On the Jordan-Hölder Theorem

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Research Report No. 11 - 1987, Department of Mathematics, IAS.

Mathematical Sciences Research Centre
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A long time ago, stimulated by a paper of D. W. Barnes: "On complemented chief factors of finite soluble groups" Bull. Austral. Math. Soc. 7 (1972) 101-104, we discussed sharpened forms of the Jordan-Hölder Theorem. The results of this discussion were written down in a letter by one of us to Barnes. The issues addressed have cropped up from time to time in conversation and correspondence with others. There has, by now, been sufficient interest in them to justify some archiving. So we reproduce the letter here in typed form (with only some minor cosmetic editing).

1980 Mathematics Subject Classification (1985) 20 E 15
Received for Research Reports 28 April 1987.
Dear Don,

Mike and I have had some long discussions, provoked by the paper you are currently publishing in the Bulletin. Here is a description of some stray results we think we have: the proofs haven't been written down yet, and I would appreciate your comments on what if any of this may be worth writing down in full.

To begin with, let us consider a possibly infinite group \( G \) with a finite chief series \( 1 = H_0 < ... < H_n = G \). By the Jordan-Hölder Theorem, every other chief series has the same length; let \( 1 = K_0 < ... < K_n = G \) be one. Let \( X/Y \in \Phi \) mean that \( X/Y \) is a Frattini chief factor, and \( X/Y \notin \Phi \) that \( X/Y \) is a non-Frattini chief factor. Our first result is that one of your theorems, with suitable modification, holds in this generality, and that the correspondence between the non-Frattini chief factors is unique:

(1) There is a permutation \( \beta \) of \( \{1, ..., n\} \) and to each \( i \) with \( H_i/H_{i-1} \in \Phi \) a maximal subgroup \( M_i \) such that \( M_i \) supplements \( H_i/H_{i-1} \) and \( K_i \beta /K_{i\beta -1} \) (that is, \( H_{i-1}K_{i\beta -1} \leq M_i \) but \( H_iM_i = K_i \beta M_i = G \)). If \( \beta' \) also has this property, then

\[
i\beta = i\beta' = \max \{ j \mid H_j/K_{j-1}/H_{j-1}K_{j-1} \in \Phi \} \text{ whenever } H_j/H_{j-1} \in \Phi.
\]

It follows that the number of Frattini chief factors in a chief series of \( G \) is independent of the chief series, for \( \beta \) must respect the \( \Phi \) or non-\( \Phi \) nature of chief factors. It would be nice to know that \( \beta \) can be chosen so as to also respect the \( G \)-isomorphism type of the chief factors: however, in general this is not possible. We have an example of a group whose normal subgroup lattice is finite and distributive, yet it has two chief series such that each has just one chief factor of a specific \( G \)-isomorphism type, in the first series this is Frattini, in the other it is non-Frattini (in fact, complemented by a maximal subgroup). What can be salvaged here? The permutation \( \beta \) can be chosen to behave perfectly for all finite or abelian chief factors, and such trouble is only possible in relation to infinite nonabelian Frattini chief factors. Also, \( \beta \) can be chosen so as to respect Frattini type and \( G \)-similarity: we call two chief
factors of $G$ similar if the automorphism groups induced on them by $G$
are isomorphic $quâ$ automorphism groups. (If $A$ is a group of
automorphisms of $X$ and $B$ a group of automorphisms of $Y$, we say $A$
and $B$ are isomorphic $quâ$ automorphism groups if there exist ordinary
isomorphisms $\varphi : A \rightarrow B$, $\psi : X \rightarrow Y$ such that
$$\forall x \in X. \forall a \in A. (x^a)\psi = (x\varphi)^{a\psi}.$$  

All this has also provoked a careful look at the Jordan-Hölder
Theorem itself. The upshot is as follows. Let $\Lambda$ be a modular lattice of
finite length (say, the normal subgroup lattice of $G$ above), and
$H_0 < \ldots < H_n$, $K_0 < \ldots < K_n$ two maximal chains in $\Lambda$. If $X, Y \in \Lambda$, write
$(X \wedge Y, X) \preceq (Y, X \vee Y)$ and say $(X \wedge Y, X)$ is perspective to $(Y, X \vee Y)$. On
the set of "chief factors" or "prime intervals" of $\Lambda$, $i.e.$ on
$$\{(U, V) \mid U, V \in \Lambda, U < V, U \leq Z \leq V \implies Z = U \text{ or } V\},$$
the relation $\preceq$ is easily seen to be a partial order; call the equivalence
relation generated by $\preceq$, projectivity. The usual statement asserts that
there is a permutation $\gamma$ on $\{1,\ldots,n\}$ such that $(H_{i-1}, H_i)$ and $(K_{i\gamma}, K_{i\gamma})$ are
projective for all $i$; and such a $\gamma$ is always given by some equivalent of
the definition $i\gamma = \max \{j \mid H_{i-1} \wedge K_{j-1} < H_i \vee K_{j-1}\}$. The $\gamma$ so defined is in
fact distinguished in a number of ways: (2) it is the only permutation for
which corresponding chief factors are projective within the sublattice
generated by the $H_i$ and $K_j$ (by Theorem 5 on p. 72 of the 1948 edition
of Birkhoff, this sublattice is finite and distributive); (3) it is the only
permutation such that to each $i$ there is an $(X_i, Y_i)$ perspective to both
$(H_{i-1}, H_i)$ and $(K_{i\gamma}, K_{i\gamma})$; (4) it is the only permutation such that to each $i$
there is a $(U_i, V_i)$ to which both $(H_{i-1}, H_i)$ and $(K_{i\gamma}, K_{i\gamma})$ are perspective.

Coming closer to the situation considered by you, assume $G$ is finite
but not necessarily soluble. It is easy to see that for two chief factors to
have a common maximal supplement (as in (1) above) it is now necessary
and sufficient that they be both perspective to a (common) non-Frattini
chief factor. This opens up the way to dualization of your condition, and
we have:-

(5) if $G$ is finite, there is a unique permutation $\beta$ such that
(5a) for each $i$ with $H_i/H_{i-1} \in \Phi$ there exists a non-Frattini $X_i/Y_i$
to which both $H_i/H_{i-1}$ and $K_{i\beta}/K_{i\beta-1}$ are perspective; and
(5b) to each $i$ with $H_i/H_{i-1} \in \Phi$, there is a Frattini $X_i/Y_i$
perspective to both $H_i/H_{i-1}$ and $K_{i\beta}/K_{i\beta-1}$.

Namely,

$$i\beta = \begin{cases} \max \{j | H_j K_{i-1}/H_{i-1} K_{i-1} \in \Phi \} & \text{if } H_i/H_{i-1} \in \Phi, \\ \min \{j | (H_i \cap K_j)/(H_{i-1} \cap K_j) \in \Phi \} & \text{if } H_i/H_{i-1} \in \Phi. \end{cases}$$

In this situation, of course, corresponding chief factors are $G$-isomorphic. (This definition of $\beta$ could also be used in the case of infinite $G$, but there we have no convenient interpretation for the relationship set up by $\beta$ between the Frattini chief factors of the two series.) Without the solubility assumption you had, one cannot quite expect common complements for corresponding non-Frattini chief factors: it is not hard to make an insoluble finite group with two chief series which have different numbers of complemented chief factors. On the other hand, a supplement of an abelian chief factor is always a complement, so (1) and (5) really generalize your Theorem 2.

Our proofs are based on the key idea in the proof of Lemma 2.6 in the Carter-Fischer-Hawkes paper, and are largely lattice theoretic. Using that, (5) is quickly translated to the context of a modular lattice $\Lambda$ of finite length, with the set of "chief factors" partially ordered by perspectivity, and the set $\Phi$ of Frattini chief factors being an arbitrary subset subject to two axioms:

(A1) If $(X, Y) \in \Phi$ and $(X, Y) \preceq (U, V)$ then $(U, V) \in \Phi$

(A2) If $(X, Y) \not\in \Phi$, $(U, V) \in \Phi$, $(X, Y) \preceq (U, V)$, and $(X, U)$ is a chief factor, then $\exists F \in \Lambda . (X, F) \in \Phi$ and $(F, V) \not\in \Phi$;

and it is this lattice version of (5) that we deal with. [Incidentally, the empty set and the set of all chief factors of $\Lambda$ both satisfy the conditions (A1) and (A2) on $\Phi$, (4) and (3) arise as special cases of the lattice version of (5)].

Before closing, perhaps I should describe the examples I mentioned. For the first one, let $B$ be a nonabelian finite simple group, $S$ the full
symmetric group on a countably infinite set \( J \) and \( W \) the unrestricted permutational wreath product:

\[
W = B \text{Wr} S = S B^J.
\]

The normal subgroup lattice of \( W \) is fairly easily found to be the following, with \( F \) the set of finitary permutations on \( J \) and \( A \) the alternating part of \( F \):

The next key fact is that, according to a result of B. H. Neumann (MZ 87 (1965)), \( B^J \) is not complemented in \( B^J \), from which one derives that \( B^J \) has no proper supplement in \( W \). Take now \( G \) to be the subgroup of the direct square \( W \times W \) of \( W \) generated by the diagonal \( W\delta \) and \( B^J \times 1 \); put

\[
H_1 = B^J \times 1, \quad K_1 = 1 \times B^J, \quad H_2 = K_2 = B^J \times B^J, \quad H_3 = K_3 = H_2(B^J\delta), \quad H_4 = K_4 = H_2(A B^J)\delta, \quad H_5 = K_5 = H_2(F B^J)\delta, \quad H_6 = K_6 = G.
\]

The only chief factor \( K_1/K_{j-1} \) which is \( G \)-isomorphic to \( H_1 \) is clearly \( K_2/K_1 \). Now, \( G = H_1(W\delta), \ H_1 \cap W\delta = 1 \), and it is easy to see that \( W\delta \) is maximal in \( G \); so \( H_1/1 \notin \Phi \). However, \( K_2/K_1 \in \Phi \). For, if \( M \) is a maximal subgroup containing \( K_1 \), then \( M/K_1 \) is a maximal subgroup of \( G/K_1 \approx W \); as in this isomorphism \( K_2/K_1 \) corresponds to \( B^J \) which has no proper supplement in \( W \), one must have \( M/K_1 \geq K_2/K_1 \). The diagram of the normal subgroup lattice of \( G \) is
The second example is much easier. Let $\mathcal{A}$ be the automorphism group of the alternating group $A_6$. As is well known, $\mathcal{A}/A_6 = C_2 \times C_2$ (we consider $A_6$, and also $S_6$, embedded in $\mathcal{A}$), and - as is perhaps less well known - $\mathcal{A}$ does not split over $A_6$. [Indeed, every element of $\mathcal{A}$ but not of $S_6$ is known to interchange the two classes of odd involutions in $S_6$, and a complement of $A_6$ in $\mathcal{A}$ would have to contain an odd involution and an element of $\mathcal{A}$, outside $S_6$, centralizing that involution.] Let $T$ denote a Sylow 2-subgroup of $\mathcal{A}$, and let $G$ be the subgroup of the direct square $\mathcal{A} \times \mathcal{A}$ of $\mathcal{A}$ generated by $A_6 \times 1$ and the diagonal copy $T\delta$ of $T$. The complemented chief factors of $G$ are the abelian non-Frattini factors and some of the nonabelian chief factors. Each chief series has just one nonabelian chief factor; a chief series through $A_6 \times 1$ has its nonabelian factor complemented (by $T\delta$, say). However, another chief series may be taken through $1 \times (T \cap A_6)$ and $A_6 \times (T \cap A_6)$, and here

$$(A_6 \times (T \cap A_6))/(1 \times (T \cap A_6))$$

is not complemented. Indeed, if $M$ were a complement, its projection into the first direct factor $\mathcal{A} \times 1$ of $\mathcal{A} \times \mathcal{A}$ would have to supplement $A_6 \times 1$, but as $A_6 \times 1$ has no complement in $\mathcal{A} \times 1$, we would have to have an element $a \times b$ in $M$ with $1 + a \in A_6$. As $G = (A_6 \times 1)(T\delta)$, we would have to have $a \times b = (c \times 1)(t \times t)$ with $c \in A_6$, $t \in T$; hence $a = ct$.
would show \( t \in A_6 \cap T \) and so \( a \times b \in A_6 \times (T \cap A_6) \cap M = 1 \times (T \cap A_6) \), contrary to \( a \neq 1 \) above.

In the hope that I have not overstated any of our claims, and that you will excuse the hurried nature of this note, I am sending it on without risking further indefinite delays by attempting to polish it up.

All the best,

Laci.