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Some theorems on wreath products

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Some theorems on wreath products

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1. *Discussion**)

The Embedding Theorem constructs, for each group G and each subgroup H of index n in G, embeddings φ of G in the (unrestricted, permutational) wreath product $H \operatorname{Wr} S_n$ of H by the relevant symmetric group. Such wreath products have a functorial property which gives for each homomorphism $\alpha\colon H \to A$ a homomorphism $\alpha \operatorname{Wr} S_n\colon H \operatorname{Wr} S_n \to A \operatorname{Wr} S_n$. The composites $\alpha\uparrow$ of φ and $\alpha \operatorname{Wr} S_n$ are of fundamental importance. For example, if n is finite and A is a general linear group, GL_k say, so α is a linear representation of H, then $\alpha\uparrow$ (composed with the obvious inclusion of $GL_k \operatorname{Wr} S_n$ in GL_{kn}) is the induced representation of G. In this sense at least, the Embedding Theorem goes back all the way to Frobenius. (For recent expositions, see § 5 in COSSEY, KEGEL, KOVÁCS [1] and § 4 in ROBINSON, WILSON [4].)

The first question considered here is: how does one recognize whether a homomorphism $G \to H \operatorname{Wr} S_n$ is one of the embeddings given by that Theorem? What distinguishes these embeddings from others?

Towards an answer we must emphasize first that the Theorem gives not just one embedding but a whole lot: one for each of the $|H|^n$ transversals of H in G. Second, the symmetric group which really occurs in the Theorem is that acting on the set of all cosets of G modulo H, while the functorial view demands that we think of S_n as the symmetric group on some set given without reference to G or H: so we have to choose one of the $n!$ possible identifications of these two sets. All told, we have $n!|H|^n$ options. It is not hard to see that the resulting embeddings differ precisely by inner automorphisms of the wreath product: if we let $\operatorname{Inn}(H \operatorname{Wr} S_n)$ act on $\operatorname{Hom}(G, H \operatorname{Wr} S_n)$ by composition, they form a single complete orbit of this action. (In general, there are some coincidences so we get fewer than $n!|H|^n$ distinct embeddings: we shall return to this point later.)

More notation is needed before we can proceed. It will *not* be assumed that n is finite. Throughout, I shall denote a fixed set of cardinality n, and for emphasis we shall often write S_I rather than S_n. The wreath product $A \operatorname{Wr} S_I$ is the semidirect product of S_I and the group A^I of all functions $I \to A$. [Permutations and

*) A condensed, preliminary version of this paper was presented at the International Colloquium on Group Theory held in memory of Tibor Szele (1918—1955) at Debrecen (Hungary), 16—20 September 1985.

functions will be written on the left and composed accordingly. The action of S_I on A^I is defined in terms of this composition but written exponentially: so

$$f^p(i) = f(pi) \quad \text{whenever} \quad f \in A^I, \quad p \in S_I, \quad i \in I.]$$

The natural projection of $A \operatorname{Wr} S_I$ onto S_I will be denoted π (or π^A when a distinction appears necessary). Given $W = A \operatorname{Wr} S_I$ and an i in I, the elements pf of W such that $pi = i$ form a subgroup, W_i say, which has an obvious direct decomposition $A \times (A \operatorname{Wr} S_{I \setminus \{i\}})$: the corresponding projection $W_i \to A$, $pf \mapsto f(i)$ will be called π_i (or π_i^A when appropriate). [Homomorphisms will be written on the right and composed accordingly.] The answer to the recognition problem above may now be expressed as follows.

Theorem 1. *A homomorphism* φ: $G \to W = H \operatorname{Wr} S_I$ *is one of the embeddings given by the Embedding Theorem if and only if*

(a) $G\varphi\pi$ *is transitive (as subgroup of* S_I*), and*
(b) there is an element 0 *in* I *such that*
  (b1) the stabilizer of 0 *in* G *with respect to the permutation representation* $\varphi\pi$ *is* H, *and*
  (b2) the restriction $\varphi\downarrow$: $H \to W_0$ *followed by* π_0 *is an inner automorphism of* H.

It must never be forgotten that here W is the group concretely constructed above, with a distinguished copy (the "top group") of S_I and a distinguished copy (the "base group") of H^I as semidirect factors, and equipped with π and the π_i. Changing to a different wreath decomposition of this group may easily spoil the result. For example, let G be a nonabelian group of order 6 and H a subgroup of index 3 in G. Then the base group has two conjugacy classes of complements in W, one being the class containing the top group; it is easy to verify that the relevant embeddings are precisely those whose whose images fall into the other class. This illustrates the sensitivity of Theorem 1 to the slightest change in the wreath decomposition: one cannot even replace the top group by another (nonconjugate) complement of the base group, without upsetting the conclusions.

This recognition problem has an obvious variant: given a homomorphism γ: $G \to W = A \operatorname{Wr} S_I$, how can one tell whether $\gamma = \alpha\uparrow$ for some suitable α?

Theorem 1′. *Let* γ: $G \to W = A \operatorname{Wr} S_I$ *be any homomorphism. There is a subgroup* H *in* G *(of index equal to the cardinality of* I*) and a homomorphism* α: $H \to A$ *such that* $\alpha\uparrow = \gamma$ *(for a suitable identification of* I *with the set of the left cosets of* G *modulo* H, *and for a suitable transversal of* H *in* G*), if and only if*

(a) $G\gamma\pi$ *is transitive (as subgroup of* S_I*), and*
(b) there is an element 0 *in* I *such that*

$$G\gamma \leqq (G\gamma \cap W_0)\pi_0 \operatorname{Wr} S_I.$$

Of course here $(G\gamma \cap W_0)\pi_0 \operatorname{Wr} S_I$ is thought of as a subgroup of $A \operatorname{Wr} S_I$ [embedded via $\beta \operatorname{Wr} S_I$ where β is the inclusion of $(G\gamma \cap W_0)\pi_0$ in A]. Note that (a), (b) do not involve γ directly, only its image $G\gamma$. Also, once (a) is assumed, the inclusion in (b) holds either for all elements of I or for none at all.

The second question of this paper also comes in two versions. One, what is the cardinality of the set of all embeddings φ constructed by the Embedding Theorem for given G and H? The discussion above leads to the conclusion that it is the index in $H \operatorname{Wr} S_I$ of the centralizer $\mathbf{C}_W(G\varphi)$ of the image of any one of these embeddings, so the real question is to determine $\mathbf{C}_W(G\varphi)$.

Theorem 2. *Let H be a subgroup of index n in a group G, and $\varphi\colon G\to W= =H \operatorname{Wr} S_n$ any one of the embeddings given by the Embedding Theorem. Then $\mathbf{C}_w(G\varphi)\cong\mathbf{C}_G(H)$. If G is finite, the number of distinct φ of this kind is therefore*

$$(n-1)!\,|H|^{n-1}|G{:}\mathbf{C}_G(H)|.$$

The second version asks: given G, H, and $\alpha\colon H\to A$, what is the cardinality of the set of all homomorphisms $\alpha\!\uparrow\colon G\to A \operatorname{Wr} S_n$ "induced" by this α? An argument similar to the discussion above yields that it is the index in W of any one $\mathbf{C}_W(G(\alpha\!\uparrow))$, except that W must be taken as $(H\alpha) \operatorname{Wr} S_n$, *not* as $A \operatorname{Wr} S_n$. In place of $\mathbf{C}_G(H)$, the answer will involve the subgroup $\mathbf{C}_G(H/\ker\alpha)$ defined as the set of those elements g of G for which the mutual commutator $[H,g]$ is contained in $\ker\alpha$: that is, those g which normalize both H and $\ker\alpha$, and whose (conjugation) action on $H/\ker\alpha$ is trivial. Of course when $A=H$ and α is the identity map, this is just $\mathbf{C}_G(H)$, and the $\alpha\!\uparrow$ are just the φ of Theorem 2. That result is therefore a special case of the following.

Theorem 2′. *Let H be a subgroup of index n in a group G, let $\alpha\colon H\to A$ be a homomorphism, and $\alpha\!\uparrow\colon G\to A \operatorname{Wr} S_n$ any one of the homomorphisms induced by α. Set $W=(H\alpha) \operatorname{Wr} S_n$: then $\mathbf{C}_W(G(\alpha\!\uparrow))\cong\mathbf{C}_G(H/\ker\alpha)/\ker\alpha$. If G is finite, the number of distinct $\alpha\!\uparrow$ induced by the given α is*

$$(n-1)!\,|H\alpha|^{n-1}|G{:}\mathbf{C}_G(H/\ker\alpha)|.$$

It may be worth noting that the proofs of Theorems 2 and 2′ yield explicit isomorphisms, not just the existence of isomorphisms.

2. *Proofs*

Theorems 1 and 1′ depend on the answer to a related question: how can one recognize whether two homomorphisms $\gamma,\gamma'\colon G\to W=A \operatorname{Wr} S_I$ are the same up to composition with an inner automorphism of W? In turn, this is an extension of the familiar question: how can one recognize whether $\gamma\pi$ and $\gamma'\pi$ are equivalent as permutation representations $G\to S_I$? The answer to that is of course classical, the essential case being that of transitive representations. Accordingly, let us narrow down our question: after correction by an inner automorphism of W induced by an element of the top group S_I, we assume that $\gamma\pi$ and $\gamma'\pi$ are equal and transitive, and ask whether γ and γ' differ only by an inner automorphism of W induced by some element of the base group A^I. The answer is: if and only if $(\gamma\!\downarrow)\pi_0$ and $(\gamma'\!\downarrow)\pi_0$ differ only by an inner automorphism of A. Here 0 is any element of I, and $\gamma\!\downarrow$, $\gamma'\!\downarrow$ are the restrictions of γ, γ', respectively, to $H\to W_0$ where H is the stabilizer of 0 with respect to $\gamma\pi$. This is contained in the Uniqueness Theorem of [2], which may be conveniently paraphrased as follows.

Uniqueness Theorem. *Let γ and γ' be homomorphisms of a group G into a wreath product A Wr S_I, such that $\gamma\pi=\gamma'\pi$ and $G\gamma\pi$ is transitive as subgroup of S_I. Consider*

$$F=\{f\in A^I \mid \gamma'=\gamma(\operatorname{inn} f)\}$$
$$=\{f\in A^I \mid g\gamma'=f^{-1}(g\gamma)f \textit{ for all } g \textit{ in } G\},$$
$$B=\{b\in A \mid (\gamma'\downarrow)\pi_0=(\gamma\downarrow)\pi_0(\operatorname{inn} b)\}$$
$$=\{b\in A \mid h\gamma'\pi_0=b^{-1}(h\gamma\pi_0)b \textit{ for all } h \textit{ in } H\}.$$

Then π_0 maps F one-to-one onto B.

Addendum. The inverse of this bijection may be described in terms of a transversal of H in G but is of course independent of that. To each i in I choose a t_i in G such that $(t_i\gamma\pi)0=i$ [equivalently, $(t_i\gamma'\pi)0=i$]. Write $t_i\gamma$ as p_if_i with p_i from the top group S_I and f_i from the base group A^I; similarly, set $t_i\gamma'=p_if_i'$. The inverse bijection maps an element b of B to the element f of F defined by

$$f(i)=f_i(0)bf_i'(0)^{-1} \quad \text{for all } i \text{ in } I.$$

Proof of Theorem 1. The "only if" claim comes straight from the proof of the Embedding Theorem and we shall not spell it out: the reader can easily elaborate details from the sketch given on p. 216 of [1]. Take 0 as the element of I identified with the trivial coset of H in G; the inner automorphism of H in question is induced by the representative of this coset in the transversal chosen.

For the "if" part, suppose (a) and (b) hold; let t_0 be an element of H which induces the inner automorphism $(\varphi\downarrow)\pi_0$. For each i in I other than this 0, choose a t_i in G such that $(t_i\varphi\pi)0=i$: this gives a transversal of H in G. Identify I with the set of the cosets of G modulo H by matching each i to t_iH. Let φ' be the embedding constructed with this choice of transversal and identification. It is obvious that $\varphi\pi=\varphi'\pi$ and that $(\varphi\downarrow)\pi_0=\operatorname{inn} t_0=(\varphi'\downarrow)\pi_0$. Invoke the Uniqueness Theorem with φ, φ', H in place of γ, γ', A, noting that now $1\in B$: hence F is also nonempty. Take any f in F: then $\varphi=\varphi'(\operatorname{inn} f^{-1})$, and of course $\varphi'(\operatorname{inn} f^{-1})$ is just an embedding constructed from a different transversal [namely, from that with $t_if(i)^{-1}$ in place of t_i]. This completes the proof of Theorem 1.

Proof of Theorem 1'. For the "only if" part, we have to show that (a) and (b) hold when $\gamma=\alpha\uparrow$. Let $\alpha\uparrow=\varphi(\alpha \text{ Wr } S_I)$ with a $\varphi\colon G\to H \text{ Wr } S_I$ given by the Embedding Theorem, and 0 an element of I such that (b1) and (b2) of Theorem 1 hold. The proof depends on the fact that π and π_0 are "natural". To express this we now distinguish π^H from π^A and π_0^H from π_0^A, but simply keep W and W_0 for the domains of π^A and π_0^A, leaving the domains of π^H and π_0^H unnamed. The naturality of π means that $(\alpha \text{ Wr } S_I)\pi^A=\pi^H$; this yields that $G(\alpha\uparrow)\pi^A=G\varphi\pi^H$, so $G(\alpha\uparrow)\pi^A$ is transitive by (a) of Theorem 1. The naturality of π_0 means that $((\alpha \text{ Wr } S_I)\downarrow)\pi_0^A=\pi_0^A\alpha$ for the relevant restriction $(\alpha \text{ Wr } S_I)\downarrow$: this yields that

$$(\alpha\uparrow\downarrow)\pi_0^A=(\varphi\downarrow)((\alpha Wr S_I)\downarrow)\pi_0^A=(\varphi\downarrow)\pi_0^H\alpha.$$

As $H(\varphi\downarrow)\pi_0^H=H$ by (b2) of Theorem 1, we have $H(\alpha\uparrow\downarrow)\pi_0^A=H(\varphi\downarrow)\pi_0^H\alpha=H\alpha$. Of course $H(\alpha\uparrow\downarrow)=H(\alpha\uparrow)$, while $H(\alpha\uparrow)\pi^A=H\varphi\pi^H$ and (b1) of Theorem 1 give that $H(\alpha\uparrow)\leqq G(\alpha\uparrow)\cap W_0$: hence by the conlusion of the previous sentence $H\alpha\leqq$

$\leqq (G(\alpha\uparrow)\cap W_0)\pi_0^A$. In view of $G(\alpha\uparrow)\leqq(H\,\mathrm{Wr}\,S_I)(\alpha\,\mathrm{Wr}\,S_I)=(H\alpha)\,\mathrm{Wr}\,S_I$, this proves the inclusion claimed in (b).

The proof of the "if" claim depends on the Addendum to the Uniqueness Theorem: so assume (a), (b), and define H as the stabilizer (with respect to the permutation representation $\gamma\pi^A$) of the 0 of (b), so $H\gamma=G\gamma\cap W_0$. Define α: $H\to A$ as $(\gamma\downarrow)\pi_0^A$; the inclusion in (b) may then be written as $G\gamma\leqq(H\alpha)\,\mathrm{Wr}\,S_I$. By (a), to each i in I one may choose a t_i in G such that $(t_i\gamma\pi^A)0=i$, and these form a transversal of H in G. Define γ' as $\alpha\uparrow$ formed with respect to such a transversal and the matching identification of i with t_iH, for each i in I. Elaborating this definition of γ' shows that $(g\gamma'\pi^A)i=j$ means $gt_iH=t_jH$; by the definition of H, this is equivalent to $((gt_i)\gamma\pi^A)0=j$. It follows that $\gamma\pi^A=\gamma'\pi^A$. Define f_i and f_i' as in the Addendum. We have seen that $G\gamma\leqq(H\alpha)\,\mathrm{Wr}\,S_I$: hence $f_i\in(H\alpha)^I$. Similarly, $f_i'\in(H\alpha)^I$ because by its definition γ' factors through $\alpha\,\mathrm{Wr}\,S_I$. Let φ be the embedding $G\to H\,\mathrm{Wr}\,S_I$ used in forming $\alpha\uparrow$: we know from the proof of Theorem 1 that $(\varphi\downarrow)\pi_0^H=\mathrm{inn}\,t_0$. As π_0 is natural,

$$(\gamma'\downarrow)\pi_0^A=(\varphi\downarrow)\big((\alpha WrS_I)\downarrow\big)\pi_0^A=(\varphi\downarrow)\pi_0^H\alpha=(\mathrm{inn}\,t_0)\alpha=\alpha(\mathrm{inn}\,t_0\alpha)=(\gamma\downarrow)\pi_0^A(\mathrm{inn}\,t_0\alpha).$$

In terms of the Uniqueness Theorem, this means that $t_0\alpha\in B$; hence by the Addendum the element f of A^I defined by

$$f(i)=f_i(0)(t_0\alpha)f_i'(0)^{-1}\quad\text{for all } i \text{ in } I$$

lies in F: that is, $\gamma=\gamma'(\mathrm{inn}\,f^{-1})$. From the foregoing we see that in fact $f(i)\in H\alpha$ for all i, so $f^{-1}\in(H\alpha)^I$. It follows that composition with $\mathrm{inn}\,f^{-1}$ merely changes γ' to an $\alpha\uparrow$ defined with reference to a different transversal. This completes the proof of Theorem 1′.

Theorems 2 and 2′ depend on the other result from [2] as strengthened in [3]. The relevant part may be paraphrased as follows.

Centralizer Theorem. *Let* γ: $G\to W=A\,\mathrm{Wr}\,S_I$ *be a homomorphism such that* $\gamma\pi$ *is a transitive permutation representation; let* H *be the stabilizer in* G *of some point,* 0 *say, of* I*; and let* S *denote the image of* H *in the (external) direct product* $G\times A$ *under the embedding given by* $h\mapsto(h, h\gamma\pi_0)$. *Then there is a homomorphism of* $\mathbf{N}_{G\times A}(S)$ *onto* $\mathbf{C}_W(G\gamma)$ *with kernel* S.

(Strictly speaking, the statement in [3] deals with the image R of $H\gamma$ in $G\gamma\times A$ under $h\gamma\mapsto(h\gamma, h\gamma\pi_0)$, and gives an explicit homomorphism ψ of $\mathbf{N}_{G\gamma\times A}(R)$ onto $\mathbf{C}_W(G\gamma)$ with kernel R. Since S contains the kernel $(\ker\gamma)\times 1$ of the homomorphism $\gamma\times 1$ of $G\times A$ onto $G\gamma\times A$ and $S(\gamma\times 1)=R$, the composite of $\gamma\times 1$ and that ψ will serve in the present version.)

We have already noted that Theorem 2 is a special case of Theorem 2′. For the proof of the latter, one may assume without loss of generality that $A=H\alpha$, and then W can be thought of as $A\,\mathrm{Wr}\,S_I$. Further, once α is given, the isomorphism type of $\mathbf{C}_W(G(\alpha\uparrow))$ is independent of the choice of $\alpha\uparrow$, so we may as well take an $\alpha\uparrow$ defined with reference to a transversal in which the trivial coset is represented by 1, and to an identification which matches that coset to 0. We know from (a) of Theorem 1′ that $(\alpha\uparrow)\pi^A$ is transitive, while the proof of the "only if" part of

Theorem 1 and the naturality of π yield that H is the stabilizer of 0. We can therefore apply the Centralizer Theorem with $\gamma=\alpha\uparrow$. By an argument used in the proof of Theorem 1′, now $(\alpha\uparrow\downarrow)\pi_0^A=\alpha$, so S is the image of $h\mapsto(h, h\alpha)$. It is easy to see that if $(g, a)\in\mathbf{N}_{G\times A}(S)$ then g must normalize both H and $\ker\alpha$, and then (using $H\alpha=A$) that $\mathbf{N}_{G\times A}(S)=\big(\mathbf{C}(H/\ker\alpha)\times 1\big)S$ with $\big(\mathbf{C}(H/\ker\alpha)\times 1\big)\cap S=(\ker\alpha)\times 1$. Consequently $\mathbf{N}_{G\times A}(S)/S\cong\mathbf{C}_G(H/\ker\alpha)/\ker\alpha$, and so the Centralizer Theorem yields Theorem 2′.

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