr')^{\lambda'} (r'^{-1})^{\lambda'}) r'} = (e^{(m^{\lambda'})})^{(r'^{-1})^{\lambda'} r'},$$

which means that $B^M = B$.

(ii) $\Longrightarrow$ (iii): First consider $M = N_H(C) \cap N_H(D)$ ($\geq N$). The action of M on its normal section D/C together with $D/C \cong_N E$ (via λ) gives rise to an extension $\lambda'\colon M \longrightarrow \mathbf{Aut}(E)$ of λ. Therefore, from (ii) we infer that $N = M$, proving the second assertion in (iii). The third statement is derived similarly, using that — as a consequence of the simplicity of the socle D/C ($\cong E$) of the almost simple group N/C — $N \cap X = C$ whenever $C < X \leq H$, $N \leq N_H(X)$, $D \nleq X$, and observing the canonical isomorphism $N/(N \cap X) \cong NX/X$.

(iii) $\Longrightarrow$ (i): We must show that $G^* = G$ if $H <\cdot\, G^* \leq G$. Consider the minimal normal subgroup $L = G^* \cap E^T$ of G^*. Let

$$L = L_1 \times \ldots \times L_k$$

be its decomposition into simple (not necessarily non-abelian) direct factors. Since $N = N_H(E)$, the projection $\pi\colon E^T \longrightarrow E$ (with kernel $E^{T\setminus\{1\}}$) clearly commutes with the actions of N on L and E. Moreover, $L^\pi = 1$ would contradict the transitivity of the action (by conjugation) of H on $\{ E^t \mid t \in T \}$ together with $L \trianglelefteq G^* = HL$. It follows that $L^\pi = (L^N)^\pi = (L^\pi)^N = E$, the only non-trivial N-invariant subgroup of E ($\cong_N D/C \cdot\!\trianglelefteq N/C$). From the above decomposition of L as a direct product of simple groups we now deduce that precisely one of the L_i — say, L_1 — projects onto E, while the others are contained in $E^{T\setminus\{1\}}$. In particular,

$$L_i \cong L_1 \cong (L_1)^\pi = E, \text{ but } (L_i)^\pi = 1 \quad (i = 2,\ldots,k).$$

Furthermore, $N = N_H(E)$ must normalize L_1: indeed, for each $n \in N$, $(L_1)^n$ is some L_j and $(L_j)^{n\pi} = (L_j)^{\pi n}$. Thus

$$L_1 \cong_N E,\ N \leq N_1 = N_H(L_1) \text{ and } C = C_H(E) \leq C_1 = C_H(L_1).$$

Since D/C acts faithfully on E, and therefore on L_1, while L_1 is centralized by C_1, the last condition in (iii) gives that

$$C = C_1.$$

$D_1 = S(N_1 \bmod C)$ ($> C$) is normalized by N and so, by virtue of the third condition in (iii), contains D. Since D/C induces in L_1 the group of all inner automorphisms of L_1 ($\cong_N E \cong_N D/C$), we see that $D/C = \mathbf{Inn}_{N_1/C}(L_1) \trianglelefteq N_1/C = N_1/C_1$. Hence $N_1 \leq N_H(C) \cap N_H(D) = N$ (by the first condition in (iii)). We conclude that

$N = N_1$, from which $G^* = G$ follows: note that $G^* \cong L_1 \wr_{N_1} H$ (by the Bercov-Lafuente Proposition) with $L_1 \cong E$.

(b) $\mathbf{Core}_G(H) = 1$ together with $H <\cdot G$ yields primitivity of G, and implies that $\mathbf{Core}_H(C) = C_H(E^T) = 1$. Put $K = \mathbf{Core}_H(D)$. By application of the last condition in (iii) (with $X = CK$) we get that either $K \leq C$ or $CK = D$.

If $K \leq C$ then $K \leq \mathbf{Core}_H(C) = 1$. Further, $G \in \mathcal{P}_{II}$, for otherwise $S(H) \leq N$, and hence $S(H)C/C \leq S(N/C) = D/C$, would follow from Baer's Theorem — but $S(H) \leq D$ would require that $S(H) \leq \mathbf{Core}_H(D) = K$.

Next observe that the above argument, when applied to any minimal normal subgroup Y of H instead of K, shows that $D \leq YC$. Since D/C is isomorphic to the non-abelian simple group E, only one such Y can exist: one checks that for any two different minimal normal subgroups Y_1 and Y_2 of H one should have that

$$D' = [(D \cap Y_1)C,(D \cap Y_2)C] \leq [Y_1,Y_2](C \cap Y_1)(C \cap Y_2)C' \leq C.$$

In fact, this argument proves:

(#) $S(H)$ *is a non-abelian minimal normal subgroup of* H *whose simple component* F *has a section isomorphic to the simple component* E *of* E^T.

It remains to deal with the case when $CK = D$. In this case

$$\bigcap_{t\in T} (K \cap C^t) = \bigcap_{t\in T} C_K(E^t) = C_K(E^T) = 1.$$

Further, for any $t \in T$,

$$K/C_K(E^t) = K/(K \cap C^t) = K/(K \cap C)^t \cong K/(K \cap C) \cong KC/C = D/C \cong E.$$

Combination of the last two statements shows that K is a subdirect product of groups isomorphic to the non-abelian simple group E. It is well known that this implies

$$K \cong E \times \ldots \times E.$$

In particular, the normal subgroup K of H is now a direct product of minimal normal subgroups of H, that is, $K \leq S(H)$. Hence (#) together with $K \neq 1$ yields that

$$S(H) = K = \mathbf{Core}_H(D).$$

Now for any $t \in T$, $KE^T/C_K(E^t)(\times_{t\neq s\in T} E^s) \cong (K/C_K(E^t))E^t$ (the semidirect product), where $K/C_K(E^t) \cong D^t/C^t$ induces in E^t the group $\mathbf{Inn}(E^t)$ of all inner automorphisms. We get that $(K/C_K(E^t))E^t \cong \mathbf{Inn}(E^t)E^t \cong E \times E$. An argument as above gives that KE^T ($= S(H)E^T \trianglelefteq HE^T = G$) is isomorphic to a direct product of copies of E, and so is contained in S(G). Therefore, $S(G) > E^T$ is not minimal normal, and $G \in \mathcal{P}_{III}$ follows from Baer's Theorem. ◊

In the course of the proof of 1.1b (see (#) there) we have made the following observation (some kind of analogue to what Baer's Theorem says about primitive groups of type III), which was first deduced in Lafuente [8] as a consequence of the above result.

1.2 COROLLARY. *If* $G \in \mathcal{P}_{II}$ *possesses a core-free maximal subgroup* H *complementing* S(G), *then* $G/S(G) \cong H \in \mathcal{P}_{II}$. *Moreover, if* $S(G) \cong E \times ... \times E$ *and* $S(H) \cong F \times ... \times F$ *with* E *and* F *simple, then* E *is isomorphic to a section of* F.

(In fact, using the notation of 1.1, in 1.2 we have that $D \leq S(H)C$.)

The next two theorems will give an explicit method for constructing all groups in $\mathcal{P}_{II}^{\prime}$. It will be convenient to distinguish the two cases $E = F$ and $E \neq F$ (where notation is as in 1.2).

1.3 LEMMA. *If* R *is a subdirect subgroup in the direct product* $G = E_1 \times ... \times E_n$ *of the non-abelian simple groups* E_i $(i = 1,...,n)$, *then* $N_G(R) = R$.

Proof. Since R is subdirect in G, so is $N = N_G(R) \geq R$. Because of the structure of G this requires that

$$N = N_1 \times ... \times N_m \text{ with } N_i \cong E_{j(i)} \text{ for suitable } j(i) \in \{1,...,n\} \ (i = 1,...,m).$$

Moreover, the normal subgroup R of N must be a direct product of some of the N_i, say, $R = N_1 \times ... \times N_k$ $(k \leq m)$. Hence $N = R \times C_N(R) = R \times C_G(R)$ and it suffices to observe that $C_G(R) = 1$: indeed, this follows from

$$[E_i, C_G(R)^{\pi_i}] = [R^{\pi_i}, C_G(R)^{\pi_i}] = [R, C_G(R)]^{\pi_i} = 1 \ (i = 1,...,m),$$

where π_i denotes the canonical projection from G to E_i. ◊

1.4 THEOREM. *Let* $H \in \mathcal{P}_{II}$ *and assume that* $S(H) \cong E^{(n)}$, *where* E *is a non-abelian simple group.*

(a) *Let* S *be a direct product of at least two of the simple components of* $S(H)$ *and consider a diagonal subgroup* B *(with respect to the direct decomposition of* S *into its simple components); so* $B \cong E$. *Assume that* $N = N_H(B)$ *acts transitively (via conjugation) on the set of simple components of* S *(i.e.,* $S \cdot\!\!\trianglelefteq NS = NS(H)$).

Then the canonical copy of H *in* $G = E \wr_N H$, *the twisted wreath product with respect to the action of* N *on* E *induced by the isomorphism* $E \cong B$, *is maximal in* G *and complements the base group* $S(G)$ $(\cong E^{(|H:N|)})$ *of* G; *in particular,* $G \in \mathcal{P}_{II}'$.

(b) *Conversely, if* H *occurs as a core-free maximal subgroup of the group* G *and complements* $S(G)$, *where* $S(G)$ *is minimal normal in* G *and a direct product of copies of* E, *then* $G \cong E \wr_N H$, *and the conditions stated in (a) are satisfied by* H *and* N *(with respect to suitable choices of* S *and* B).

Remark. Our proof will show that the groups S and B in 1.4b can be recovered from G $(= E \wr_N H)$, H, N, C $(= C_H(E))$, D $(= S(N \bmod C))$, E as follows:

S is the unique minimal normal subgroup of $NS(H)$ not contained in C,

and $B = D \cap S$.

Proof. (a) Put $C = C_H(B) = C_N(B)$ and $D = B \times C$. We will check that H, N, C, D, E satisfy condition (iii) of 1.1a.

Clearly, $D/C \cong_N B \cong_N E$ (and therefore, in particular, $D = S(N \bmod C)$) and $N = N_H(B) \le N_H(C) \cap N_H(D)$. Using 1.3, we see that

$$S(H) = S \times C_{S(H)}(B).$$

Since $C_{S(H)}(B) = C \cap S(H)$ and $D = B \times C$, it follows that $D \cap S(H) = B \times C_{S(H)}(B)$. Further, since $N_H(C) \cap N_H(D)$ normalizes both $C \cap S(H)$ and $D \cap S(H)$, from $B \cong E$ and $C_{S(H)}(B) \cong E^{(k)}$ $(k \in \mathbb{N})$ we now infer that $N_H(C) \cap N_H(D) \le N_H(B)$. We have shown that $N = N_H(C) \cap N_H(D)$.

Next, assume that $C < X \le H$ and $N \le N_H(X)$. Since N/C is isomorphic to a subgroup of **Aut**(E) containing **Inn**(E), the normal subgroup $(X \cap N)/C$ of N/C either contains the unique minimal normal subgroup D/C of N/C or is 1. We have to exclude the latter case. First note that

$$X \cap S(H) = (X \cap S) \times C_{S(H)}(B),$$

(for $C_{S(H)}(B) = C \cap S(H) \leq X \cap S(H)$) and that $D \not\leq X$ is equivalent to $B \not\leq X$. Therefore, as the simple subgroup B of N normalizes X and as $B \leq S$, in the case that $X \cap N = C$ we must have that $B \cap (X \cap S) = 1$.

Let L be any simple component of S (i.e., any simple component of S(H) contained in S); and let $\pi\colon S \longrightarrow L$ denote the corresponding projection map. Then L (which is B^{π}, for B is subdirect in S) normalizes $(X \cap S)^{\pi}$, whence the latter group is 1 or L. It follows from $N \leq N_H(X)$ together with our hypothesis that by conjugation with the elements of N the simple components of S are permuted transitively that, unless $X \cap S = 1$, $(X \cap S)^{\pi} = L$ for all such L. In view of $B \leq N_S(X)$ together with $B \cap (X \cap S) = 1$ we now obtain from 1.3 that $X \cap S = 1$. Consequently, $C_{S(H)}(B) = X \cap S(H) \trianglelefteq X$.

Now consider the subgroup BX of H. Observe that

$$BX \cap S(H) = B(X \cap S(H)) = B \times C_{S(H)}(B);$$

this last group is normalized by X. Since X also normalizes $C_{S(H)}(B)$, by an argument as before we see that X normalizes B; i.e., $X \leq N_H(B) = N$. Having assumed that $X \cap N = C$, we obtain a contradiction against our assumption that $C < X$, thus completing the verification of condition (iii) of 1.1a. Hence that result yields that H is maximal in $G = E \wr_N H$.

Moreover, H cleary complements the base group, which is obviously minimal normal in G. Also, $D \cap S(H) = B(C \cap S(H)) = B \times C_{S(H)}(B) < S \times C_{S(H)}(B) = S(H)$, whence 1.1b yields that $G \in \mathcal{P}_{II}'$: note that $\mathbf{Core}_G(H) = \mathbf{Core}_H(N) = 1$ follows from $H \in \mathcal{P}_{II}$ and $S(H) \not\leq N$ — $S(H) \leq N$ would require that $S(H) \leq S(N \bmod C) = D$.

(b) By hypothesis, $G = HS(G) \in \mathcal{P}_{II}'$, where H is a maximal subgroup of G such that $H \cap S(G) = 1$ and $S(G) \cong E^{(m)}$ for some $m \in \mathbb{N}$. By the Aschbacher-Scott Lemma, $m > 1$. Moreover, by the Bercov-Lafuente Proposition, $G = E \wr_N H$ with E a simple component of S(G) and $N = N_H(E)$. Let C and D be as in 1.1a.

From $S(H) \cong E^{(n)}$ with E non-abelian simple we deduce that $S(H)$ is a direct product of minimal normal subgroups of $NS(H)$. If K denotes any one of these, we may apply 1.1a to get that either $K \leq C \cap S(H)$, or else $D = C(K \cap D)$ — in which case $D \cap S(H) = (C \cap S(H))(K \cap D)$ —: take $X = CK$ in condition (iii) of 1.1a. Since $S(H) \nleq C$ (recall that $\mathbf{Core}_H(C) \leq \mathbf{Core}_H(D) = 1$ by 1.1b), there is a minimal normal subgroup S ($\leq S(H)$) of $NS(H)$ not contained in C; and thus the perfectness of D/C ($\cong E$ — cf. 1.1a) yields that the remaining minimal normal subgroups of $NS(H)$ in $S(H)$ are contained in C. Consequently,

$$C \cap S(H) = (C \cap S) \times S^*,$$

where S^* denotes the product of all minimal normal subgroups of $NS(H)$ in $S(H)$ other than S. Now we note that $C \cap S \lhd D \cap S$ and that S is a non-abelian minimal normal subgroup of $NS(H)$, whence $C \cap S = 1$ follows by means of an argument as in the third last paragraph of our proof of (a); in fact, the argument from there also shows that $D \cap S$ is a subdirect subgroup of S. Thus $C \cap S(H) = S^*$ and

$$D \cap S(H) = (C \cap S(H)) \times B, \text{ where } B = D \cap S.$$

Clearly,

$$B \cong (D \cap S(H))/(C \cap S(H)) \cong (D \cap S(H))C/C = D/C \cong E \text{ and } S \cong E^{(k)}.$$

In fact, B must be a proper subgroup of S, as follows from $\mathbf{Core}_H(D) = 1$ (see 1.1b) together with $S(H) = S \times S^* = S \times (C \cap S(H))$. Hence $S \cong E^{(k)}$ with $k > 1$, and B is a diagonal subgroup of S.

Now observe that $C \leq D \leq CS$, $C \cap S = 1$ and $B = D \cap S \trianglelefteq D$ yield that

$$D = C \times B; \text{ in particular, } C \leq C_H(B).$$

Consequently, we may again apply an argument used in part (a) of this proof to deduce that N ($= N_H(C) \cap N_H(D)$ — see 1.1a) normalizes B; so it remains to verify that $N_H(B) \leq N$. To prove that $C = C_H(B)$, we apply condition (iii) of 1.1a with $X = C_H(B)$ ($\geq C$); note that $D \nleq X$ follows from $D \geq B \nleq X$. Finally, from $C = C_H(B)$ and $D = C \times B$ we conclude that $N_H(B) \leq N_H(C) \cap N_H(D) = N$; and it is now obvious that $B \cong_N D/C \cong_N E$. ◊

As an analogue to the above result for the (easier) case when the simple component of $S(G)$ is a proper section of the simple component of $S(H)$ we record the following stronger version of Lafuente's Theorem from [9] together with a converse.

1.5 THEOREM. *Let* $H \in \mathcal{P}_{\mathrm{II}}$ *with* $S(H) \cong F^{(n)}$ *for some non-abelian simple group* F, *and let* E *be a non-abelian simple group isomorphic to a proper section of* F.

(a) *Let* $A \trianglelefteq B \leq S(H)$ *be such that* $B/A \cong E$ *and put* $N = N_H(A) \cap N_H(B)$. *Suppose that* $A^* = A$ *whenever* $A^* \leq S(H)$ *with* $A^* \cap B = A$ *and* $N \leq N_H(A^*)$.

Then the canonical copy of H *in* $G = E \wr_N H$, *the twisted wreath product with respect to the action of* N *on* E *induced by the isomorphism* $E \cong B/A$, *is maximal in* G *and complements the base group* $S(G)$ $(\cong E^{(|H:N|)})$; *in particular,* $G \in \mathcal{P}'_{\mathrm{II}}$.

(b) *Conversely, if* H *occurs as a core-free maximal subgroup of the group* G *and complements* $S(G)$, *where* $S(G)$ *is minimal normal in* G *and a direct product of copies of* E, *then* $G \cong E \wr_N H$, *and the conditions stated in (a) are satisfied by* H *and* N (*with respect to suitable choices of* A *and* B).

Remark. A section B/A of $S(H)$ as in 1.5a can always be found provided only that $S(H)$ has some section $B^\#/A^\#$ isomorphic to E: choose a subgroup A of $S(H)$ maximal with respect to the properties that $A^\# \leq A$, $B^\# \leq N_{S(H)}(A)$, $B^\# \cap A = A^\#$, and set $B = B^\# A$.

Using ideas from the proof of Lafuente's Theorem from [9], the proof of 1.5 is straightforward from 1.1.

In view of Lafuente's Lemma and the remark after the statement of 1.5, the following is immediate from 1.4 and 1.5.

1.6 COROLLARY. *Let* $H \in \mathcal{P}_{\mathrm{II}}$ *with* $S(H) \cong F^{(n)}$ *for some non-abelian simple group* F. *Then* $G/S(G) \cong H$ *for some* $G \in \mathcal{P}'_{\mathrm{II}}$ *satisfying* $S(G) \cong E^{(m)}$ *with* E *non-abelian simple if and only if one of the following two conditions holds:*

(i) E *is isomorphic to a proper section of* F;

(ii) $E \cong F$, $n > 1$, *and there exists a (simple) diagonal subgroup* B *of some non-simple direct factor* R *of* $S(H)$ *such that the action of* $N_H(B)$ *on the set of simple direct factors of* R *(via conjugation) is transitive.*

In particular, a primitive group H of type II is isomorphic to $G/S(G)$ for some $G \in \mathcal{P}'_{II}$ unless, perhaps, the simple component of $S(H)$ is a minimal simple group. In the latter case, exceptions do occur, even when $n > 1$: for an example of a primitive group H of type II not satisfying condition (ii) above (with respect to $E \cong F$ and E minimal simple), but with $n > 1$, the reader is referred to [9].

§2 On the inclusion problem for $\mathcal{P}'_{II}$-groups

In this section we comment on the following

Question. *If* $H \lessdot G = E \wr_N H \in \mathcal{P}'_{II}$ (with E, as always, non-abelian simple), *what are the proper subgroups* G^* *of* G *containing* $S(G)$ *such that* $H \cap G^* \lessdot G^*$?

(Observe that $H \cap G^*$ is automatically core-free in G^*, provided only that $H \cap G^* \lessdot G^*$ and $S(G) \leq G^*$. Also note that, in the language of permutation groups, this question asks for the primitive "permutation subgroups" of the primitive permutation group G, i.e., for those subgroups G^* of G which are primitive with respect to the "same" faithful permutation representation — namely, the one on the cosets of H in G.)

Assume that $S(G) \leq G^* < G = E \wr_N H \in \mathcal{P}'_{II}$ with $H \lessdot G$. Put $H^* = H \cap G^*$ and $N^* = N \cap G^*$. Clearly, if $H^* \lessdot G^*$ then $S(G) \mathrel{\cdot\!\trianglelefteq} G^*$ and $G^* = E \wr_{N^*} H^*$, so this question amounts to asking:

When is $H^* <\cdot E \wr_{N^*} H^*$ *and* $S(G) = S(E \wr_{N^*} H^*)$?

Consider a series of subgroups $G^* = G_0 <\cdot G_1 <\cdot \ldots <\cdot G_k = G$ with $S(G) \leq G$. Clearly, if $H^*\ (= H \cap G^*) <\cdot G^*$ then for $i = 1,\ldots,k$, $H \cap G_i$ is a core-free maximal subgroup of G_i. Conversely, if $H \cap G_i <\cdot G_i$ for $i = 0,1,\ldots,k$, then G^* is a subgroup of G with the requested properties. Hence we may as well restrict attention to maximal subgroups of G; in fact, the answer to our question we are about to give for maximal subgroups does not easily translate into a result for general subgroups of G, other than by saying that it provides a recursive procedure to find those subgroups G^* of G we are interested in here.

In the remainder of this section we will deal with the case when $G^* <\cdot G$. In order that $S(G)$ be minimal normal in G^*, it is necessary that H^* acts transitively on the set of simple components of $S(G)$, which means that $H = H^*N$. From this observation together with 1.2 and 1.4 we obtain a criterion for $H^* <\cdot G^*$ in the case where the simple components of $S(H)$ and $S(G)$ are isomorphic:

2.1 PROPOSITION. *Suppose that*

$$H <\cdot G = E \wr_N H \in \mathcal{P}_{II}', \text{ where } H\ (\in \mathcal{P}_{II}) \text{ has } E \text{ as simple component of } S(H),$$

and define C *and* D *in terms of* N *and* E *as in 1.1. Assume that* $S(G) \leq G^* <\cdot G$, *and put* $H^* = H \cap G^*\ (<\cdot H)$ *and* $N^* = N \cap G^*$.

Then $H^* <\cdot G^*$ *if, and only if, the following holds:*

(i) $H = H^*N$; (ii) $D \leq H^*C$; (iii) $H^* \in \mathcal{P}_{II}$;

(iv) N^* *acts transitively (by conjugation) on the set of simple components of* S^*, *the product of all simple components of* $S(H^*)$ *not contained in* N^*.

For somewhat more detailed information we have to distinguish several cases (for each of which we will give an example to show that it can actually arise). We will retain the notation from the preceding Proposition.

Case 1: $S(H) \leq H^*$.

It is obvious from 2.1 and 1.4 that in this case $H^* <\cdot\, G^*$ if, and only if, in addition to (i) the following two conditions are satisfied:

(iii') $S(H)$ is minimal normal in H^*;

(iv') N^* acts transitively on the set of simple components of S.

2.2 EXAMPLE 1. Take H as the (non-twisted) wreath product $E \wr S_n$ with respect to the natural permutation representation of S_n for some $n \geq 4$, write its base group as $E^{(n)} = E_1 \times ... \times E_n$ ($E_i \cong E$ for $i = 1,...,n$), choose $S = E_1 \times ... \times E_{n-1}$, and let B be $\Delta(S)$, the canonical diagonal subgroup of S; clearly, $N_H(B) = S_{n-1}(B \times E_n)$ with the point stabilizer S_{n-1} of n in S_n. Then the subgroup of H defined by $H^* = A_n E^{(n)}$ satisfies (i,iii',iv'), for $N^* = A_{n-1}(B \times E_n)$.

Case 2: $1 < H^* \cap S(H) < S(H)$.

From 2.1 we get that in this case $H^* <\cdot\, G^*$ if, and only if, in addition to (i+ii+iv) the following condition is satisfied:

(iii") $H^* \cap S(H)$ is minimal normal in H^* (cf. the argument below);

Put $T = H^* \cap S(H)$. Clearly,

$$T = \langle (H^* \cap E^{\#})^h \mid h \in H^* \rangle,$$

where $E^{\#}$ is a product of m of the simple components of $S(H)$, and $H^* \cap E^{\#}$ is a diagonal subgroup of $E^{\#}$ isomorphic to E. Here $T \cdot\!\trianglelefteq H^*$, and hence $T \cong E^{(n)}$ for some $n \in \mathbb{N}$, requires that $m > 1$.

2.3 EXAMPLE 2. Consider the wreath product $K = A \wr C_2$ with $C_2 = \langle c \rangle$ of order 2 and A a group of outer automorphisms of a copy F of our given non- abelian simple group E (i.e., $A \leq \mathbf{Aut}(F)$ and $A \cap \mathbf{Inn}(F) = 1$) such that $|A| = 3$. Write K as semidirect product $\langle c \rangle(A \times A^c)$. Form the corresponding twisted wreath product

$$H = F \wr_A K = F \wr_A [\langle c \rangle(A \times A^c)].$$

By our remark towards the end of the introduction, this is a primitive group of type

II with $S(H) = F^T$ (the base group) for the transversal $T = A^c\langle c\rangle$ of A in K.

We write $T = T' \cup T''$ with $T' = A^c$ and $T'' = A^c c$ and define a subgroup H^* of H containing K by forming the canonical diagonal subgroup D of $F^{T'}$ isomorphic to F and putting

$$H^* = K(D \times D^c);$$

note that $D \times D^c$ is a K-invariant subgroup of $F^{T'} \times F^{T''}$ $(= F^T)$, and that

$$H^* \cong (AF) \wr \langle c\rangle \text{ (}AF\text{ denoting the semidirect product);}$$

in particular, $H^* \in \mathcal{P}_{II}$ with $S(H^*) = D \times D^c \cong E \times E$.

Further, let B be the canonical diagonal subgroup of the subgroup $F^1 \times F^c$ of F^T. Then B is c-invariant, whence 1.4 applies: setting

$$G = E \wr_N H, \text{ where } N = N_H(B) \text{ and } E \cong_N B,$$

provides us with a primitive group G of type II such that $H \lessdot G$ and $H \cap S(G) = 1$.

Moreover, by definition of B, $N \cap F^T$ comprises all components F^t with t ranging through $T \setminus \{1,c\}$. Since D and D^c are subdirect subgroups of $F^{T'}$ and $F^{T'c}$, respectively, from $1 \in T'$ we infer that

$$F^T = (H^* \cap F^T)(N \cap F^T).$$

It follows that $H = KF^T = H^*N$. A similar argument shows that $D \leq H^*C$ (where D and C are defined as before): indeed, $BC_{F^T}(B) \leq F^T = (H^* \cap F^T)C_{F^T}(B)$.

Finally, put $G^* = H^*S(G)$ and observe that N^* $(= N \cap G^*)$ interchanges the two factors E and E^c, for c is contained in both N and H^*.

We have now verified all the conditions imposed on the groups E, H, G, H^*, G^* by our remarks preceding this example.

Case 3: $H^* \cap S(H) = 1$.

In this case the conditions (i-iv) from 2.1 do not simplify any further.

2.4 EXAMPLE 3. Here we begin by choosing the candidate for H^*. Let C_4 be a cyclic group of order 4, take a copy F^* of our given simple group E and set

$$H^* = F^* \wr C_4, \text{ the standard wreath product.}$$

To establish some notation, suppose that $C_4 = \langle c \rangle$ with $c = (1,3,2,4) \in S_4$, and that $F^* \wr C_4$ is written as a semidirect product $C_4(F_1^* \times F_2^* \times F_3^* \times F_4^*)$ with F_i^* being the i-th component of the base group $(F^*)^{(4)}$ and S_4 acting on the latter in the canonical manner by permuting the components. Furthermore, for $I \subseteq \{1,2,3,4\}$, $\Delta[\times_{i \in I} F_i^*]$ will always denote the canonical diagonal subgroup of $\times_{i \in I} F_i^*$ isomorphic to F^*.

Observe that $\Delta[F_1^* \times F_2^*] \times \Delta[F_3^* \times F_4^*]$ is centralized by the subgroup C_2 of C_4, so the following is a subgroup of H^*:

$$M^* = C_2 \times (\Delta[F_1^* \times F_2^*] \times \Delta[F_3^* \times F_4^*]).$$

Consider its normal subgroup $K^* = C_2 \times \Delta[F_3^* \times F_4^*]$. Since the corresponding quotient is isomorphic to F^*, we may take yet another copy F of E and form the twisted wreath product

$$H = F \wr_{M^*} H^*$$

with respect to the action of M^* on F, with kernel K^*, induced by $F \cong \Delta[F_1^* \times F_2^*]$ and $M^*/K^* \cong \Delta[F_1^* \times F_2^*] \cong \mathbf{Inn}(\Delta[F_1^* \times F_2^*])$. Clearly, both H^* and H are monolithic primitive groups: note that $\mathbf{Core}_{H^*}(M^*) = 1$.

We choose a right a transversal T of M^* in H^* containing 1 and c, write the base group of H as F^T, and use a similar convention for the "diagonal subgroup operator" Δ as before. Now let $B = \Delta[F^1 \times F^c]$ $(\cong F \cong E)$ and $N = N_H(B)$ and note that $\langle c \rangle \leq N$. Hence 1.4 applies: the twisted wreath product G defined by

$$G = E \wr_N H$$

is such that its canonical subgroup H complementing the base group is a corefree maximal subgroup.

For a proof of $H = H^*N$, first observe that $B = \Delta[F^1 \times F^c]$ requires that

$$N \leq N_H(F^1 \times F^c) = N_{H^*}(F^1 \times F^c)F^T = C_4(\Delta[F_1^* \times F_2^*] \times \Delta[F_3^* \times F_4^*])F^T,$$

for

$$N_{H^*}(F^1 \times F^c) = N_{H^*}(F^1)\langle c\rangle = M^*\langle c\rangle = C_4(\Delta[F_1^* \times F_2^*] \times \Delta[F_3^* \times F_4^*]).$$

Since obviously

$$C_4 \times \Delta[F_1^* \times F_2^* \times F_3^* \times F_4^*] \leq N,$$

it readily follows that

$$N \cap F^T = B \times (\times_{t \in T\setminus\{1,c\}} F^t) \text{ and } N \cap H^* = N^*,$$

where

$$N^* = C_4 \times \Delta[F_1^* \times F_2^* \times F_3^* \times F_4^*].$$

However, one also checks that

$$\Delta(F^1 \times \Delta[F_3^* \times F_4^*]) \leq N.$$

Consequently,

$$|H:H^*N| = \frac{|H|}{|H^*N|} = \frac{|H^*|\cdot|F^T|\cdot|H^* \cap N|}{|H^*|\cdot|N|} = \frac{|F^T|\cdot|H^* \cap N|}{|N \cap F^T|\cdot|N:N \cap F^T|}$$

$$= \frac{|F^T|\cdot|H^* \cap N|}{|N \cap F^T|\cdot|N:(H^* \cap N)(N \cap F^T)|\cdot|H^* \cap N|} \leq \frac{|E|}{|E|} = 1,$$

and we get that

$$H = H^*N.$$

Next, we note that from

$$H^* \cap N = N^* = C_4 \times \Delta[F_1^* \times F_2^* \times F_3^* \times F_4^*]$$

we get that N^* covers D/C: in fact, $\Delta[F_1^* \times F_2^* \times F_3^* \times F_4^*]$ induces the group of all inner automorphisms in B.

Finally, we observe that $H^* \in \mathcal{P}_{II}$ holds trivially, and N^* (which contains c) clearly acts transitively on $\{F^1, F^c\}$, and so the group G^* defined by

$$G^* = E \wr_{N^*} H^*$$

is a subgroup of G satisfying $H^* <\cdot\, G^*$.

We leave it to the reader to carry out a corresponding analysis for the case when the simple component of S(G) is a proper section of the simple component of S(H).

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