SOME FITTING FORMATIONS OF FINITE
SOLUBLE GROUPS

L.G. Kovács

RESEARCH REPORT No. 37 - 1982

Mathematics Research Report
SOME FITTING FORMATIONS OF FINITE SOLUBLE GROUPS

L.G. Kovács

In his recent thesis [9], C.L. Kanes constructed an interesting new family of Fitting formations and proved that not all members of this family are saturated. His attempt to find a necessary and sufficient condition of saturation in this context was only partly successful. This note presents such a condition.

Standard terminology will be used: see for instance Gaschütz [5]. Let $F$ be an algebraically closed field of prime characteristic $q$; let $\pi$ be a set of primes (with $\pi'$ the complementary set), $X$ a Fitting formation in the product class $S_\pi S_{\pi'}$, and $H^\pi_q(X)$ the class of all those groups $G$ in $S_q S_{\pi'} X$ which satisfy the following condition: if $U$ is an irreducible submodule of $F \otimes V$ for some $q$-chieffactor $V$ of $G$, then $U$ regarded as $0_q q^\pi(G)$-module is homogeneous (that is, a direct sum of pairwise isomorphic irreducible modules). By Theorem 5.2.2 of Kanes [9], each such class $H^\pi_q(X)$ is a Fitting formation. His Theorem 5.3.4 states in effect that when $q \notin \pi$, this formation is saturated if and only if $X$ lies in the class $S_\pi \vee S_{\pi'}$, of all direct products of $\pi$-groups and $\pi'$-groups. The general version is the following.
THEOREM. The formation $H^\pi_q(X)$ is saturated if and only if 
$X \subseteq S_q(S_{\pi} \vee S_{\pi'})$, and in that case $H^\pi_q(X) = S_qS_qX$.

Thus when $H^\pi_q(X)$ is really a "new" Fitting formation (rather than an "old" one easier described as a product), it is never saturated. Note that in the case $q \nmid \pi$, the assumption $X \subseteq S_qS_{\pi}$, ensures that $X \subseteq S_q(S_{\pi} \vee S_{\pi'})$ is equivalent to $X \subseteq S_{\pi} \vee S_{\pi'}$, so this result is in agreement with the theorem of Kanes paraphrased above. The relevant special case of the proof given here is essentially his.

It is obvious that if $X \subseteq S_q(S_{\pi} \vee S_{\pi'})$ then 
$H^\pi_q(X)$ is the whole class $S_qS_qX$ and of course this formation is always saturated. Suppose that $X \nsubseteq S_q(S_{\pi} \vee S_{\pi'})$ yet $H^\pi_q(X)$ is saturated: the Theorem will be proved by showing that this leads to a contradiction.

The argument is carried out in three steps. The central step is the construction of two groups, $G_1$ and $G_2$ say, with the following properties. First, $G_1 \in X \cap H^\pi_q(X)$, 
$O_{\pi'}(G_1) = 1$, and $G_2$ lies in the formation generated by $G_1$. Second, there exists a faithful irreducible $F\Phi_{\pi}(G_1)$-module $W_1$ which is $G_1$-invariant (that is, isomorphic to each of its conjugates by elements of $G_1$). Third, there is an irreducible $F\Phi_{\pi}(G_2)$-module $W_2$ which is not $G_2$-invariant. Once we are in possession of these groups and modules, the final step goes easily, as follows.

By Theorem 7.1 of Dade [4] (see also Isaacs [8]), $W_1$ is the restriction of some $F\Phi_{\lambda}$-module $U_1$ (which is faithful since
\[0_{π}(G_1) = 1, \text{ and of course irreducible}; \text{ in turn, } U_1 \text{ is a}
\text{submodule of } F \otimes V_1 \text{ for some faithful irreducible } F_{q_1}\text{-module}
V_1. \text{ One readily sees that the semidirect product } V_1 G_1 \text{ lies}
in \text{ } H_q^{π}(X). \text{ On the other hand, } W_2 \text{ is a submodule of the}
\text{restriction of some irreducible } F_{q_2}\text{-module } U_2, \text{ and as } W_2 \text{ is}
\text{not } G_2\text{-invariant that restriction is certainly not homogeneous.}
\text{Take an irreducible } F_{q_2}\text{-module } V_2 \text{ such that } F \otimes V_2 \text{ contains}
U_2; \text{ then } V_2 G_2 \not\subset H_q^{π}(X). \text{ Since our formation is saturated, it}
\text{has a "full integrated local definition" (see Carter and Hawkes}
[3]); \text{ the formation } F \text{ corresponding to the prime } q \text{ in that is}
such that } F = S_q F \subset H_q^{π}(X) \subset S_q, F. \text{ As } V_1 \text{ is faithful,}
0_q'(V_1 G_1) = 1, \text{ so } V_1 G_1 \in H_q^{π}(X) \subset S_q, F \text{ implies that } V_1 G_1 \in F.\n\text{Because } G_2 \text{ lies in the formation generated by } G_1, \text{ we get that}
G_2 \in F; \text{ thus } V_2 G_2 \in S_q F \subset H_q^{π}(X), \text{ and we have the desired}
\text{contradiction.}

The first step is to consider a group } H \text{ of}
\text{minimal order among all groups contained in } X \text{ but not in}
S_q\left( S_{π} \vee S_{π'} \right). \text{ Such an } H \text{ must clearly have a unique maximal}
\text{normal subgroup, say } M, \text{ and a unique minimal normal subgroup,}
say } P. \text{ Let } p^k \text{ denote the order of } P, \text{ with } p \text{ prime. Since}
H/P \text{ lies in } S_q\left( S_{π} \vee S_{π'} \right) \text{ but } H \text{ does not, } p \neq q; \text{ since}
H \in X \subset S_{π} S_{π'}, \text{ but } H \not\subset S_{π'}, \text{ we must have } P \leq O_q(H), \text{ so}
p \in π. \text{ Thus in turn we obtain that } 0_q(H) = 1, \quad 0_q(M) = 1,
M = 0_{π}(M) \times O_{π',1}(M), \text{ but also } 0_{π}(H) = 1, \quad 0_{π}(M) = 1; \text{ so } M
\text{is a } π\text{-group. As } H \not\subset S_{π}, \text{ the prime index, say } r, \text{ of } M \text{ in}
H must lie in \( \pi' \). Thus H has no nontrivial \( \pi \)-quotient while \( H/P \notin S_{q}(S_{\pi} S_{\pi'}) \), so in fact \( H/P \notin S_{\pi} S_{\pi'} \). It follows that \( M/P \notin S_{\pi} \cap S_{q} S_{\pi'} \): in other words, \( M/P \) is a \( q \)-group which can only be nontrivial if \( q \in \pi \).

If \( M/P = 1 \), we may now proceed with the central step as Kanes did in the case \( q \notin \pi \). Set \( G_2 = H \) and \( W_2 \) any nontrivial irreducible FP-module. For \( G_1 \) take a semidirect product of an extraspecial group of order \( p^{2k+1} \) and a group of order \( r \), the latter acting on the Frattini quotient of the first as a Sylow \( r \)-subgroup of \( H \) acts on the direct sum of \( P \) with its contragredient, and so that \( G_1 \) have centre of order \( p \).

The Frattini quotient of \( G_1 \) is then a subdirect square of \( H \) and so lies in the metanilpotent Fitting formation

\[ X \cap H^\pi(X) \cap S_{q} S_{\pi'} : \text{by Hawkes [6]} \] (see also Bryce and Cossey [2]),

all such formations are saturated, so \( G_1 \in X \cap H^\pi(X) \). Faithful irreducible modules of extraspecial groups are invariant under group automorphisms which act trivially on the centre (see Huppert [7], V.16.14): hence any such \( P_{0}(G_1) \)-module will serve as \( W_1 \).

It is somewhat harder to deal with the case \( M/P \neq 1 \).

As noted above, in this case \( q \notin \pi \); so \( 0_{\pi}(H) = M \) and \( r \neq q \).

Let \( Q \) be a Sylow \( q \)-subgroup of \( M \) and \( R \) a Sylow \( r \)-subgroup of \( H \): by the Frattini argument, \( R \) can be chosen to normalize \( Q \).

As \( PQ \) is the only maximal normal subgroup, the mutual commutator subgroup \( [Q, R] \) must then be \( Q \). Note also that the centralizer \( C_{p}(Q) \) must be trivial, else \( P \) would not be the only minimal
normal subgroup.

If \([P, R] \subset P\), set \(G_1 = H\). Let \(P_0\) be a maximal subgroup of \(P\) containing \([P, R]\); let \(Q_0\) be the largest subgroup of \(Q\) which normalizes \(P_0\) and acts trivially on \(P/P_0\). By our choice of \(P_0\) we know that \(P_0\), and hence also \(Q_0\), is normalized by \(R\); also, \(P_0Q_0\) is normal in \(PQ_0\). Let \(W_1\) be the FM-module induced from any 1-dimensional \(FPQ_0\)-module \(W\) with kernel \(P_0Q_0\). The maximal choice of \(Q_0\) ensures that \(PQ_0\) is the "inertia subgroup" in \(M\) of the restriction of \(W\) to \(P\); hence \(W_1\) is irreducible. As \(R\) acts trivially on \(PQ_0/P_0Q_0\) we know that \(W\) is \(R\)-invariant: hence so is \(W_1\). It follows that the kernel of \(W_1\) is normal in \(G\); as it does not contain \(P\), it must be trivial.

If \([P, R] = P\), we construct \(G_1\) as a group which is like \(H\) in every relevant respect except this. (Of course \(G_1\) will not necessarily retain the minimal property of \(H\), either. The construction will make no use of \([P, R] = P\), and could be performed in any case; the only reason for handling \([P, R] \subset P\) above separately was to explain, before getting submerged in other complications, just how the property of \(G_1\) corresponding to \([P, R] \subset P\) will be exploited.) Let \(1 \neq h \in R\) so \(R = \langle h \rangle\), and set \(h_1 = \{h^{\pm 1}, h\} \in QR \times QR\), \(Q_1 = Q \times Q \leq QR \times QR\), \(R_1 = \langle h_1 \rangle \leq QR \times QR\): then \(Q_1R_1\) is normal in the direct product \(QR \times QR\), and \([Q_1, R_1] = Q_1\) is easy to verify. Write \(P \not\subset P\) for \(P \otimes P\) viewed as \(QR \times QR\)-module, and note the following facts. As \(QR \times 1\) or \(1 \times QR\)
module, \( P \# P \) is a direct sum of "isomorphic copies" of \( P \).

It follows from \( C_P(Q) = 1 \) and \( Q \times 1 \leq Q_1 \) that \( C_{P\#P}(Q_1) = 1 \).

On the other hand, \( C_{P\#P}(R_1) > 1 \), for the Kronecker product of a matrix and its inverse must have at least one eigenvalue \( 1 \).

Now consider the semidirect product \((P\#P)(QR \times QR)\). This is a product of two normal subgroups, \((P\#P)(QR \times 1)\) and \((P\#P)(1 \times QR)\), each of which is a subdirect power of \( H \); therefore it lies in the Fitting formation \( X \cap H^\pi_q(X) \). Consequently, so does its normal subgroup \((P\#P)Q_1 R_1\). By Maschke's Theorem, \( P \# P \) is completely reducible as \( Q_1 R_1 \)-module: so the observation

\( C_{P\#P}(R_1) > 1 \) above yields that \( P \# P \) has an irreducible \( Q_1 R_1 \)-submodule \( P_1 \) such that \( C_{P_1}(R_1) > 1 \). Let \( S \) be a \( Q_1 R_1 \)-submodule complementing \( P_1 \) in \( P \# P \), and \( T \) a normal subgroup of \((P\#P)Q_1 R_1\) maximal with respect to containing \( S \) but avoiding \( P_1 \). We set \( G_1 = (P\#P)Q_1 R_1 / T \). Clearly, \( P_1 T / T \) is operator-isomorphic to \( P_1 \) and is both the unique Sylow \( p \)-subgroup and the unique minimal normal subgroup of \( G_1 \). As \( C_{P\#P}(Q_1) = 1 \), we have \([P_1, Q_1] = P_1 \), so \( Q_1 \not\leq T \); thus by \([Q_1, R_1] = Q_1 \) we also have \( R_1 \not\leq T \). On the other hand, \( C_{P_1}(R_1) > 1 \) ensures that \([P_1 T / T, R_1 T / T] < P_1 T / T \). These facts guarantee that the present \( G_1 \) has all the relevant properties of the \( G_1 \) of the previous paragraph: one may choose \( W_1 \) as before.

To simplify the description of \( G_2 \) and \( W_2 \), we change notation so from now on \( P, Q, R \) will stand for appropriate Sylow subgroups of \( G_1 \). If \( P \) has a maximal subgroup \( P_2 \) not
containing any conjugate of \([P, R]\), we can take \(G_2 = G_1\).

Indeed, in that case let \(W\) be a 1-dimensional FP-module with kernel \(P_2\), and \(W_2\) any irreducible FPQ-module whose restriction to \(P\) contains \(W_2\) (that is, any irreducible quotient of the PQ-module induced from \(W\)). By its choice, \(W\) is not invariant under any conjugate of \(R\): so the number of isomorphism types of \(G_2\)-conjugates of \(W\) is divisible by \(r\). If \(W_2\) were \(G_2\)-invariant, the set of isomorphism types of irreducible submodules of the restriction of \(W_2\) to \(P\) would be \(G_2\)-invariant; in any case, by Clifford's Theorem, that set is a single \(Q\)-orbit: thus the cardinality of this set would have to be a power of \(q\) divisible by \(r\). As this is impossible, \(W_2\) is not \(G_2\)-invariant, and we are done.

If \(P\) does not have such a maximal subgroup \(P_2\), we choose \(G_2\) differently: not as \(G_1\), the semidirect product of \(QR\) with the QR-module \(P\), but as the semidirect product of \(QR\) with the \(k\)-fold direct power \(P^k\) of this QR-module. Then \(G_2\) is a subdirect power of \(G_1\), so it lies in the formation generated by \(G_1\). If we can find a maximal subgroup \(P_2\) in \(P^k\) not containing any conjugate of \([P^k, R]\), we can choose \(W_2\) as before. Recall that \(k\) was chosen so that \(P\) itself is the \(k\)-fold direct power of a group \(C\) of order \(p\); if \(\varphi_1, \ldots, \varphi_k\) are the corresponding coordinate-projections of \(P\) onto \(C\), then the intersection of the kernels of these homomorphisms is trivial. Define a homomorphism \(\varphi\) from \(P^k\) to \(C\) by \((x_1, \ldots, x_k)\varphi = \prod x_i \varphi_i\); we claim the kernel of \(\varphi\) can serve as
$P_2$. As $[P, R] \neq 1$ (else the normal closure of $R$, which includes $Q$ because $[Q, R] = Q$, would also act trivially on $P$, contrary to $C_P(Q) = 1$), to each element $g$ of $QR$ there is an index $i(g)$ such that $[P, R]^{\theta_{i(g)}} \neq 1$. Accordingly, there is an element $y_g$ in $P$ such that $[-y_g, h^g]^{\theta_{i(g)}} \neq 1$ (here, as before $\langle h \rangle = R$). Let $z_g$ be the element of $P^k$ whose components are all trivial except the $i(g)$-component which is $y_g$: then all components of $[z_g, h^g]$ are trivial except the $i(g)$-component which is $[-y_g, h^g]$, so.

$[z_g, h^g]^{\theta} = [-y_g, h^g]^{\theta_{i(g)}} \neq 1$. This shows that $[P^k, R^g]$, that is, $[P^k, R]^g$, is not contained in $P_2$, and the proof of the Theorem is complete.

In conclusion, note that when $q = 2$, $\pi = \{2, 13\}$, and $X = S_1^3S_2^3S_3^3 \cap V_{13}^3$ where $V_{13}^3$ is as defined in the paper [1] of Berger and Cossey, the smallest group $H$ in $X$ but outside $S_q(S_{\pi} \cap S_{\pi})$ has $H/P \cong SL(2, 3)$ and $[P, R] = P$; so the hard version of the case $M/P \neq 1$ genuinely does arise and must be coped with. On the other hand, when $q = 31$, $\pi = \{5, 31\}$, and $X = S_5^5S_{31}^5S_3^3 \cap V_{5}^3$, we find that $H/P$ is the nonabelian group of order $93$ and each of the $31$ maximal subgroups of the group $P$ of order $5^3$ is a conjugate of $[P, R]$; so the easy choice of $G_2$ is not always available, either.
References


[9] C.L. Kanes, Constructions for Fitting formations (thesis submitted to the Australian National University in September, 1982.)