Groups with uniform automorphisms

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Estratto

ATTI DEL CONVEGNO INTERNAZIONALE DI
TEORIA DEI GRUPPI E GEOMETRIA COMBINATORIA
FIRENZE, 23-25 OTTOBRE 1986

Supplemento ai Rendiconti del Circolo Matematico di Palermo

Serie II - numero 19 - 1988

Via Archirafi, 34 - 90123 Palermo (Italia)

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1. In 1957 and 1958 Graham Higman [3] and Guido Zappa [12], [13] published independent proofs of the result that if a finite soluble group has a fixed-point-free automorphism of prime order than the group is nilpotent. This seems to have been folklore for some time by then (Huppert and Wielandt [5] attributed it to Witt; see also the first paragraph in Zappa [13]), but it was attracting attention again. The thesis of Thompson (see [10]) created a sensation by showing that the result holds even if solubility is not assumed. Higman [3] relaxed finiteness instead: he proved that there is a function k such that each locally nilpotent group with a fixed-point-free automotphism of prime order p is nilpotent of class at most $k(p)$. On the other hand, a free group of rank p has fixed-point-free automorphisms of order p (any automorphism which permutes some free generating set cyclically), so one cannot omit both solubility and finiteness. Indeed, the infinite dihedral group has a fixed-point-free automorphism of order 2, so solubility without some suitable finiteness condition is still not enough for nilpotence.

In 1958, Zappa [13] proposed another way to avoid the assumption of finiteness. It is very easy to see that an automorphism α is fixed-point-free if and only if the map $x \mapsto x^{-1}x^{\alpha}$ is one-to-one. A map of a finite set to itself is one-to-one if and only if it is onto. With this in mind, Zappa called an automorphism α of a group G uniform if to each element g of G there is an x in G such that $x^{-1}x^{\alpha} = g$. To this day, no-one seems to have found a nonnilpotent group with a uniform automorphism of prime order. Zappa himself proved in [13] that all polycyclic groups which have such automorphisms must be nilpotent. In 1960, Curzio [1] gave a series of similar theorems. In my 1961 thesis [8], I extended their results by proving that a group with a uniform automorphism of prime order must be nilpotent if *residually* it is finitely generated soluble, or soluble with minimum condition for subgroups, or locally normal. However, these were still assumptions involving finiteness conditions which I thought unnecessary: the theorem I really wanted to prove then had eluded me unitil recently.

Recall that (in the terminology of Kurosh [9]) an SI*-group is a group having an ascending invariant chain with abelian quotiens: that is, a group all of whose nontrivial quotients have nontrivial abelian normal subgroups.

Theorem 1. *Every locally residually* SI*-*group which admits a uniform automorphism of prime order must be nilpotent*.

This vindicates Zappa's approach by showing that a very weak solubility condition (satisfied even by the free groups), without any kind of finiteness condition, is sufficient for the nilpotence of groups with uniform automorphisms of prime order. As we have seen, fixed-point-free automorphisms of prime order do not restrict group structure in the same way.

In view of Higman's theorem quoted above, one may also want to compare nilpotent groups with uniform automorphisms of order p and nilpotent groups with fixed-point-free automorphisms of order p. For example, do the former also have their nilpotency class bounded by k(p)? The answer lies in considering nilpotent groups G with an automorphism α of order p such that α is both fixed-point-free and uniform. By Higman's theorem, the class of such a G is at most k(p). If H is an α-admissible subgroup of G, then the restriction $\alpha \downarrow H$ is obviously a fixed-point-free automorphism of H. If N is an α-admissible normal subgroup of G, then the automorphism induced by α on G/N (which we shall denote by α/N) is obviously uniform.

Theorem 2. *All nilpotent groups with fixed-point-free automorphisms of order* p *occur as admissible sugroups* H *in such* G, *the relevant automorphism of* H *being* $\alpha \downarrow H$. *All nilpotent groups with uniform automorphisms of order* p *occur as admissible quotients* G/N *(even with central* N) *of such* G, *the relevant automorphism of* G/N *being* α/N.

It is a well known and almost trivial observation that every nilpotent group with a fixed-point-free automorphism of order p is p-torsionfree (that is, has no element of order p). We shall see that *every nilpotent group with a uniform automorphism of order* p *is p-radicable* (that is, a group in which each element has a pth root). It follows that *the Sylow p-subgroup of such a group is central*.

2. To explain what was missing from the proof of Theorem 1 until recently, let us say that an automorphism α of a group G is p-<u>splitting</u> if

$$\alpha^p = 1 \ , \ \text{and} \ \ g g^{\alpha} g^{\alpha^2} \ldots g^{\alpha^{p-1}} = 1 \ \text{ for all } g \text{ in } G.$$

What makes p-splitting automorphisms relevant here is that every uniform automorphism of order p is obviously p-splitting. Moreover, the restriction of a p-splitting automorphism to an admissible subgroup is p-splitting, but restrictions of uniform automorphisms need not be uniform. Splitting automorphisms have also occured in the context of what became known as the Hughes problem or the H_p-problem. Building on Thompson's [10] and on a paper [4] of Hughes and Thompson, Kegel [6] proved that if a finite group has a p-splitting automorphism then it must be nilpotent. I also proved this in [8] (using [10] but independently of [4] and [6]). Another key reult in [8] was Theorem 6.1.2: if a locally SI*-group G has a p-splitting automorphism, then G has a normal Sylow p-subgroup P, and G/P is nilpotent of class at most k(p). It is easy to see that if the p-splitting automorphism of G is in fact uniform, then so is its restriction to P. Being a locally SI* p-group, P is of course locally nilpotent [because any finitely generated subgroup of P is finite, as a routine induction on the SI*-length of that subgroup shows]. The trouble was that I could say nothing more about P. This deadlock was broken by a recent theorem of Khukhro [7].

<u>Khukhro's Theorem</u>. <u>There is a function</u> f <u>such that all d-generator nilpotent groups with p-splitting automorphisms have nilpotency class at most</u> f(d,p).

This is a very deep theorem. (For example, since the identity automorphism of a group of exponent p is p-splitting, it includes Kostrikin's celebrated therem on the Restricted Burnside Problem). It enabled me to deduce from Theorem 6.1.2. of [8] a slightly weaker version of Theorem 1, namley one in which the <u>locally residually</u> SI* assumption was strengthened to <u>residually locally</u> SI*.

This paper was written while I enjoyed the hospitality of the Istituto Matematico "Ulisse Dini" at Firenze. In a discussion there, Dr. J.S. Wilson and I observed that Khukhro's Theorem also allows one to strengthen Theorem 6.1.2 to the following.

<u>Theorem 3</u>. <u>Every locally residually SI*-group with a p-splitting automorphism is locally nilpotent, and the quotient modulo its Sylow p-subgroup is nilpotent of</u>

class at most $k(p)$.

The present strong form of Theorem 1 makes use of this instead.

The proof of Thorem 3 will be given in the next section. The penultimate section will then be devoted to the proof of Theorem 1, and the last section will contain a sketch of the proof of Theorem 2.

3. For the proof of Theorem 3, it is convenient to work with a species of (universal) algebras other than groups: namely, with groups which have an extra unary operation and are subject to extra axioms which ensure that the extra operation is a p-splitting group automorphism. Khukhro's Theorem means that is this variety of algebras the locally nilpotent algebras form a subvariety: call that $\underset{\sim}{X}$. As for any variety, it is clear that if an algebra is residually or locally in $\underset{\sim}{X}$ then it belongs to $\underset{\sim}{X}$. For the first statement of Theorem 3, it therefore suffices to prove the nilpotence of a finitely generated - say, a d-generator - SI*-group G with a p-splitting automorphism. Let X be the smallest admissible normal subgroup of G such that G/X lies in $\underset{\sim}{X}$. By Khukhro's Theorem, G/X is then nilpotent of class at most $f(d,p)$, and the lower central series of G terminates in X. Thus X is the normal closure of the left-normed commutators of weight $1+f(d,p)$ with entries from a given finite generating set of G. If $X \neq 1$, let M be maximal among the admissible normal subgroups of G not containing X; then MX/M is a minimal normal admissible subgroup in the SI*-group G/M, so MX/M is abelian, and G/M is a finitely generated abelian-by-nilpotent group with a p-splitting automorphism. By a well known theorem of P.Hall [2], G/M is residually finite; hence by the theorem of Kegel [6] (or by Theorem 7.1.6 of [8]), G/M is residually nilpotent; and then by Khukhro's Theorem G/M is nilpotent [of class at most $f(d,p)$] . Thus in fact $X=1$, and G is nilpotent as required.

For the second statement of Theorem 3, note that the quotient of a locally nilpotent group modulo its Sylow p-subgroup is p-torsionfree, and that a p-splitting automorphism of a p-torsionfree group is always fixed-point-free: so the theorem of Higman [3] which we have quoted completes the proof.

4. My thesis [8] contained a number of results which had arisen in discussions I had had with Dr. I.D.Macdonald. One of these was Lemma 3.4.2, of which we shall need the following variant.

Lemma. Let H be a nilpotent group generated by two infinite sequences *of elements*

$$x_1, x_2, \dots \qquad \text{and} \qquad y_1, y_2 \dots$$

and having an automorphism β *of order* p *such that*

$$x_{i+1}^{\beta} = x_{i+1}x_i \quad \text{and} \quad y_{i+1}^{\beta} = y_{i+1}y_i \quad \text{for each positive integer } i .$$

The H *is p-radicable.*

Proof. Suppose at first that H is actually abelian. Then $\beta-1$ is surjective andomorphism of H. Further, in the endomorphism ring of H one has that

$$(\beta - 1)^p = \begin{cases} p(1-\beta) & \text{when } p = 2 , \\ p\sum_{i=1}^{p-1} k_i\beta^i(-1)^{p-i} & \text{when } p > 2 \end{cases}$$

where k_i denotes the integer defined by $k_i = (p-1)!/i!\,(p-i)!$ (exploiting the fact that the relevant binomial coefficients are all divisible by p). This shows that in either case $(\beta-1)^p$ is the composite of $h \mapsto h^p$ and of another endormorphism of H. Since $(\beta-1)^p$ is surjective, it follows that $h \mapsto h^p$ must also be surjective: that is, H is p-radicable.

In the general case, this first step shows that the factor group H/H' of H over the commutator subgroup H' is p-radicable, and from this the p-radicability of H follows by a standard result (Theorem 4.6 in Warfield [11]).

We are now ready to prove Theorem 1. Consider a locally residually SI*-group G with a uniform automorphism α of prime order p. Then α is p-splitting, so Theorem 3 gives that G is locally nilpotent, the elements of p-power order in G form a normal subgroup P, and G/P is nilpotent of class at most $k(p)$.

Let x_1 be any element of P and y_1 any element of G. Since α is uniform, one can choose two infinite sequences

$$x_1, x_2, \dots \qquad \text{and} \qquad y_1, y_2, \dots$$

of elements of G inductively so that

$$x_{i+1}^{\alpha} = x_{i+1}x_i \quad \text{and} \quad y_{i+1}^{\alpha} = y_{i+1}y_i \quad \text{for each positive integer } i$$

For each such i, let H_i be the smallest α-admissible subgroup to contain x_i and y_i: that is,

$$H_i = \langle x_i, x_i^{\alpha}, \ldots, x_i^{\alpha^{p-1}}, y_i, y_i^{\alpha}, \ldots, y_i^{\alpha^{p-1}} \rangle$$

Since H_{i+1} is α-admissible and contains x_i and y_i, it contains H_i: so the H_i form an ascending chain of subgroups. Let H denote the union of the H_i. Next we claim that if $i > j \geq 1$ then

In all superscripts on this page, replace i by j

$$x_i^{\alpha^i} \in \langle x_i, x_{i-1}, \ldots, x_{i-j} \rangle \quad \text{and} \quad y_i^{\alpha^i} \in \langle y_i, y_{i-1}, \ldots, y_{i-j} \rangle \ .$$

This clearly holds when $j = 1$, while if in fact $i > 1 + j$ then it implies that

$$\begin{aligned} x_i^{\alpha^{i+1}} = (x_i^{\alpha^i})^{\alpha} &\in \langle x_i, x_{i-1}, \ldots, x_{i-j} \rangle^{\alpha} \\ &= \langle x_i^{\alpha}, x_{i-1}^{\alpha}, \ldots, x_{i-j}^{\alpha} \rangle \\ &\leq \langle x_i, x_{i-1}, \ldots, x_{i-j}, x_{i-(j+1)} \rangle \ , \end{aligned}$$

so the claim follows by induction on j. The point of it is that therefore

$$H_i \leq \langle x_i, x_{i-1}, \ldots, x_{i-p+1}, y_i, y_{i-1}, \ldots, y_{i-p+1} \rangle$$

whenever $i \geq p$, whence $\langle x_1, x_2, \ldots, y_1, y_2, \ldots \rangle$ contains all the H_i and so contains H. The converse of the last inclusion is obvious, so we have that

$$\langle x_1, x_2, \ldots, y_1, y_2, \ldots \rangle = H \ .$$

Recall that G is locally nilpotent, thus by Khukhro's Theorem each H_i is nilpotent of class at most $f(2p, p)$, and therefore so is H. By the Lemma (applied with $\beta = \alpha{\downarrow}H$), H is p-radicable. A well-known theorem of Černikov (Theorem 4.12 in Warfield [11]) says that in a radicable nilpotent group the torsion subgroup is central. By the same argument, in a p-radicable nilpotent group all elements of p-power order are central. Thus x_1 is central in H. We conclude that the arbitrary element x_1 of P commutes with the arbitrary element y_1 of G, so P is central in G. Since G/P is nilpotent, this proves the nilpotence of G.

This argument has incidentally also established that G is p-radicable and its Sylow p-subgroup is central.

5. The first half of Theorem 2 is a special case of Theorem 5.2.1 of [8]. From its proof, we shall need (special cases of) two steps. By Lemma 5.1.1 of [8], a fixed-point-free automorphism of prime order p of a nilpotent group is always p-splitting. By Lemma 5.1.3 of [8], a p-splitting automorphism of a p-radicable nilpotent group is always uniform.

(The deduction of the first half of Theorem 2 from these steps may be sketched as follows. If H is a nilpotent group with a fixed-point-free automorphism β of prime order, then H is p-torsionfree. Like each p-torsionfree nilpotent group, H can be embedded in a certain p-radicable nilpotent group G which one might call the Mal'cev p-completion (or the p'-localization) of H. This G is determined by H up to isomorphism, and each automorphism of H extends uniquely to an automorphism of G. In particular, G has an automorphism α of order p such that $\alpha \downarrow H = \beta$. Each nontrivial element of G has a nontrivial power in H, so α must be fixed-point-free. The two steps quoted then show that α is also uniform.)

For a sketch of the proof of the second half of Theorem 2, let G be a nilpotent group, say, of class c, with a uniform automorphism α of order p. Let X be any generating set of G. Take a family of copies of the additive group $\mathbb{Z}[1/p]^+$ of the ring $\mathbb{Z}[1/p]$ generated by the reciprocal of p, the family being indexed by the cartesian product of X and $\{0,1, \ldots, p-1\}$, and a corresponding family of homomorphisms of these groups into G so that the image of the homomorphism with index (x,j) contains $x\alpha^j$. This can be done because G is p-radicable. Form the free product F of this family of groups and the homomorphism $\phi : F \to G$ determined by this family of homomorphisms : then ϕ is surjective. Moreover, F has an automorphism ψ of order p which, for each x in X, permutes cyclically the free factors indexed by $(x, 0)$, $(x, 1), \ldots, (x, p-1)$, and is such that $\psi\phi = \phi\alpha$. Let A be the largest quotient of F which is nilpotent of class c : then ψ induces an automorphism π of order p on A, and ϕ factors through a homomorphism γ of A onto

G such that $\pi\gamma = \gamma\alpha$. In particular, the kernel C of γ is admissible under π .

Since the free factors of F are locally cyclic, F is locally free. In a free group, if f^n is a product of commutators of weight $c + 1$ then so is f itself. It follows that A is torsionfree. In a p-torsionfree nilpotent group, $x^p = y^p$ implies $x = y$ (see Theorem 4.10 in Warfield [11]), so the quotient modulo a p-radicable normal subgroup is p-torsionfree. The largest abelian quotient of A is obviously p-radicable, and therefore so is A (see Theorem 4.6 in Warfield [11]). Of course A/C is isomorphic to G, so the Sylow p-subgroup B/C of A/C is central in A/C. Thus the mutual commutator subgroup $[A, B]$ is contained in C; it is p-radicable (see Corollary-Exercise 4.15 in Warfield [11]), and hence $A/[A, B]$ is p-torsionfree. Let $D/|A, B|$ denote the unique largest p-radicable subgroup of the abelian group $C/[A, B]$. Then B/D is abelian, p-torsionfree, p-radicable, and C/D has no nontrivial p-radicable subgroup.

Of course $B, [A, B], D$ all admit π. Consider the automorphism π/D induced on A/D by π. Clearly, $(\pi/D)^p = 1$. Let E/D denote the subgroup of A/D consisting of the fixed points of π/D . If $e \in E$, then $e^{-1}e^{\pi} \in D \le C$ so eC is a fixed point of π/C ; on the other hand π/C is p-splitting (because it corresponds to α under the isomorphism $A/C \cong G$ induced by γ) and therefore its fixed points have order dividing p; so $eC \in B/C$. This proves that $E \le B$. If $b \in B$ and $b^p \in E$ so $(b^p)^{\pi} \equiv b^p \bmod D$, then $(b^{-1}b^{\pi})^p \in D$ follows (becasue B/D is abelian), so $b \equiv b^{\pi} \bmod D$ (because B/D is p-torsionfree) : that is, $b \in E$. Thus B/E is p-torsionfree; as B/D is p-radicable, it follows that E/D is p-radicable.

Suppose that $E/D > 1$: we shall show that this leads to a contradiction. Since C/D has no nontrivial p-radicable subgroup, now $E/D \not\le C/D$, whence $EC/C > 1$; being a homomorphic image of the p-radicable E/D , this EC/C is p-radicable. On the other hand, EC/C consists of fixed points of the p-splitting automorphism π/C and so must have exponent dividing p . This is impossible, so $E/D = 1$: that is, π/D is fixed-point-free.

We have reached the conclusion that G occurs as the quotient $(A/D)/(C/D)$ of a p-radicable nilpotent group A/D with a fixed-point-free automorphism π/D , with C/D central in A/D and α being $(\pi/D)/(C/D)$. All that remains to see is that π/D is also uniform, but that now follows directly from the two steps quoted from [8] at the beginning of this section.

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