JOURNAL OF ALGEBRA 99, 114–131 (1986)

Maximal Subgroups in Composite Finite Groups

L. G. KOVÁCS

Department of Mathematics, Australian National University, Institute of Advanced Studies, Canberra ACT 2601, Australia

Communicated by B. Huppert

Received May 30, 1984

1. INTRODUCTION

The purpose of this paper is to present a method for translating the problem of finding all maximal subgroups of finite groups into questions concerning groups that are nearly simple. (A finite group is called nearly simple if it has only one minimal normal subgroup and that it nonabelian and simple.) In view of the recently announced classification of all finite simple groups this seems to be a useful reduction, though it must be acknowledged (see, e.g., Scott [6]) that there are still enormous obstacles on the way to understanding even just the maximal subgroups of the simple groups.

Let G be a finite group and M a minimal normal subgroup of G. The maximal subgroups of G containing M are of course in bijective correspondence with the maximal subgroups of the smaller group G/M, and so we need not concern ourselves with those. If M is abelian, the maximal subgroups of G not containing M are precisely the complements of M in G; the number of conjugacy classes of these is 0 or the order of the first cohomology group $H^1(G/M, M)$. The case of M nonabelian and nonsimple is the principal part of this paper. If neither reduction is applicable, then all minimal normal subgroups of G are nonabelian simple groups: this is dealt with in the entirely straightforward penultimate section of the paper.

These reductions are all "canonical" or "natural" in a sense which could perhaps be expressed in the language of categories and functors, but here we prefer to stay with older conventions. (In particular, we usually do not distinguish between a homomorphism and that obtained from it by restricting the codomain.) Nevertheless, the interested reader will observe that much of the strength of the results lies precisely in their canonical nature, implicit as it may remain in this exposition.

While I do not know of any explicit statement of this reduction

0021-8693/86 $3.00

elsewhere,[1] some of it must have been folklore for quite some time. Applications have appeared in the context of primitive permutation groups, associated with the names of O'Nan and Scott: see Cameron [2] and the final Appendix of Scott [6]. Further comment on this application is to be found in the Postscript to this paper.

The main results, very much more specific and somewhat more general than one can indicate without numerous further definitions, appear as Theorems 3.03 and 4.3. They depend heavily on a recent paper [3] of Fletcher Gross and the author. Indeed, this work has grown out of the collaboration which was reported in that paper and for which I remain indebted to Professor Gross.

2. Trivialities from Extension Theory

The theory of extensions of a group with trivial centre is a trivial special case of extension theory: see for instance 11.4.21 in Robinson [5] whose terminology is largely followed here, or Mac Lane [4] who mentions that for this case direct and bare-handed proofs can be readily improvised. We shall need more detail than what is usually given. Throughout this section, S denotes a (not necessarily finite) group whose centre is trivial.

An extension of S is a short exact sequence of homomorphisms

$$1 \to S \xrightarrow{\varepsilon} E \xrightarrow{\xi} X \to 1. \tag{2.1}$$

Another extension

$$1 \to S \xrightarrow{\varepsilon^*} E^* \xrightarrow{\xi^*} X \to 1 \tag{2.2}$$

of S by the same group X is said to be equivalent to 2.1 if there is an isomorphism φ of E^* onto E which makes the following diagram commute:

$$\begin{tikzcd}
& E^* \arrow[dr, "\xi^*"] \arrow[dd, "\varphi"] & \\
S \arrow[ur, "\varepsilon^*"] \arrow[dr, "\varepsilon"'] & & X \\
& E \arrow[ur, "\xi"'] &
\end{tikzcd}$$

[1] After the draft of this paper was completed, I learned that a forthcoming paper [1] by Aschbacher and Scott will address the same issues. I am grateful for the opportunity to see a draft of [1]. While the conclusions naturally have several common components, the approaches and expositions differ so much that detailed reconciliation is a taxing exercise and will not be attempted here. The difficulties involved strongly suggest that both versions of the story are worth telling.

Such an isomorphism is called an equivalence of 2.2 to 2.1. The triviality of the centre of S implies that, given 2.1 and 2.2, there is at most one such isomorphism.

A coupling of X to S is a homomorphism $\chi: X \to \operatorname{Out} S$. In the direct product $X \times \operatorname{Aut} S$, let E be the subgroup consisting of all (x, σ) such that $\sigma \in \chi(x)$ [recall that $\chi(x)$ is a coset of $\operatorname{Inn} S$ in $\operatorname{Aut} S$, so this makes sense]. For each s in S, let $\operatorname{inn} s$ denote the inner automorphism of S defined as $s' \mapsto ss's^{-1}$, and define $\varepsilon: S \to E$ by $\varepsilon: s \mapsto (1, \operatorname{inn} s)$. Note also that $\xi: (x, \sigma) \mapsto x$ is a homomorphism of E onto X with kernel $1 \times \operatorname{Inn} S$. Thus E with the ε and ξ so defined forms an extension 2.1. Conversely, any extension 2.2 gives rise to a coupling $\chi^*: X \to \operatorname{Out} S$ where $\xi^*(x)$ is the set of those automorphisms σ of S to which there exists a y in E^* such that $\xi^*(y) = x$ and $\varepsilon^*(\sigma(s)) = y\varepsilon^*(s)y^{-1}$ for all s in S. It is immediate to see that if 2.2 is equivalent to the 2.1 constructed above from χ then $\chi^* = \chi$. Conversely, if $\chi^* = \chi$ then 2.1 and 2.2 are equivalent. Thus there is a bijection between the set $\operatorname{Hom}(X, \operatorname{Out} S)$ and the set of all equivalence classes of extensions of S by X. Moreover this family of bijections, one for each X, is natural in the sense that if χ corresponds to 2.1 and Y is a subgroup of X, then the restriction of χ to Y corresponds to the extension $1 \to S \to F \to Y \to 1$ where F is the complete inverse image of Y in E under ξ and the maps are the restrictions of ε and ξ.

Let us return to 2.1 defined by χ with $E \leqslant X \times \operatorname{Aut} S$ as above, and 2.2 an extension with an equivalence $\varphi: E^* \to E$ to 2.1. If Z is a complement of $\varepsilon^*(S)$ in E^*, then $\varphi(Z)$ is a complement of $\varepsilon(S)$ in E, and to each x in X there is a unique z in Z such that $\xi\varphi(z) = x$, that is, $\varphi(z) = (x, \sigma)$ for some σ in $\operatorname{Aut} S$. The map ζ which takes x to this σ is a homomorphism of X into $\operatorname{Aut} S$, such that $\omega\zeta = \chi$ where ω is the natural homomorphism of $\operatorname{Aut} S$ onto $\operatorname{Out} S$. Conversely, if ζ_1 is a homomorphism with $\omega\zeta_1 = \chi$ then

$$Z_1 = \{\phi^{-1}(x, \zeta_1(x)) \mid x \in X\}$$

defines a complement Z_1 of $\varepsilon^*(S)$ in E^*, and $\zeta = \zeta_1$ if and only if $Z = Z_1$. Thus the set of all complements of $\varepsilon^*(S)$ in E^* is bijective to the set[2]

$$\{\zeta \in \operatorname{Hom}(X, \operatorname{Aut} S) \mid \omega\zeta = \chi\}.$$

Moreover, the family of these bijections, one for each χ, is also natural with respect to restriction to any subgroup Y of any X.

[2] This set admits a natural permutation action by $\operatorname{Inn} S$ (if $\sigma \in \operatorname{Inn} S$, let it take ζ to the composite of ζ and $\operatorname{inn} \sigma$); the orbits of this action match the conjugacy classes of complements in E^*.

Note once more that this sketch can be filled out with easy proofs from first principles, without any reference to cohomology or free presentations or to any other substantial ingredient of general extension theory.

3. Full Subgroups

In a previous paper [3] we discussed at length certain consequences of the following hypothesis involving four groups G, M, K, N:

(*) We have $K \trianglelefteq M \trianglelefteq G$ and N is the normalizer $\mathbb{N}_G(K)$ of K in G (so that, whenever t ranges through a right transversal T of N in G, $t^{-1}Kt$ ranges just once through all the conjugates of K in G); moreover, $M = \prod_{t \in T} M/t^{-1}Kt$ in the sense that the obvious homomorphism of M into the unrestricted direct product of the quotients $M/t^{-1}Kt$ is an isomorphism of M onto this product.

(Note that in this hypothesis T only occurs as an index set and its choice is irrelevant to the validity of the hypothesis.)

Let now H be a supplement of M in G: that is, a subgroup such that $HM = G$. We then ask whether (*) holds with H, $H \cap M$, $H \cap K$, $H \cap N$ in place of G, M, K, N. If the answer is affirmative or if $H \cap M = 1$, we shall here call H *high* (with respect to K). Let $\mathscr{H}$ stand for the set of all conjugacy classes of high subgroups of G. Inclusion order on the set of high subgroups induces a pre-order on $\mathscr{H}$ (that is, a reflexive and transitive relation which only fails to be a partial order if, as can happen in a "badly" infinite group, each of two high subgroups is contained in a conjugate of the other without the two subgroups being conjugate to one another). We paraphrase here the relevant part of Theorem 4.2 of [3] and its Corollaries, as follows.

Theorem 3.01. *If (*) holds and $\mathscr{H}$ is as above, then $\mathscr{H}$ is order-isomorphic to the similarly pre-ordered set of all conjugacy classes of supplements of M/K in N/K, one such order-isomorphism being induced by the map which takes H to $(H \cap N)K/K$.*

In particular, a high H is maximal among the high subgroups of G if and only if $(H \cap N)K/K$ is maximal in N/K. This is a small start towards an understanding of the maximal subgroups of G: the aim of this section is to carry it on.

The most obvious way that a supplement H can fail to be high is to have $H \cap M > 1$ and $H \cap N < \mathbb{N}_H(H \cap K)$ [clearly we always have $H \cap N \leqslant \mathbb{N}_H(H \cap K)$]. With this in mind one can ask, instead of the previous question, whether (*) holds for H, $H \cap M$, $H \cap K$, $\mathbb{N}_H(H \cap K)$. It would be reasonable to attempt to expand our understanding of supplements of M

by investigating all those H for which the answer to this question is affirmative. The complexity of that problem forces us to restrict attention to subgroups for which $(H \cap M)K = M$ also holds [together with $HM = G$, this is equivalent to $HK = G$]. Thus we make the following definition: a subgroup F of G is *full* (with respect to K) if $FK = G$ and (*) holds with F, $F \cap M$, $F \cap K$, $\mathbb{N}_F(F \cap K)$ in place of G, M, K, N. (The change from H to F reflects the observation that the only subgroup which is both high and full is G itself: see Lemma 3.9.)

Now consider a supplement F of M such that (*) holds for F, $F \cap M$, $F \cap K$, $\mathbb{N}_F(F \cap K)$. By Lemma 2.2 of [3], as t ranges through a right transversal T of N in G, the intersection of the products $Ft^{-1}Kt$ (which are not assumed to be subgroups) is a subgroup H which is high with respect to K. In particular, (*) holds for H, $H \cap M$, $H \cap K$, $H \cap N$. Moreover, now F is full in H with respect to $H \cap K$ [indeed, $H = F(H \cap K)$ follows from $F \leqslant H \subseteq FK$]. Thus the problem abandoned above as too complex would be completely solved if one could handle full subgroups in general.

Unfortunately, the full story of full subgroups would take too long to elaborate here. What we are about to give deals with full subgroups under the assumption that the centre of M/K is trivial: we shall only hint at the changes that become necessary if this assumption is abandoned. As no condition on M/K can ensure the triviality of the centre of $(H \cap M)/(H \cap K)$ for each high H, the case we deal with does not fit well with the previous paragraph. On the other hand, in the notation used there, if F is maximal in G then $F < H < G$ is excluded so F is either high or full in G itself. Thus the reduction outlined there is not needed for what is our real aim here: the understanding of the maximal subgroups of G.

Our principal tool in this section will be the second part of Theorem 4.1 of [3], which we restate in a somewhat weaker form. First, we add to the notation involved in (*) the following. The natural homomorphisms of G onto G/M and of N onto N/K will be denoted σ and e, respectively. The composite of the natural homomorphism of N/K onto N/M with the inclusion of N/M in G/M will be written as $\alpha\colon N/K \to G/M$.

THEOREM 3.02. *Suppose* (*) *holds. Let τ be a homomorphism of a group G^* onto G/M, let N^* be the complete inverse image of N/M under τ, and β a homomorphism of N^* into N/K such that $\alpha\beta$ is the restriction τ_{N^*}. Then there exists a homomorphism γ of G^* into G, unique up to composition with certain inner automorphisms of G, such that $\sigma\gamma = \tau$ and $\beta = e\gamma_{N^*}$.*

To prepare for the statement of the main result of this section, assume (*) holds and F is a subgroup of G such that $FK = G$. We also assume that the centre of M/K is trivial so the previous section is applicable throughout, though it will be some time before we make any substantial use of this restriction. Now $FK=G$ yields $M=(F\cap M)K$. The natural isomorphism $M/K\cong(F\cap M)/(F\cap K)$ is then an embedding of M/K into $\mathbb{N}_F(F\cap K)/(F\cap K)$, the image of the embedding being the kernel of the natural homomorphism of $\mathbb{N}_F(F\cap K)/(F\cap K)$ onto $\mathbb{N}_F(F\cap K)M/M$. Thus we have an extension

$$(F)\quad 1\to M/K\to \mathbb{N}_F(F\cap K)/(F\cap K)\to \mathbb{N}_F(F\cap K)M/M\to 1;$$

explicitly, the embedding takes Km to $F\cap Km$, and the surjection takes $(F\cap K)x$ to Mx. The restriction of (F) to N/M is

$$1\to M/K\to (F\cap N)/(F\cap K)\to N/M\to 1,$$

and one sees readily that the natural homomorphism of $(F\cap N)/(F\cap K)$ onto $(F\cap N)K/K=N/K$ is an equivalence of this restriction to the extension

$$(N)\quad 1\to M/K\to N/K\to N/M\to 1$$

formed by the natural maps. Let

$$\chi_F\colon\ \mathbb{N}_F(F\cap K)M/M\to \operatorname{Out} M/K,$$
$$\chi_N\colon\ N/M\to \operatorname{Out} M/K$$

be the couplings defined by (F) and (N), respectively. It follows from the discussion above that χ_N is the restriction of χ_F to N/M. Moreover, if F is replaced by a conjugate, on account of $G=FK$ that can be taken as $k^{-1}Fk$ for some k in K, and one readily sees that conjugation by k induces an equivalence of the extension (F) to the similarly defined extension $(k^{-1}Fk)$. In particular, the subgroup $\mathbb{N}_F(F\cap K)M/M$ of G/M and the coupling χ_F depends only on the conjugacy class $\mathbb{F}$ of F in G: accordingly, the subgroup will be denoted $D_{\mathbb{F}}$ and the coupling $\chi_{\mathbb{F}}$.

Next suppose $E\leqslant F$ and $EK=G$. Note that $E\cap\mathbb{N}_F(F\cap K)$ normalizes both E and $F\cap K$, hence it also normalizes their intersection which is just $E\cap K$: thus

$$E\cap\mathbb{N}_F(F\cap K)\leqslant\mathbb{N}_E(E\cap K).$$

On the other hand, $E\leqslant F\leqslant EK$ yields $F=E(F\cap K)$ and so in turn

$$F\cap K\leqslant\mathbb{N}_F(F\cap K)\leqslant F=E(F\cap K)$$

yields

$$\mathbb{N}_F(F\cap K)=[E\cap\mathbb{N}_F(F\cap K)](F\cap K).$$

The conclusions of the last two sentences combine to give $D_{\mathbb{F}} \leqslant D_{\mathbb{E}}$. The middle term of the restriction of (E) to $D_{\mathbb{F}}$ has a natural isomorphism onto the middle term of (F), and this isomorphism is readily seen to be an equivalence: thus $\chi_{\mathbb{F}}$ is the restriction of $\chi_{\mathbb{E}}$ to $D_{\mathbb{F}}$.

The main result of this section is the following.

THEOREM 3.03. *Suppose* $(*)$ *holds and the centre of* M/K *is trivial. Let* $\mathscr{F}$ *be the set of all conjugacy classes of full subgroups of* G (*with respect to* K), *with its pre-order inherited from inclusion order on the set of full subgroups. For each subgroup* D *of* G/M *containing* N/M, *consider the set* $\mathscr{X}_D$ *of all those homomorphisms of* D *into* Out M/K *whose restriction to* N/M *is* χ_N, *and let* $\mathscr{X}$ *be the union of the* $\mathscr{X}_D$, *partially ordered by setting* $\chi \leqslant \psi$ *if* χ *is a restriction of* ψ. *The map* $\mathbb{F} \mapsto \chi_{\mathbb{F}}$ *gives an order-anti-isomorphism of* $\mathscr{F}$ *onto* $\mathscr{X}$.

(The theorem implies that the pre-order on $\mathscr{F}$ is always a partial order, however infinite our group may be.)

The proof begins by establishing the most important special case. By a complement of K in G, we mean a subgroup F of G such that $FK = G$ and $F \cap K = 1$. (We do not intend to suggest that either of F or K should normalize the other: in fact, neither can.) Clearly, each complement F of K in G is full, and for its class $\mathbb{F}$ we have $D_{\mathbb{F}} = G/M$; conversely, if F is full and $D_{\mathbb{F}} = G/M$, then F is a complement of K in G. Let $\mathscr{F}_{G/M}$ denote the set of all conjugacy classes of complements of K. The following statement is then a part of (3.03).

(3.04) The map $\mathbb{F} \mapsto \chi_{\mathbb{F}}$ gives a bijection of $\mathscr{F}_{G/M}$ onto $\mathscr{X}_{G/M}$.

To see this, define a map $\mathscr{X}_{G/M} \to \mathscr{F}_{G/M}$ as follows. For each χ in $\mathscr{X}_{G/M}$, let

$$(G^*) \quad 1 \to M/K \to G^* \xrightarrow{\tau} G/M \to 1$$

be an extension affording χ: we know χ determines (G^*) up to equivalence, because M/K is assumed to have trivial centre. The restriction, say

$$(N^*) \quad 1 \to M/K \to N^* \to N/M \to 1,$$

of (G^*) to N/M is, by the definition of $\mathscr{X}_{G/M}$, equivalent to (N): thus there is an isomorphism β of N^* onto N/K which is this equivalence, again unique because M/K has trivial centre. Now (3.02) yields the existence of a homomorphism $\gamma\colon G^* \to G$ such that $\tau = \sigma\gamma$ where σ is the natural homomorphism of G onto G/M, and $\beta = e\gamma_{N^*}$ where e is the natural homomorphism of N onto N/K and γ_{N^*} is the relevant restriction of γ. The first of these conditions implies that $\gamma(G^*)M = G$, the second yields

$\gamma(N^*)K=N$; together, these guarantee $\gamma(G^*)K=G$. Let $x\in G^*$ and $\gamma(x)\in K$. Since N^* is a complete inverse image under τ and $\tau(x)=\sigma\gamma(x)=1$, we have $x\in N^*$; thus $\beta(x)=e\gamma(x)=1$; hence, as β is an isomorphism, we conclude $x=1$. It follows that γ maps G^* isomorphically onto $\gamma(G^*)$, and $\gamma(G^*)$ is a complement of K. Write $\mathbb{F}$ for the conjugacy class of $\gamma(G^*)$ in G; then $\mathbb{F}\in\mathscr{F}_{G/M}$. Recall from (3.02) that γ is unique up to composition with certain inner automorphisms of G: thus $\mathbb{F}$ is uniquely determined by (G^*). Finally, if (G^*) were replaced by another extension affording χ, say, by

$$1\to M/K\to G^\dagger\to G/M\to 1$$

and a $\gamma^\dagger\colon G^\dagger\to G$ was obtained as $\gamma\colon G^*\to G$ was, we would have an equivalence $\delta\colon G^*\to G^\dagger$ of (G^*) to this extension, and the composite $\gamma^\dagger\delta$ would enjoy all the relevant properties of γ to enable us to invoke (3.02) with the conclusion that $\gamma^\dagger\delta$ is the composite of γ and an inner automorphism of G. As $\gamma^\dagger$, like γ, would have to be one-to-one, we could conclude that $\gamma^\dagger(G^\dagger)\in\mathbb{F}$. This proves that $\mathbb{F}$ depends only on χ, not on the choice of (G^*) [or γ or β].

So we have a well-defined map $\chi\mapsto\mathbb{F}$ from $\mathscr{X}_{G/D}$ to $\mathscr{F}_{G/D}$. Set $F=\gamma(G^*)$; note that now $\mathbb{N}_F(F\cap K)=F$ so (F) is of the form $1\to M/K\to F\to G/M\to 1$; and deduce from the defining properties of γ that it is an equivalence of (G^*) to (F). It follows that $\chi_{\mathbb{F}}=\chi$, so the composite $\chi\mapsto\mathbb{F}\to\chi_{\mathbb{F}}$ is the identity on $\mathscr{X}_{G/D}$.

Finally, start with an $\mathbb{F}$ in $\mathscr{F}_{G/D}$; we can now take the (G^*) affording $\chi_{\mathbb{F}}$ as (F) for an arbitrary F in $\mathbb{F}$; then $G^*=F$, and $\gamma\colon G^*\to G$ may be compared with the inclusion of F in G: yet another application of the uniqueness part of (3.02) now tells us that $\gamma(G^*)\in\mathbb{F}$. This completes the proof of (3.04).

This seems the place to indicate the changes that would become necessary if we abandoned the trivial centre assumption. First, of course, we would have no bijection (indeed, not even an injection or surjection) to $\mathscr{X}_{G/M}$ from the set $\mathscr{E}$ of all equivalence classes of those extensions of M/K by G/M whose restriction to N/M is equivalent to (N): so we would have to replace $\mathscr{X}_{G/M}$ by $\mathscr{E}$. More awkwardly, we would also lose the uniqueness of the equivalences β of those restrictions to (N). Consequently the cardinality of $\mathscr{F}_{G/M}$, instead of being the cardinality of $\mathscr{X}_{G/M}$ or at least that of $\mathscr{E}$, would be a sum over $\mathscr{E}$, with the summand corresponding to the class of an extension (G^*) defined as follows. Consider the group of all those automorphisms of N^* which are self-equivalences of the extension (N^*): that is, of the automorphisms which are trivial both on the kernel of τ and modulo that kernel. [This is isomorphic to the group of all derivations from N/M into the centre of M/K, that centre being a G/M-module with an action obtained from (G^*).] Within this group, take the subgroup consisting of the restrictions to N^* of those automorphisms of G^* which are self-equivalences of (G^*). The index of this subgroup is the summand corresponding to the equivalence class of (G^*). This much can be proved by adapting the arguments above; to complete a reasonable generalization of (3.04), one would then invoke general (as opposed to trivial) extension theory, and calculate the cardinality of $\mathscr{F}_{G/M}$ in terms of the cohomology of G/M and N/M with coefficients in the centre of M/K. While this is quite straightforward, it makes the complexity of a similar generalization of (3.03) look rather formidable: we certainly shall not attempt to pursue the matter here.

The proof of (3.03) combines applications of (3.01) and (3.04). To set the scene for these, consider an arbitrary subgroup $D=Q/M$ of G/M containing N/M, and let P denote the intersection of the conjugates of K in Q. It is straightforward to see that (*) is satisfied both by G, M, P, Q and by Q/P, M/P, K/P, N/P. Let $\mathbb{F}$ be any conjugacy class of subgroups of G, and F a member of $\mathbb{F}$. The next point we have to establish is the following: note that it does not depend on the assumption that the centre of M/K is trivial.

(3.05) We have $\mathbb{F}\in\mathscr{F}$ with $D_{\mathbb{F}}=D$ if and only if F is high with respect to P and $(F\cap Q)P/P$ is a complement of K/P in Q/P.

First, assume that $\mathbb{F}\in\mathscr{F}$ and $D_{\mathbb{F}}=D$: then $FK=G$ and $\mathbb{N}_F(F\cap K)M=Q$. The latter implies that $F\cap Q=\mathbb{N}_F(F\cap K)(F\cap M)$; since $F\cap M$ obviously normalizes both F and K, this means that

$$F\cap Q=\mathbb{N}_F(F\cap K). \tag{3.06}$$

On the other hand, $FK=G$ yields that $Q=(F\cap Q)K$. Thus each conjugate of K in Q may be written as $t^{-1}Kt$ with $t\in\mathbb{N}_F(F\cap K)$, showing that $F\cap t^{-1}Kt=t^{-1}(F\cap K)t=F\cap K$. It follows that

$$F\cap P=F\cap K, \tag{3.07}$$

and thus also

$$F\cap Q=\mathbb{N}_F(F\cap P). \tag{3.08}$$

The two direct product conditions on $F\cap M$ involved in F being full with respect to K and high with respect to P, are therefore the same. Finally, $(F\cap Q)P/P$ avoids K/P because $F\cap K\leqslant P$ yields that $(F\cap Q)P\cap K=(F\cap Q\cap K)P=P$, and it supplements K/P by (3.01) applied to G, M, P, Q.

Conversely, suppose that F is high with respect to P and $(F\cap Q)P/P$ is a complement of K/P in Q/P. Now $FK = F(F \cap Q)K = FQ \geqslant FM = G$: thus $F \cap M \neq 1$ and (3.08) holds because F is high. On the other hand,

$$F \cap K = F \cap Q \cap K \leqslant (F \cap Q)P \cap K = P$$

because $(F \cap Q)P/P$ is assumed to avoid K/P: hence (3.07), and therefore also (3.06), now follows. The two direct product conditions again coincide, so F is full. Finally, $D_{\mathbb{F}} = Q/M$ follows from (3.06) as $Q = (F \cap Q)M$ because of $M \leqslant Q \leqslant G = FM$. This completes the proof of (3.05).

Now, consider the subset $\mathscr{F}_D$ of those $\mathbb{F}$ in $\mathscr{F}$ for which $D_{\mathbb{F}} = D = Q/M$. Because of (3.05), one can apply (3.01) to G, M, P, Q and deduce that $F \mapsto (F \cap Q)P/P$ induces a bijection of $\mathscr{F}_D$ onto the set of all conjugacy classes of complements of K/P in Q/P. Next, apply (3.04) to Q/P, M/P, K/P, N/P (legitimately, because the centre of $M/P/K/P$ is trivial), to obtain a bijection of this set onto the set $\mathscr{Y}$ of all those homomorphisms ψ of $Q/P/M/P$ into $\operatorname{Out} M/P/K/P$ whose restrictions to $N/P/M/P$ equal the coupling defined by the extension

$$(N/P) \quad 1 \to M/P/K/P \to N/P/K/P \to N/P/M/P \to 1.$$

The composite is then a bijection of $\mathscr{F}_D$ onto $\mathscr{Y}$, given as $\mathbb{F} \mapsto \psi_{\mathbb{F}}$, where $\psi_{\mathbb{F}}$ is the coupling afforded by the extension

$$1 \to M/P/K/P \to (F \cap Q)P/P \to Q/P/M/P \to 1$$

formed from $(F \cap Q)P/P$ in the same way as (F) was formed from F [recall from the penultimate paragraph of the proof of (3.04) that for complements of K/P the relevant extension has this simple form]. Consider next the natural identification of $M/P/K/P$ with M/K, and of $Q/P/M/P$ with $Q/M = D$. This takes $\operatorname{Hom}(Q/P/M/P, \operatorname{Out} M/PP/K/P)$ bijectively onto $\operatorname{Hom}(D, \operatorname{Out} M/K)$ in such a way that $\mathscr{Y}$ goes onto $\mathscr{X}_D$. On the other hand, it takes the last displayed extension to one that is equivalent, via the natural isomorphism $(F \cap Q)P/P \cong (F \cap Q)/(F \cap P) = \mathbb{N}_F(F \cap K)/(F \cap K)$, to (F): so our bijection of $\mathscr{Y}$ onto $\mathscr{X}_D$ takes $\psi_{\mathbb{F}}$ to $\chi_{\mathbb{F}}$. This proves that $\mathbb{F} \mapsto \chi_{\mathbb{F}}$ maps $\mathscr{F}_D$ bijectively onto $\mathscr{X}_D$, and hence we have a bijection of $\mathscr{F}$ onto $\mathscr{X}$.

It remains to show that this bijection is an order-anti-isomorphism. We have already seen, before the statement of (3.03), that the map itself reverses pre-order; so what is outstanding is that the inverse map also does that. In preparation for the proof of this, note that if $F \in \mathbb{F} \in \mathscr{F}$, then F, $F \cap M$, $F \cap K$, $\mathbb{N}_F(F \cap K)$ satisfy (*) and a subgroup E of F is full in F with respect to $F \cap K$ if and only if it is full in G with respect to K. Thus the part of

(3.03) already established may be applied to full subgroups E of F in two different ways. Once, as before, it associated with each such E a homomorphism

$$\chi_E: \mathbb{N}_E(E \cap K)M/M \to \text{Out } M/K$$

wich depends only on the conjugacy class of E in G and which restricts to χ_N. The second application, with reference to F in place of G, associates with E a homomorphism

$$\varphi_E: \mathbb{N}_E(E \cap K)(F \cap M)/(F \cap M) \to \text{Out}[(F \cap M)/(F \cap K)]$$

which depends only on the conjugacy class of E in F and restricts to the corresponding φ_N. Let ρ be the isomorphism of Out M/K onto Out$[(F \cap M)/(F \cap K)]$ obtained from the natural isomorphism $M/K = (F \cap M)K/K \cong (F \cap M)/(F \cap K)$, and θ the natural isomorphism $F/(F \cap M) \cong FM/M = G/M$; then φ_E is the appropriate restriction of θ followed by χ_E followed by ρ.

Remark. We may now conclude that if E, E_1, F are full in G with E and E_1 both in F and conjugate in G, they are already conjugate in F: for, conjugacy in G is equivalent to $\chi_E = \chi_{E_1}$ while conjugacy in F is equivalent to $\varphi_E = \varphi_{E_1}$.

Now suppose E is a full subgroup of G such that $D_{\mathbb{E}} \geqslant D_{\mathbb{F}}$ and χ_F is the restriction of χ_E: we want to prove that some conjugate of E lies in F. Consider the subgroup $\theta^{-1}(D_{\mathbb{E}})$ of $F/F \cap M$ and define a homomorphism φ of this subgroup into Out$[(F \cap M/(F \cap K)]$ as the restriction of θ followed by χ_E followed by ρ. By what we have already proved, there is a full subgroup E_1 in F with respect to $F \cap K$ such that $\varphi_{E_1} = \varphi$. This E_1 is then full in G with respect to K and is clearly such that $\chi_{E_1} = \chi_E$: hence E_1 is a conjugate of E. This completes the proof of Theorem(3.03).

We conclude this section with two simple lemmas needed later.

LEMMA 3.09. *If* (*) *holds, H is high, F is full, and $H \geqslant F$, then $H = G$.*

Proof. Since H is high, by part 2(a) of Lemma 2.1 of [3] we have $H = \bigcap Ht^{-1}Kt$ where t ranges through some right transversal T of N in G. As also $NH = G$, this T may as well be chosen within H. If F is full and contained in H, then $FK = G$ and hence $HK = G$: now $Ht^{-1}Kt = t^{-1}HKt = G$ for all t, and $H = G$ follows.

LEMMA 3.10. *If* (*) *holds, H is high, F is full, $H \leqslant F$, and $H \cap M > 1$, then $F = G$.*

Proof. Let T be a right transversal of $F \cap N$ in F and hence also of N in G, such that $1 \in T$. Set $S = \bigcap_{1 \neq t \in T} t^{-1}Kt$: by (*), M is the direct product of S and K. Since H is full and $H \cap M > 1$, we also have $(H \cap M)K/K \cong (H \cap M)/(H \cap K) > 1$; thus $(H \cap M)K \cap S > 1$. On the other hand, $(H \cap M)K \cap S \leqslant \bigcap Ht^{-1}Kt$ so, again by part 2(a) of Lemma 2.1 of [3], $(H \cap M)K \cap S \leqslant H$. Therefore $H \cap S > 1$. It follows that $\mathbb{N}_F(F \cap K)$ can contain no nontrivial element of T: for, if t were such an element, we would have

$$F \cap K = t^{-1}(F \cap K)t = F \cap t^{-1}Kt \geqslant F \cap S \geqslant H \cap S > 1,$$

contrary to $K \cap S = 1$. As T is a transversal of $F \cap N$ in F, this means that $\mathbb{N}_F(F \cap K) = F \cap N$; in turn, this means that the full F is in fact high: thus by (3.09) we have $F = G$.

4. Maximal Subgroups

The critical fact for this section is the following variant of the Lemma on p. 328 of Scott's [6].

Lemma 4.1. *Suppose* (*) *holds, M/K is nonabelian and simple, and the index $|N:K|$ is finite. Then a subgroup F of G is full if (and only if) $FK = G$.*

Proof. This follows from the well-known fact that if X is any group and $Y_1, \ldots, Y_n$ are pairwise distinct normal subgroups of X with each X/Y_i nonabelian and simple, then $X/\bigcap_{i=1}^n Y_i = \prod_{i=1}^n X/Y_i$. Indeed, $FK = G$ implies that $(F \cap M)/(F \cap K) \cong (F \cap M)K/K = M/K$ and the finiteness of $|G:N|$ implies that $F \cap K$ has only finitely many distinct conjugates, say $Y_1, \ldots, Y_n$ in F: thus the above result, with $X = F \cap M$, yields our claim.

The point of this for us lies in the next result.

Lemma 4.2. *Suppose* (*) *holds, M/K is nonabelian and simple, and $|G:N|$ is finite. If V is a proper subgroup of G supplementing M in G, then either V is full, or V is contained in some high proper subgroup.*

Proof. By Lemma 2.2 of [3], the intersection H of the subsets $Vt^{-1}Kt$ (with t ranging through some right transversal of N in G) is a high subgroup, which obviously contains V. If H is not proper, then 4.1 ensures that V is full.

For ease of expression, we give the main result of the paper as a counting theorem. Although it could be rephrased as asserting the (set-theoretic) equivalence of two sets, what the proof really shows is not only the existence of such an equivalence but the "natural" equivalence of the two sets, in an appropriate sense which could only be made rigorous in the language of categories instead of sets.

THEOREM 4.3. *Suppose* $(*)$ *holds,* M/K *is nonabelian and simple, and* $|G:N|$ *is finite. Then the cadinality of the set of the conjugacy classes of the maximal subgroups of* G *not containing* M *is* $a+b+c$, *where*

> *a is the sum, over all subgroups* D *of* G/M *minimal with respect to properly containing* N/M, *of the cardinalities of the sets* $\mathscr{X}_D$ *of all those homomorphisms of* D *into* Out M/K *whose restrictions to* N/K *are equal to the coupling* χ_N *afforded by the extension*
>
> $$(N) \quad 1 \to M/K \to N/K \to N/M \to 1;$$
>
> *b is the cardinality of the set of the conjugacy classes in* N/K *of those maximal subgroups of* N/K *which neither avoid nor contain* M/K; *and*
>
> *c is the cardinality of the set of the conjugacy classes of those maximal subgroups of* N/K *which complement* M/K *and have the property that the homomorphisms of* N/M *into* Aut M/K *determined by them are not restrictions of homomorphisms into* Aut M/K *from any subgroup of* G/M *properly containing* N/M.

Remark (added in proof). Alternatively, one might say that b *is the number of conjugacy classes of those maximal subgroups in the nearly simple group* N/Z *which neither avoid nor contain the simple normal subgroup* MZ/Z: here Z is defined by $Z/K = \mathbb{C}_{N/K}(M/K)$. Indeed, if L/Z is such a subgroup, then L/K is obviously maximal in L/K and can neither avoid nor contain M/K. Conversely, suppose L/K is such a subgroup in N/K: if we can show that $L \geqslant Z$ it will be obvious that L/Z has the required properties. Let L_0/K be the largest normal subgroup of N/K contained in L/K. As L_0/K cannot contain, it must avoid and hence centralize the minimal normal subgroup M/K: so $L_0 \leqslant Z$. If $L_0 = Z$, we are done. If not, then Z/L_0 and ML_0/L_0 form a disjoint pair of nontrivial normal subgroups in the group N/L_0 with a corefree maximal subgroup L/L_0, so L/L_0 must complement ML_0/L_0 on account of Lemma 5.1; but this is impossible since we have assumed that $L/K \cap M/K > 1$.

Proof of Theorem 4.3. It follows from (4.2) that a maximal subgroup of G not containing M is either high or full. Towards the converse, note first that if H is maximal among the high subgroups of G and $H \cap M > 1$, then (4.2) and (3.10) yield that H is maximal in G. Second, note that, by

(4.2) and (3.09), if F is maximal among the full subgroups of G, then F is maximal in G. Thus (3.01) and (3.03) justify the roles of b and a in (4.3). To complete the proof, it suffices to show that a complement H of M in G is contained in some full proper subgroup F of G if and only if the homomorphism $\lambda: N/M \to \operatorname{Aut} M/K$ defined by the complement $(H \cap N)K/K$ of M/K in N/K is the restriction of a homomorphism $\zeta: D \to \operatorname{Aut} M/K$ for some D with $N/M < D \leqslant G/M$.

To see this, first suppose that H is contained in such an F. Set $Z = [H \cap \mathbb{N}_F(F \cap K)](F \cap K)$, and observe that $Z/(F \cap K)$ is then a complement of $(F \cap M)/(F \cap K)$ in $\mathbb{N}_F(F \cap K)/(F \cap K)$. In the context of the extension

$$(F) \quad 1 \to M/K \to \mathbb{N}_F(F \cap K)/(F \cap K) \to D_F \to 1$$

discussed before, this gives rise to a homomorphism $\zeta: D_F \to \operatorname{Aut} M/K$. As we saw there, the natural isomorphism of $(F \cap N)/(F \cap K)$ onto N/K is an equivalence from the restriction (to N/M) of (F) to the extension (N), so the restriction of ζ to N/M is indeed λ. In view of (3.03), the assumption $F < G$ implies $D_F > N/M$.

For the converse, suppose λ is the restriction of a ζ. Let ω stand for the natural homomorphism of $\operatorname{Aut} M/K$ onto $\operatorname{Out} M/K$. Since the restriction of $\omega\zeta$ is $\omega\gamma = \chi_N$, we have $\omega\zeta \in \mathscr{X}_D$. By (3.03), there exists then a full proper subgroup F with $\chi_F = \omega\zeta$. Accordingly, in (F), $(F \cap M)/(F \cap K)$ has a complement $Z/(F \cap K)$ in $\mathbb{N}_F(F \cap K)/(F \cap K)$ which affords ζ: retracing the steps above, we see that the restriction of ζ being λ means precisely that $(Z \cap N)K = (H \cap N)K$. Now apply (3.01) to F, $F \cap M$, $F \cap K$, $\mathbb{N}_F(F \cap K)$, to conclude that $F \cap M$ must have a complement H_1 in F such that $[H_1 \cap \mathbb{N}_F(F \cap K)](F \cap K) = Z$. This H_1 also complements M in G, and has the property that $(H_1 \cap N)K = (Z \cap N)K = (H \cap N)K$. Finally, apply (3.01) in the original setting: H_1 and H must be conjugate in G, so H must lie in some conjugate of F. This completes the proof of Theorem 4.3.

Remark. It is straigtforward to see that the index (in G) of a full subgroup F whose conjugacy class corresponds to a coupling in $\mathscr{X}_{Q/M}$ is $|M/K|^{|G:N| - |G:Q|}$, while the index of a high subgroup H is given by $|M : (H \cap M)K|^{|G:N|}$.

5. Corefree Maximal Subgroups

In our search for an overview of the maximal subgroups of an arbitrary finite group G, we could have started on a different tack. Namely, suppose we know the maximal subgroups of all proper factor-groups of G; by the inclusion-exclusion principle, this accounts for all those maximal subgroups of G which contain at least one nontrivial normal subgroup of G. The problem is then to find the corefree maximal subgroups: those which contain no nontrivial normal subgroup of G. Towards the solution of this, one may exploit the following general facts which seem to be well known and whose proofs (from first principles and without any reference to the foregoing) are so straightforward that to spell them out would be an insult to the reader.

LEMMA 5.1. *Let M_1 and M_2 be nontrivial normal subgroups of a (not necessarily finite) group G, and μ_i the natural homomorphism of G/M_i onto G/M_1M_2. Suppose that $M_1 \cap M_2 = 1$. Then G has corefree maximal subgroups if and only if the following three conditions are satisfied:*

(i) *the M_i are nonabelian;*

(ii) *each nontrivial normal subgroup of G/M_i contains M_1M_2/M_i;*

(iii) *there exist isomorphisms γ of G/M_1 onto G/M_2 such that $\mu_2\gamma = \mu_1$.*

Moreover, if (i) *and* (ii) *hold, the set of all corefree maximal subgroups of G is equivalent to the set of all these isomorphisms, in such a way that two subgroups are conjugate if and only if the quotient of the corresponding isomorphisms is an inner automorphism induced by some element of M_1M_2/M_2. (Explicitly: a corefree maximal subgroup H of G must complement each M_i in G, and so $\gamma(M_1h) = M_2h$ defines an isomorphism; let this γ correspond to H.)*

In the program outlined in the Introduction, we do not need to apply Lemma 5.1 until the problem has been reduced to the following setting.

COROLLARY 5.2. *Let G be a finite group in which at least one normal subgroup is nonabelian simple. If that is the only minimal normal subgroup, G is nearly simple: suppose this is not the case.*

Then G has no corefree maximal subgroups unless there are precisely two minimal normal subgroups, say M_1 and M_2, and the factor groups G/M_1, G/M_2 are isomorphic nearly simple groups. In this exceptional case, the number of corefree maximal subgroups in G is the number of isomorphisms γ of G/M_1 onto G/M_2 such that $\mu_2\gamma = \mu_1$.

Differently put: G has no corefree maximal subgroups unless G is the subdirect square of some nearly simple group R defined by the pullback diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & R \arrow[d, "\omega"] \\ R \arrow[r, "\omega"] & R/S \end{tikzcd}$$

where S is the simple normal subgroup of R and ω the natural homomorphism of R onto R/S; in this exceptional case, the number of conjugacy classes of corefree maximal subgroups in G is the order of the centralizer of R/S in Out S. (*Yet another way to construct this exceptional G is as the semidirect product of S by R given by the composite of $r \mapsto \operatorname{inn} r$ and the restriction map* $\operatorname{Inn} R \to \operatorname{Aut} S$: *then G appears as a subgroup of the holomorph of S.*)

This completes the program set down in the Introduction.

6. Postscript

Any overview of the conjugacy classes of maximal subgroups in a group G may be thought of as a description of the primitive permutation representations of G: this is, of course, how our exposition is related to the O'Nan–Scott context. We quoted from [3], as (3.01) here, that if G, M, K, N satisfy (*) then the set $\mathscr{H}$ of conjugacy classes of high subgroups H of G is equivalent to the set of conjugacy classes of supplements L/K of M/K in N/K, the natural equivalence being induced by $H \mapsto (H \cap N)K/K$. We now recall (from the first paragraph of the proof of Theorem 4.2 in [3]) also the definition of the inverse of this map: G is embedded in the (unrestricted, permutational) wreath product of N/K by a certain (transitive) permutation group; the wreath product of L/K by the same permutation group is viewed as a subgroup of W, and its intersection with G is a high subgroup whose conjugacy class corresponds to that of L/K. In other words, the permutation representation of G on the set of cosets of a high subgroup H is the restriction to G of the permutation representation of the wreath product W on the set of cosets of the wreath product of $(H \cap N)K/K$ with the same permutation group used in forming W. Such permutation representations of wreath products have become known as "product actions".

Consider now the case of a group G with a nonsimple minimal normal subgroup M that is the direct product of finitely many nonabelian simple groups. Let K be the product of all but one of these simple direct factors of M, and N the normalizer of K in G: then G, M, N, K satisfy (*) and Theorem 4.3 applies. Using also (3.05), we highlight some of the conclusions in terms of those primitive permutation representations of G whose kernels do not contain M. If N is not maximal in G, then each such representation is the restriction of a primitive product action of some wreath product W containing G [namely, W is either the wreath product of N/K by the permutation group induced by G on the set of cosets of N, or the wreath product of Q/P by the permutation group induced by G on the set of cosets of Q where Q is minimal among those subgroups of G which properly contain N and P is the intersection of the conjugates of K in Q]. If N is maximal in G, there may also be such representations which are not restrictions of product actions of wreath products [namely, up to equivalence of permutation representations, precisely one for each homomorphism, if any, of G/M into Out M/K whose restriction to N/M agrees with the coupling defined by the extension (N) discussed in Sect. 3].

The actual context of the O'Nan–Scott Theorem is that of permutation groups rather than of permutation representations of groups. To complete the translation, we should therefore say just how many of the maximal subgroups counted in Theorem 4.3 are corefree: in other words, which of the primitive permutation representations are faithful. The first part of the answer is: they all are if every nontrivial normal subgroup of G contains M; since M is assumed nonabelian and minimal in G, this is equivalent to requiring that the centralizer $\mathbb{C}_G(M)$ of M in G be trivial. In the spirit of [3] we mention that one can recognize, purely in terms of the homomorphism $\alpha: A = N/K \to B = G/M$, whether this is the case: check whether the image under α, of the centralizer in A of the kernel of α, is corefree in B. [In particular, in (3.01) it can well happen that all high maximal subgroups H are corefree in G even if none of the corresponding maximal subgroups $(H \cap N)K/K$ of N/K are corefree in N/K.] The second part of the answer is that if α fails this test then $\mathbb{C}_G(M) > 1$ and so Lemma 5.1 becomes applicable. Write S for the kernel M/K of α, and D for the intersection of the conjugates of $\alpha(\mathbb{C}_A(S))$ in B. [The failure of the first test means precisely that $D > 1$.] What Lemma 5.1 yields is that now G has no corefree maximal subgroups unless the following hold. The centralizer $\mathbb{C}_B(D)$ is trivial; there is a subgroup E in D such that $B, D, E, \alpha(A)$ satisfy (*); and there is a homomorphism π of A onto $\alpha(A)/E$ such that

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\pi"'] & \alpha(A) \arrow[d] \\ \alpha(A)/E \arrow[r] & \alpha(A)/D \end{tikzcd}$$

is a pullback where the unnamed maps are the natural homomorphisms of $\alpha(A)$ and of $\alpha(A)/E$ onto $\alpha(A)/D$. Differently put: the exceptional cases are precisely those which can be built by starting with any B, D, E, C satisfying (*) and such that D/E is nonabelian simple, $|B : C|$ is finite, and $\mathbb{C}_B(D) = 1$ (the latter being recognizable in terms of the homomorphism $C/E \to B/D$ alone); defining A and α with $\alpha(A) = C$ as constituents of the pullback above; and finally building the "induced extension" G, M, K, N defined by $\alpha: A \to B$ as in Section 3 of [3]. In this exceptional case, the number of conjugacy classes of corefree maximal subgroups of G is the order of $\mathbb{C}_{\mathrm{Out}\, D}(B/D)$. The corefree maximal subgroups H now complement M in G, so their conjugacy classes are among the c counted last in Theorem 4.3. One can also say, in terms of the corresponding maximal subgroups $(H \cap N)K/K$ complementing M/K in N/K, just which maximal complement H is corefree in G. To do this satisfactorily, one observes first that the subgroup $\mathbb{C}_G(M)K/K$ of $A = N/K$ is the intersection of $\mathbb{C}_A(S)$ with the complete inverse image of D under α, so this subgroup is recognizable in terms of α alone. A maximal subgroup H complementing M in G is corefree if and only if $(H \cap N)K/K$ does not contain $\mathbb{C}_G(M)K/K$.

It remains to acknowledge some minor conflicts with the two printed versions of the O'Nan–Scott Teorem. From part (a) of the theorem on p. 328 of [6], the word "prime" should be omitted; and consequently we cannot see why, in case (e) of the Theorem of the next page, the parameter p should not need to range over composite numbers as well as primes.[3] In part (ii)(a) of Theorem 4.1 of [2], "the socle" should be replaced by "a minimal normal subgroup": I am indebted to Dr T. M. Gagen for directing my attention to this point.

References

1. M. Aschbacher and L. Scott, Maximal subgroups of finite groups, *J. Algebra* **92** (1985), 44–80.
2. P. J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* **13** (1981), 1–22.
3. F. Gross and L. G. Kovács, On normal subgroups of groups which are direct products, *J. Algebra* **90** (1984), 133–168.
4. S. Mac Lane, "Homology," Die Grundlehren der Mathematischen Wissenschaften, 114. Springer-Verlag, Berlin/Göttingen/Heidelberg, 1963.
5. Derek J. S. Robinson, "A Course in the Theory of Groups," Graduate Texts in Mathematics, 80. Springer-Verlag, New York/Heidelberg/Berlin, 1982.
6. Leonard L. Scott, Representations in characteristic p, *in* "The Santa Cruz Conference on Finite Groups," (Bruce Cooperstein and Geoffrey Mason, Eds.), pp. 319–331, Proc. Symposia in Pure Maths, No. 37, Amer. Math. Soc., Providence, R. I., 1980.

[3] These have been corrected in the Appendix of Aschbacher and Scott [1].