ISRAL JOURNAL OF MATHEMATICS, Vol. 63, No. 1, 1988

PRIMITIVE PERMUTATION GROUPS OF SIMPLE DIAGONAL TYPE

BY

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ABSTRACT

Let G be a finite primitive group such that there is only one minimal normal subgroup M in G, this M is nonabelian and nonsimple, and a maximal normal subgroup of M is regular. Further, let H be a point stabilizer in G. Then $H \cap M$ is a (nonabelian simple) common complement in M to all the maximal normal subgroups of M, and there is a natural identification of M with a direct power T^m of a nonabelian simple group T in which $H \cap M$ becomes the "diagonal" subgroup of T^m: this is the origin of the title. It is proved here that two abstractly isomorphic primitive groups of this type are permutationally isomorphic if (and obviously only if) their point stabilizers are abstractly isomorphic.

Given T^m, consider first the set of all permutational isomorphism classes of those primitive groups of this type whose minimal normal subgroups are abstractly isomorphic to T^m. Secondly, form the direct product $S_m \times \mathrm{Out}\, T$ of the symmetric group of degree m and the outer automorphism group of T (so $\mathrm{Out}\, T = \mathrm{Aut}\, T/\mathrm{Inn}\, T$), and consider the set of the conjugacy classes of those subgroups in $S_m \times \mathrm{Out}\, T$ whose projections in S_m are primitive. The second result of the paper is that there is a bijection between these two sets.

The third issue discussed concerns the number of distinct permutational isomorphism classes of groups of this type, which can fall into a single abstract isomorphism class.

The aim of this paper is to investigate primitive permutation groups G on *finite* sets Ω, such that G has a unique minimal normal subgroup M, this M is nonabelian and nonsimple, and a maximal normal subgroup K of M is regular. Such groups are sometimes called *primitive groups of simple diagonal type*.

Let H be a point stabilizer in such a G. Then $H \cap M$ is a (nonabelian simple) common complement in M to all the maximal normal subgroups of M, and there is a natural way of identifying M with a direct power T^m of a nonabelian simple group T in such a way that $H \cap M$ becomes *the* diagonal diag T^m. [Let $K_1, \ldots, K_m$ be the maximal normal subgroups of M. To each element x of M,

Received September 3, 1987

let $f\colon \{1, 2, \ldots, m\} \to H \cap M$ be the function whose value $f(i)$ at i is the unique element of $H \cap M$ congruent to x modulo K_i. This function is constant if and only if $x \in H \cap M$. The map $x \mapsto f$ is an isomorphism of M onto $(H \cap M)^m$.] This is the origin of the phrase "simple diagonal type". Since the normalizer $\mathrm{N}_G(H \cap M)$ is H, one may identify Ω with the conjugacy class of $H \cap M$ in G; better still, exploiting also that $HM = G$, identify Ω with the conjugacy class of diag T^m in T^m. The uniqueness of M in G means that the centralizer $\mathrm{C}_G(M)$ is 1; hence we may change our point of view once more, to have G (abstractly) embedded in Aut T^m in such a way that the action of G on Ω is the action of the relevant automorphisms of T^m on the conjugacy class of diag T^m. In particular, the degree of G is $|T|^{m-1}$.

Of course Aut $T^m = (\mathrm{Aut}\, T) \,\mathrm{Wr}\, S_m = S_m(\mathrm{Aut}\, T)^m$; call this (permutational) wreath product W for short. Now $M = (\mathrm{Inn}\, T)^m$ and $H \cap M = \mathrm{diag}(\mathrm{Inn}\, T)^m$. It is easy to see that

$$\mathrm{N}_W(\mathrm{diag}(\mathrm{Inn}\, T)^m) = S_m \times \mathrm{diag}(\mathrm{Aut}\, T)^m;$$

hence

$$H = \mathrm{N}_G(H \cap M) = G \cap (S_m \times \mathrm{diag}(\mathrm{Aut}\, T)^m).$$

Now

$$G/M \leqq W/M = \mathrm{Out}\, T^m = (\mathrm{Out}\, T) \,\mathrm{Wr}\, S_m = S_m(\mathrm{Out}\, T)^m$$

and indeed

$$G/M = HM/M \leqq S_m \times \mathrm{diag}(\mathrm{Out}\, T)^m \cong S_m \times \mathrm{Out}\, T.$$

This embedding of G/M in $S_m \times \mathrm{Out}\, T$ depended only on the choice of H (unique up to conjugacy in G), on the indexing bijection from $\{1, \ldots, m\}$ to the set of maximal normal subgroups of M, and on the (implicit) isomorphism between $H \cap M$ and T: hence the copy of G/M in $S_m \times \mathrm{Out}\, T$ is determined up to conjugacy in the latter group by the permutational isomorphism class of G. Moreover, G and H can be recovered from this copy of G/M: namely, H as the complete inverse image in $S_m \times \mathrm{diag}(\mathrm{Aut}\, T)^m$ of this copy of G/M, and then G as $H(\mathrm{Inn}\, T)^m$.

We shall show that the relevant subgroups of $S_m \times \mathrm{Out}\, T$ are precisely those whose projections in S_m (in the direct decomposition $S_m \times \mathrm{Out}\, T$) are primitive (as subgroups of S_m). In view of the foregoing, this will establish the first part of the following.

Theorem 1. *There is a bijection between the set of permutational isomorphism classes of those primitive groups G of simple diagonal type whose unique* *minimal normal subgroups are abstractly isomorphic to T^m, and the set of the conjugacy classes of those subgroups of $S_m \times \mathrm{Out}\, T$ whose projections in S_m are primitive. Two such G are abstractly isomorphic if and only if the corresponding subgroups of $S_m \times \mathrm{diag}(\mathrm{Out}\, T)^m$ are conjugate in* $(\mathrm{Out}\, T)\,\mathrm{Wr}\, S_m$.

REMARK 1. The set of these permutational isomorphism classes may be partially ordered by calling two of them comparable if a representative of one contains a subgroup belonging to the other. The set of the relevant conjugacy classes of subgroups of $S_m \times \mathrm{Out}\, T$ may also be partially ordered in this way. It is clear that the bijection of Theorem 1 is then an order-isomorphism.

For the "only if" part of the second statement of Theorem 1, consider the two G in question embedded in the same W; restrict to the (now) common M an abstract isomorphism of the two G, to obtain an automorphism of M; the corresponding automorphism of T^m will conjugate one G into the other. The "if" part is obvious.

The promised identification of the relevant subgroups of $S_m \times \mathrm{Out}\, T$ lies much deeper. Let H be a subgroup of $S_m \times \mathrm{diag}(\mathrm{Aut}\, T)^m$ containing $\mathrm{diag}(\mathrm{Inn}\, T)^m$; set $\bar{H} = S_m \cap H(\mathrm{Aut}\, T)^m$, and define G as HM with $M = (\mathrm{Inn}\, T)^m$. Transitivity of $\bar{H}$ (as subgroup of S_m) is clearly equivalent to the minimality of M among the normal subgroups of G, so we may as well restrict attention to the H which satisfy this much. What we have to prove is that then the primitivity of $\bar{H}$ is equivalent to the maximality of H in G. Let K be a maximal normal subgroup (the direct product of all but one of the simple direct factors) of M, and $N = \mathbf{N}_G(K)$. It is easy to see that the primitivity of $\bar{H}$ is equivalent to the maximality of N in G. On the other hand, $HK = G$ is immediate from the definitions: so, by Lemma 4.1 of [4], in the terminology of that paper H is full in G (with respect to K). Lemmas 4.2 and 3.09 of [4] give that a full subgroup is maximal if, and obviously only if, it is maximal among the full subgroups. Theorem 3.03 of [4] now readily yields that H is maximal among the full subgroups if and only if N is maximal in G. This completes the proof of Theorem 1.

COROLLARY (see Remark 2 on p. 6 of Cameron [1]). *In a primitive group of simple diagonal type, the normalizers of the simple direct factors of the unique minimal normal subgroup are maximal subgroups.*

For if S is a simple direct factor of M, then $\mathbf{C}_M(S)$ is a maximal normal subgroup K of M and $S = \mathbf{C}_M(K)$, so $\mathbf{N}_G(S) = \mathbf{N}_G(K)$.

THEOREM 2. *Two abstractly isomorphic primitive groups of simple diagonal type are permutationally isomorphic if and only if their point stabilizers are abstractly isomorphic.*

This is next in the natural order of results, but not in the logical order of the present argument: so the proof must be deferred.

A primitive permutation representation (of a finite group) is said to be of simple diagonal type if its image as primitive group is of simple diagonal type, and a *maximal subgroup* is said to be *of simple diagonal type* if the natural primitive permutation representation on the corresponding coset space is of simple diagonal type. A finite group obviously cannot have a corefree maximal subgroup of simple diagonal type unless it has a unique minimal normal subgroup M and that is nonabelian and nonsimple, and (cf. the Corollary above) unless the normalizer N of a maximal normal subgroup K of M is maximal in G: so *suppose all these conditions hold.*

THEOREM 3. *The set of conjugacy classes of corefree maximal subgroups of simple diagonal type in such a group G is bijective with the set of all those homomorphisms $G/M \to \operatorname{Out} M/K$ whose restriction to N/M is the coupling belonging to the extension $M/K \hookrightarrow N/K \twoheadrightarrow N/M$.*

In view of our assumptions and the foregoing discussion, this is a straightforward consequence of Theorem 3.03 of [4]. (Section 2 of that paper is a summary of the relevant facts on extensions.) In fact there is a "natural" bijection between the two sets in question, and this may be described as follows. The homomorphism $G/M \to \operatorname{Out} M/K$ corresponding to a corefree maximal H of simple diagonal type is obtained from the coupling

$$H/(H \cap M) \to \operatorname{Out}(H \cap M)$$

belonging to the extension

$$H \cap M \hookrightarrow H \twoheadrightarrow H/(H \cap M)$$

via the natural isomorphisms

$$H/(H \cap M) \cong HM/M = G/M \quad \text{and} \quad H \cap M \cong (H \cap M)K/K = M/K.$$

For an alternative description in terms of the lead-up to Theorem 1, choose T as M/K, and the isomorphism of $H \cap M$ onto M/K in the obvious way [so in $M \cong (M/K)^m$, $x \mapsto f$, $f(i)$ is the coset of K containing the unique element of $H \cap M$ congruent to x modulo K_i]. The resulting embedding of G/M in

$S_m \times \operatorname{Out} M/K$, followed by the projection of this direct product onto its second direct factor, yields the relevant homomorphism.

A measure of the strength of Theorem 3 is that it is not easy for homomorphisms χ of G/M to differ when they are required to agree on the maximal subgroup N/M. Let χ_N denote the prescribed common restriction. With Z defined by $Z/K = \mathrm{C}_{N/K}(M/K)$, we have $\ker \chi_N = ZM/M$. Consider first the case when ZM/M is *not* normal in G/M. Then the normal closure of ZM/M supplements N/M and must lie in the kernel of χ: so there is precisely one such χ if that normal closure meets N/M in ZM/M, and there is no χ otherwise. Suppose next that ZM/M *is* normal in G/M. Then ZM/M lies in the kernel of the permutation representation of G/M on its coset space modulo N/M, as well as in the kernel of any χ of the required kind: hence also in the kernel of the embedding

$$G/M \hookrightarrow S_m \times \operatorname{Out} M/K$$

that the H corresponding to such a χ would give rise to. Thus there can be no such χ except perhaps when $ZM/M = 1$ so $Z = K$: that is, when N/K is nearly simple. (Here, as in [4], a group is called *nearly simple* if it has only one minimal normal subgroup and that is nonabelian and simple.)

Let us examine closely this exceptional case. Now there is precisely one χ for each normal complement (if any) of N/M in G/M (with the normal complement as $\ker \chi$), and there may be other χ with trivial kernels. Of course N/M is soluble because $\operatorname{Out} M/K$ is, the Schreier Conjecture having been confirmed by the classification of finite simple groups. If N/M does have a normal complement in G/M then the primitive image P of G/M in S_m (modulo the normal core of N/M) has a regular normal subgroup and its point stabilizers are soluble: so (by Theorem 1 of Aschbacher and Scott [1]) it must be soluble, and hence G/M is also soluble (being embeddable in $P \times \operatorname{Out} M/K$). Such a normal complement will be unique unless it is N/M-isomorphic to some (minimal) normal subgroup of N/M, in which case its order, the index $|G:N|$, is a prime-power divisor of $|\operatorname{Out} M/K|$. Of course if there is a χ with trivial kernel then it embeds G/M in the soluble group $\operatorname{Out} M/K$, and again we have that $|G:N|$ is a prime power dividing $|\operatorname{Out} M/K|$. We have proved the following.

COROLLARY. *A group G of the kind considered in Theorem* 3 *has at most one conjugacy class of corefree maximal subgroups of simple diagonal type except perhaps if N/K is nearly simple, G/M is soluble, and m is a prime-power divisor of* $|\operatorname{Out} M/K|$.

REMARK 2. To each finite nonabelian simple group T, at most finitely many (isomorphism classes of) G with $M/K \cong T$ can be exceptional in this Corollary (for they have to be subgroups of Aut T^m with m a prime-power divisor of $|\mathrm{Out}\ T|$). Given the classification, it should be a plausible exercise to list for each isomorphism type of T the exceptional G corresponding to it, and to count the conjugacy classes of corefree maximals of simple diagonal type in each G. (More conveniently, one would list the isomorphism types of the groups which occur as Out T: for each of these, the discussion would be uniform.) This would enable one to replace this Corollary by an entirely conclusive theorem of the kind: "Up to abstract isomorphism, the finite groups G with more than one equivalence class of faithful primitive representations of simple diagonal type are precisely those in the following list, which also gives for each such G all the relevant representations".

Before proceeding we note without proof an elementary fact.

LEMMA. *Two embeddings of a nearly simple group into the automorphism group of the relevant simple group can only differ by an inner automorphism of that automorphism group.*

Now we are ready to turn to the permutational isomorphism problem for images of faithful primitive simple diagonal type representations of a given group G. There is no problem unless G is of the exceptional kind discussed in the Corollary of Theorem 3: we continue the discussion from the paragraph which led up to that Corollary.

The kernel of the coupling belonging to $H \cap M \hookrightarrow H \twoheadrightarrow H/(H \cap M)$ is $X/(H \cap M)$ where X consists of those elements of H which induce inner automorphisms on $H \cap M$: that is, $X = (H \cap M) \times \mathbf{C}_H(H \cap M)$. The kernel of the corresponding $\chi : G/M \to \mathrm{Out}\, M/K$ is the subgroup of G/M which matches $X/(H \cap M)$ in the natural isomorphism $H/(H \cap M) \cong HM/M = G/M$: so $\ker \chi = XM/M = \mathbf{C}_H(H \cap M)M/M$.

In case $\ker \chi = 1$, we therefore have $\mathbf{C}_H(H \cap M) = 1$ so H is nearly simple; conjugation action on $H \cap M$ embeds H in $\mathrm{Aut}(H \cap M)$, whence via the natural isomorphism $H \cap M \cong (H \cap M)K/K = M/K$ we obtain an embedding μ of H in Aut M/K, and this induces an embedding of $H/(H \cap M)$ in Out M/K which is the composite of χ with the natural isomorphism $H/(H \cap M) \cong G/M$. Suppose now that H_1, χ_1, μ_1, are chosen similarly, and that φ is an (abstract) isomorphism of H onto H_1. Then $\varphi\mu_1$ is another embedding of H in Aut M/K. By the Lemma, there is an inner automorphism α of Aut M/K such that

$\mu\alpha = \varphi\mu_1$. Let $\bar{\alpha}$ be the inner automorphism of Out M/K obtained from α, and $\bar{\varphi}$ the automorphism of G/M obtained from φ and the natural isomorphisms $G/M \cong H/H \cap M$ and $H_1/H_1 \cap M \cong G/M$: then $\chi\bar{\alpha} = \bar{\varphi}\chi_1$. As the homomorphisms of G/M into S_m involved in the corresponding embeddings of G/M in $S_m \times \mathrm{Out}\, M/K$ are equivalent as permutation representations (both having N/M as a point stabilizer), we conclude that the two embeddings have conjugate images. By Theorem 1, the images of the primitive representations of G on its coset spaces modulo H and H_1, respectively, are therefore permutationally isomorphic.

Consider next the case when $\ker\chi$ is a normal complement, Y/M say, of N/M in G/M. This Y/M may be regarded an N/K-module via the natural homomorphism of N/K onto N/M and the conjugation action of N/M on Y/M. The natural isomorphisms

$$N/K = (H \cap N)K/K \cong H \cap N$$

and

$$Y/M = XM/M = \mathbf{C}_H(H \cap M)M/M \cong \mathbf{C}_H(H \cap M)$$

are coherent with this action of N/K on Y/M and with conjugation action of $H \cap N$ on $\mathbf{C}_H(H \cap M)$. It follows that H is isomorphic to the (external) semidirect product of Y/M by N/K. In particular, now H is not nearly simple and so cannot be isomorphic to any H with $\ker\chi = 1$.

To complete the proof of Theorem 2, it will suffice to show that if H_1, χ_1 are chosen similarly (so $\ker\chi_1$ is another normal complement of N/M in G/M) then the copies of G/M in $S_m \times \mathrm{Out}\, M/K$ corresponding to H and H_1 are conjugate. We shall use once more that the two embeddings of G/M followed by projection into the first direct factor yield equivalent permutation representations: so, at the cost of composing one embedding with an inner automorphism of $S_m \times \mathrm{Out}\, M/K$ induced by an element of S_m, we may assume that these two representations are one: call it ρ, say. The embedded copies of G/M now consist of the pairs $(x\rho, x\chi)$ and $(x\rho, x\chi_1)$, respectively, with x ranging through G/M. As the restrictions to N/M of χ and χ_1 are equal and one-to-one, the two embeddings now agree on N/M and their common image, V say, avoids the first direct factor S_m. The image of ρ is a soluble primitive subgroup of S_m, so it has a unique regular normal subgroup U; therefore the image of $\ker\chi$ in the first embedding, and that of $\ker\chi_1$ in the second, is this U. It follows that both embedded copies of G/M must equal UV.

This completes the proof of Theorem 2. It may be worth noting a further consequence of this argument.

REMARK 3. If a finite group G has corefree maximal subgroups of simple diagonal type which are not nearly simple, then these subgroups form a single characteristic class of subgroups of G. Equivalently, if two primitive groups of simple diagonal type are abstractly isomorphic and their point stabilisers are not nearly simple, then the two primitive groups are permutationally isomorphic.

To conclude the paper, let us consider some examples which show that the complications permitted by the results do in fact arise. For outer automorphisms of simple groups, see the Atlas [3].

First, consider a simple Chevalley group T of type D_4 over a field of odd prime-square order, so $\mathrm{Out}\, T \cong C_2 \times S_4$, and choose $m = 2$. The relevant primitive groups G of simple diagonal type have the direct square of T as a subgroup of index 2. By Theorem 1, the number of permutational isomorphism classes of such G is the number of conjugacy classes of those subgroups of order 2 in $S_2 \times (C_2 \times S_4)$ whose projection in the first direct factor S_2 is nontrivial: thus it is the number of conjugacy classes of elements of order dividing 2 in $C_2 \times S_4$, to wit, 6. (Is this the largest one can have with $m = 2$?) The next thing is to think of $S_2 \times (C_2 \times S_4)$ as $S_2 \times \mathrm{diag}(C_2 \times S_4)^2$, and verify that the six conjugacy classes fuse into one in the wreath product $(\mathrm{Aut}\, T) \mathrm{Wr}\, S_2$ (because the relevant subgroups all complement the base group, and this happens to be a standard wreath product): so the six permutational isomorphism classes lie in a single abstract isomorphism class. An obvious representative is $T \mathrm{Wr}\, S_2$. By Theorem 2, we now expect to find 6 abstract isomorphism classes of corefree maximal subgroups of simple diagonal type in $T \mathrm{Wr}\, S_2$; according to Remark 3, all but at most one of these must consist of nearly simple groups. As $S_2 \times \mathrm{diag}\, T^2$ is corefree maximal of simple diagonal type and not nearly simple, we expect 5 nearly simple isomorphism types. Indeed: $C_2 \times S_4$ has 5 conjugacy classes of subgroups of order 2, so there are precisely 5 abstract isomorphism classes of nearly simple groups containing a copy of T as a subgroup of index 2, and by the Embedding Theorem each of these is embeddable in $T \mathrm{Wr}\, S_2$. Finally we come to applying Theorem 3. In $G = T \mathrm{Wr}\, S_2$ we have $N = M = T^2$, so the number of conjugacy classes of corefree maximal subgroups of simple diagonal type is simply the order of $\mathrm{Hom}(S_2, C_2 \times S_4)$, that is, 20. These 20 will have to fall into 6 sets, those in any one set consisting of (abstractly) isomorphic groups and the corresponding primitive representations of $T \mathrm{Wr}\, S_2$ having permutationally isomorphic images. (The conjugacy class corresponding to the zero homomorphism $S_2 \to C_2 \times S_4$ consists of copies of $S_2 \times T$ and is in one set by itself. The other 19 classes split into 5 sets matching the division of the 19 subgroups of order 2 in $C_2 \times S_4$ into conjugacy classes.)

Next consider $T = \mathrm{PSL}(p, q^p)$ with p and q odd primes and $q \equiv 1 \bmod p$. The group of diagonal automorphisms of T now has order p; it is normalized by the group of field automorphisms which also has order p; and there is just one nontrivial graph automorphism: so $C_p \times C_p$ is a subgroup of index 2 in $\mathrm{Out}\, T$, with $C_p \times 1$ central and $1 \times C_p$ normal but not central (so $\mathrm{Out}\, T \cong C_p \times D_{2p} \cong C_p \,\mathrm{Wr}\, C_2$ with D_{2p} dihedral of order $2p$). Take $m = p$ and carry out the exercise on the above pattern: it is a shade easier. Dirichlet's Theorem ensures that there is no bound (independent of p) on the number of divisors of $p - 1$, so there is no such bound even on the number of soluble primitive subgroups of S_p: so Theorem 1 yields that there is no bound independent of T and m on the number of abstract isomorphism classes of primitive groups of simple diagonal type with unique minimal normal T^m. As $\mathrm{Out}\, T$ has $(p+3)/2$ conjugacy classes of subgroups of order p, there are $1 + (p+3)/2$ permutational isomorphism classes of primitive groups of simple diagonal type abstractly isomorphic to $T \,\mathrm{Wr}\, C_p$: so there is no overall bound on the number of permutational isomorphism classes of primitive groups of simple diagonal type in any one abstract isomorphism class (though I expect there *is* one if we exclude the case of the simple group being a projective special linear group).

Acknowledgement

The author is indebted to Professor Cheryl E. Praeger for many stimulating discussions on this subject.

References

1. M. Aschbacher and L. Scott, *Maximal subgroups of finite groups*, J. Algebra **92** (1985), 44–80.

2. P. J. Cameron, *Finite permutation groups and finite simple groups*, Bull. London Math. Soc. **13** (1981), 1–22.

3. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

4. L. G. Kovács, *Maximal subgroups in composite finite groups*, J. Algebra **99** (1986), 114–131.